Decision-Feedback Equalization Achieves Full Diversity for Finite Delay Spread Channels

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Abstract — **We consider a SIMO or SISO communication link. In the case of white noise, the Matched Filter Bound (MFB) is proportional to the total channel energy. Hence all diversity sources present in the channel show up in the MFB. The MFB usually is a close approximation for the performance of Maximum Likelihood Sequence Detection (MLSD) and represents an upper bound for the performance of any receiver (Rx). In this paper we consider the diversity performance of suboptimal Rx's of the decision feedback equalization (DFE) type. Two DFE designs are considered: Minimum Mean Squared Error (MMSE) or MMSE Zero-Forcing (MMSE-ZF). The SNR at the detection point of a MMSE(-ZF) DFE exhibits a performance loss w.r.t. the MFB, a loss that is determined by the energy in the feedback filter. It is shown that there exists an upperbound for this loss that is channel independent. Hence the DFE enjoys as much diversity as the MFB.**

Consider a linear modulation scheme and transmission over a Single Input Single Output (SISO) or Multiple (N_r) Output (SIMO) linear channel with additive white noise. After a Rx filter (possibly noise whitening [1, Chap. 8]) and sampling, we obtain a discrete-time system at symbol rate with a possible vectorial Rx signal due to the use of multiple receive antennas, oversampling w.r.t. the symbol rate etc.

$$\underbrace{\boldsymbol{y}_k}_{N_r\times 1} = \underbrace{\mathbf{h}[q]}_{N_r\times 1}\,\underbrace{a_k}_{1\times 1} + \underbrace{\boldsymbol{v}_k}_{N_r\times 1}\,,\quad \mathbf{h}[z]=\sum_{i=0}^{L}\mathbf{h}_i\,z^{-i} \tag{1}$$

where the noise power spectral density matrix is $S_{\boldsymbol{vv}}(z)=\sigma_v^2\,I$, $q^{-1}\,a_k=a_{k-1}$, q^{-1} is the unit sample delay operator, and $\mathbf{h}[z]$ is the channel transfer function in the z domain. The channel delay spread is L symbol periods. In the Fourier domain we get $\boldsymbol{h}(f)=\mathbf{h}[e^{j2\pi f}]$. We get for the MFB $=\rho\,\|\mathbf{h}\|^2$, $\rho=\dfrac{\sigma_a^2}{\sigma_v^2}$ with $\|\mathbf{h}\|^2=\sum_{i=0}^{L}\|\mathbf{h}_i\|_2^2$ where e.g. $\|\mathbf{h}_i\|_2^2=\mathbf{h}_i^H\mathbf{h}_i$ and $\mathbf{h}^\dagger[z]=\mathbf{h}^H[1/z^*]$ denotes the paraconjugate (matched filter). As $\|\mathbf{h}\|^2$ is the sum of all magnitude squared channel coefficients, the MFB benefits from all diversity sources that are resolvable spatially and/or temporally by the channel coefficients. Now consider a suboptimal detector, namely a DFE, which needs to suppress Inter-Symbol Interference (ISI) and one may fear that some diversity sources may be sacrificed due to this (consider e.g. the V-BLAST Rx). Let $\alpha=1$ for an Unbiased MMSE (UMMSE) design and $\alpha=0$ for MMSE-ZF ("optimal ZF" in [1, Chap. 8]), then the SNR at the DFE output satisfies

$$\int_{-\frac{1}{2}}^{-\frac{1}{2}}\ln\left(\alpha+\rho\,\|\mathbf{h}(f)\|_2^2\right)\,df=\ln\left(\alpha+\mathrm{SNR}_{DFE}^{\alpha}\right)\,. \tag{2}$$

In the case of UMMSE, (2) leads to the label "canonical receiver" since it shows that the SIMO channel capacity is that of an AWGN channel equal to the cascade of SIMO channel and DFE with same SNR. Straightforwardly, $\mathrm{SNR}_{DFE}^{\alpha}\le$ MFB. Now consider the following spectral factorization

$$S[z]=\alpha+\rho\,\mathbf{h}^\dagger[z]\,\mathbf{h}[z]=\beta\,p^\dagger[z]\,p[z] \tag{3}$$

where $p[z]$ is a scalar monic minimum phase polynomial in z^{-1} of order L (the normalized spectral factor of the spectrum $S[z]$). Since β is the infinite order prediction error variance of the spectrum $S[z]$, we have from (2)

$$\beta=\alpha+\mathrm{SNR}_{DFE}^{\alpha} \tag{4}$$

where β captures in fact all diversity sources as we shall show. The feedforward filter for both designs is $\mathbf{f}(z)=\frac{\rho\mathbf{h}^\dagger[z]}{\beta p^\dagger[z]}$ and the feedback filter $b[z]-1=p[z]-1$. Note that the ZF designs are far from unique since for ZF it suffices to satisfy $\mathbf{f}[z]\mathbf{h}[z]=b[z]$ (causal monic). By integrating both sides of (3), we get

$$\alpha+\mathrm{MFB}=\alpha+\rho\int_{-\frac{1}{2}}^{-\frac{1}{2}}\|\mathbf{h}(f)\|_2^2\,df=\beta\int_{-\frac{1}{2}}^{-\frac{1}{2}}|p(f)|^2\,df$$
$$=\beta\,\|p\|^2\le\beta\,c_L\,,\quad c_L=\sum_{l=0}^{L}\left(\begin{array}{c} l\\ L\end{array}\right)^2$$

since minimum phase filter coeffients are bounded. We can now summarize upper and lower bounds as

$$\frac{\alpha+\mathrm{MFB}}{c_L}\;\le\;\alpha+\mathrm{SNR}_{DFE}^{\alpha}\;\le\;\alpha+\mathrm{MFB}\;. \tag{5}$$

Since diversity is a property at high SNR, we get as $\rho\to\infty$

$$\frac{1}{c_L}\,\mathrm{MFB}\;\le\;\mathrm{SNR}_{DFE}^{\alpha}\;\le\;\mathrm{MFB} \tag{6}$$

which shows that the (MMSE) DFE design enjoys as much diversity as the MFB. The DFE average Pairwise Error Probability (PEP) of transmitting a and deciding a' ($d=\frac{|a-a'|}{\sigma_a}\neq 0$), is

$$P(a\to a')=\mathrm{E}_{\mathbf{h}}\,Q\Big(\sqrt{\tfrac{\mathrm{SNR}_{DFE}^{\alpha}d^2}{2}}\Big)\le\mathrm{E}_{\mathbf{h}}\,e^{-\frac{\mathrm{SNR}_{DFE}^{\alpha}d^2}{4}}$$
$$\le\mathrm{E}_{\mathbf{h}}\,e^{-\frac{\rho\|\mathbf{h}\|^2d^2}{4c_L}}=(1+\tfrac{\rho d^2}{4c_L}\rho)^{-N_r(L+1)}=\mathcal{O}\left(\rho^{-N_r(L+1)}\right)$$

where the last two equalities hold for i.i.d. Gaussian channel coefficients ($N_r(L+1)$ diversity sources). The exponent of ρ at high SNR is called the diversity order. Since the MFB case corresponds to $c_L=1$, the DFE and the MFB enjoy the same diversity order regardless of channel distribution. Note that symbol error probability or DFE error propagation simply lead to different multiplicative constants in the error probability [1].

References

[1] J.R. Barry, E.A. Lee and D.G. Messerschmitt, *Digital Communications*, third edition, Kluwer Academic Publishers, 2004.

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