# Efficient Implementation of Multiuser Detectors for Asynchronous CDMA*

Laura Cottatellucci, Ralf R. Müller

Forschungszentrum Telekommunikation Wien
Tech Gate Vienna Donau-City-Straße 1
A-1220 Vienna, Austria
{cottatellucci, mueller} @ftw.at

Mérouane Debbah

Institut Eurecom
2229 Route des Cretes B.P. 193 06904
Sophia Antipolis cedex, France
debbah@eurecom.fr

## Abstract

We consider a linear multistage detector with universal (large system) weighting for large asynchronous code-division multiple-access (CDMA). A convenient choice of the basis of the projection subspace allows a joint projection of all users' signals and avoids truncation effects due to finite time windowing. Thanks to the joint projection the complexity per bit of the proposed detector scales linearly with the number of users. Under the assumption that the system is symbol asynchronous but chip synchronous the detector performance is analyzed analytically and shown to be identical to synchronous CDMA in the large system limit thanks to the absence of truncation effects.

The proposed detector is compared to the finite-window linear minimum mean-square error detector (LMMSE). While in the synchronous case the LMMSE detector always outperforms weighted multistage detectors, we show that, with a sufficiently large delay, the proposed multistage detector can outperform the LMMSE detector in asynchronous CDMA due to constraints on the observation window.

## 1 Introduction

The asymptotic analysis of linear multiuser detectors under the assumption of random spreading sequences is mainly focused on synchronous CDMA systems and only few works analyze linear detectors in asynchronous scenarios [1, 2, 3]. In [1, 3] the effects of chip asynchronicity are analyzed. It is shown that the performance of the LMMSE detector for chip asynchronous systems equals the performance of the same detector in chip synchronous scenarios as the observation window size tends to infinity. In [2] a corresponding statement is shown for chip-synchronous but symbol asynchronous and symbol synchronous CDMA respectively. Additionally, [2] gives the large system signal-to-interference and noise ratio (SINR) for a symbol centered in an observation window of length equal to the symbol interval $T_s$. A loose lower bound of the SINR is also known for any linear MMSE detector with observation window lengths multiple of $T_s$. However, the mismatch between the lower bound in [2] and simulation results is quite large.

Multistage detectors for synchronous CDMA have been proposed and analyzed in several works [4, 5, 6, 7]. They consist of a projector onto a subspace and a subsequent

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filter. The complexity reduction promised by the use of asymptotic filter coefficients (optimum weighting as the number of users and spreading sequences tend to infinity with constant ratio) in [8] inspired studies to design asymptotic weighting in different scenarios [12, 9, 13, 14]. Although with asymptotic weighting the detector complexity is determined by the complexity of the projection onto the subspaces, the choice of the basis of the subspace received little attention. In the reverse link, the use of an appropriate basis, that supports the joint processing of all users allows to reduce the, in general, quadratic complexity order per bit of the multistage detectors in [12, 14], to linear complexity per bit [9].

In this work we focus our attention on chip synchronous but symbol asynchronous CDMA with multistage detectors. By choosing a convenient basis and allowing for a detection delay equal to the number of stages we propose a multistage detector structure, with linear complexity order per bit, which does not suffer from windowing effects and performs as well as the multistage detector for synchronous systems. We present also an algorithm to calculate an arbitrarily tight lower bound on the SINR of LMMSE detectors. This algorithm allows an arbitrarily close approximation of the asymptotic LMMSE detector SINR for any symbol impinging the received signal. We show that the proposed multistage detector can outperform the full rank windowed MMSE detector. The rationale behind this fact is that the observation window of the proposed multistage detector increases automatically with the number of stages while being fixed for the full rank LMMSE detector. The sliding observation window used in the multistage detector implementation allows constant performance for all transmitted symbols in contrast to the full rank LMMSE detector, whose performance depends on the detected symbol position in the observation window.

Effects of chip asynchronicity on multistage detectors are analyzed in [15].

## 2   System Model and Notations

In the following, upper and lower boldface symbols will be used, respectively, for matrices and vectors corresponding to signal transmitted in a specific symbol interval $n$. Matrices and vectors describing signals spanning more than one symbol interval are denoted by upper calligraphic letters.

Let us consider a direct-sequence CDMA system with $K$ users and spreading factor $N$. We focus on asynchronous systems in the reverse link. However, to make the analysis tractable we will assume the system to be chip-synchronous as in [2]. User 1 is the reference user. Without loss of generality we can assume that the time shift between any user and user 1 is, at most, one symbol and the users are ranked in ascending order of time shift with respect to the reference user. Let $\mathbf{y}(n) \in \mathbb{C}^N$ and $\mathbf{b}(n) \in \mathbb{C}^K$ be, respectively, the observed column vector synchronized to the reference user and the column vector of the transmitted user modulation symbols at time $n$. $\mathbf{S}(n) \in \mathbb{C}^{2N \times K}$ is the spreading matrix containing the users' spreading sequences at time $n$, opportunely shifted and zero elsewhere. $\mathbf{A} = \mathrm{diag}(a_1, a_2, \ldots a_K)$ is the $K \times K$ matrix of complex received amplitudes and $\boldsymbol{H}(n) = \boldsymbol{S}(n)\mathbf{A}$. For notation reasons we split the matrix $\boldsymbol{H}(n)$ in two matrices[1] $\boldsymbol{H}_u(n), \boldsymbol{H}_d(n) \in \mathbb{C}^{N \times K}$ such that $\boldsymbol{H}(n) = [\boldsymbol{H}_u^T(n), \boldsymbol{H}_d^T(n)]^T$.

Then, the baseband discrete-time asynchronous system in the reverse link is described

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[1]The indices $u$ and $d$ are used to denote, respectively, the upper and lower block in which we split the matrix $\boldsymbol{H}(n)$

by

$$\mathcal{Y} = \mathcal{H}\mathcal{B} + \mathcal{N} \tag{1}$$

where $\mathcal{Y} = [\dots, \mathbf{y}^T(n-1), \mathbf{y}^T(n), \mathbf{y}^T(n+1)\dots]^T$, $\mathcal{B} = [\dots, \mathbf{b}^T(n-1), \mathbf{b}^T(n), \mathbf{b}^T(n+1)\dots]^T$. $\mathcal{N}$ is the additive white gaussian noise with variance $\sigma^2$. The matrix $\mathcal{H}$ is a bi-diagonal block matrix with infinite block rows

$$\begin{bmatrix} \dots & \mathbf{0} & \boldsymbol{H}_d(n-1) & \boldsymbol{H}_u(n) & \mathbf{0} & \dots \end{bmatrix} \tag{2}$$

for $n \in (-\infty, +\infty)$. Throughout this work we assume that the nonzero elements of all matrices $\boldsymbol{S}(n)$, $\forall n \in (-\infty, +\infty)$, are independent and identically distributed (i.i.d.). Additionally, $E\{s_{ij}\} = 0$, $E\{|s_{ij}|^2\} = \frac{1}{N}$.

The sequence of the empirical eigenvalue distribution of $\mathbf{A}\mathbf{A}^H$ converges almost surely, as $K \to \infty$, to a non-random distribution function with upper bounded support. $\beta = \frac{K}{N}$ is the system load, i.e. the number of physical users per chip. The time shifts are i.i.d. distributed among the users. The time shift normalized to the symbol interval $T_s$, $\tau$, has probability mass function (p.m.f) $\mathrm{P}_N(\tau)$. The support of $\mathrm{P}_N(\tau)$ is $[0, \gamma]$, with $\gamma \leq 1$. As $N \to \infty$ the sequence $\{\mathrm{P}_N(\tau)\}$ converges to the p.d.f $p_\tau(\tau)$. We will also consider the system corresponding to a finite observation window of length $T$ symbols centered in the $n$-th transmitted symbol of the reference user. In order to keep the notation simple we assume $T$ to be integer and odd. However, the result will hold for any $T$ such that $TN \in \mathbb{Z}$. In this case, the model has the following form:

$$\underbrace{\begin{bmatrix} \mathbf{y}(n-\frac{T-1}{2}) \\ \vdots \\ \mathbf{y}(n) \\ \vdots \\ \mathbf{y}(n+\frac{T-1}{2}) \end{bmatrix}}_{\mathcal{Y}_{N,T}(n)} = \underbrace{\begin{bmatrix} \boldsymbol{H}_d(n-\frac{T-1}{2}) & \boldsymbol{H}_u(n-\frac{T+1}{2}) & & \mathbf{0} \\ & \ddots & \ddots & \mathbf{0} \\ & & \ddots & \ddots \\ \mathbf{0} & & \ddots & \ddots \\ \mathbf{0} & & \boldsymbol{H}_d(n+\frac{T-1}{2}) & \boldsymbol{H}_u(n+\frac{T+1}{2}) \end{bmatrix}}_{\mathcal{H}_{N,T}(n)} \underbrace{\begin{bmatrix} \boldsymbol{b}(n-\frac{T-1}{2}) \\ \vdots \\ \boldsymbol{b}(n) \\ \vdots \\ \boldsymbol{b}(n+\frac{T-1}{2}) \end{bmatrix}}_{\mathcal{B}_{N,T}(n)} + \underbrace{\begin{bmatrix} \boldsymbol{n}(n-\frac{T-1}{2}) \\ \vdots \\ \boldsymbol{n}(n) \\ \vdots \\ \boldsymbol{n}(n+\frac{T-1}{2}) \end{bmatrix}}_{\mathcal{N}_{N,T}(n)} \tag{3}$$

# 3   Multistage Detector

## 3.1   Choice of the basis of the projection subspace

In order to introduce the concept of multistage detectors we consider a synchronous system. In this case, $\boldsymbol{H}_d(n)=\mathbf{0}$, $\forall n$. $\mathcal{H}$ becomes a block diagonal matrix and the detection of the transmitted symbols at time instant $n$ is independent from the transmitted symbols at different time instants. Then, defining $\overline{\boldsymbol{H}}(n)=\boldsymbol{H}_u(n)$, the system model is given by

$$\mathbf{y}(n) = \overline{\boldsymbol{H}}(n)\boldsymbol{b}(n) + \boldsymbol{n}(n). \tag{4}$$

A linear multistage detector of order $M$ for user $k$ is a multiuser detector performing

- a projection of the observed signal onto the Krylov subspace

$$\chi_{M,k,n}(\overline{\boldsymbol{H}}(n)) = \mathrm{span}\{\boldsymbol{T}_{\sim k}^m(n)\boldsymbol{h}_k(n)\}_{m=0}^M \tag{5}$$

$$= \mathrm{span}\{\boldsymbol{T}^m(n)\boldsymbol{h}_k(n)\}_{m=0}^M, \tag{6}$$

where $\boldsymbol{h}_k$ denotes the $k$-th column of $\overline{\boldsymbol{H}}(n)$, $\boldsymbol{T}(n) = \overline{\boldsymbol{H}}(n)\overline{\boldsymbol{H}}(n)^H$, $\boldsymbol{T}_{\sim k}(n) = \overline{\boldsymbol{H}}_{\sim k}(n)\overline{\boldsymbol{H}}_{\sim k}(n)^H$, and $\overline{\boldsymbol{H}}_{\sim k}(n)$ is the $N \times (K-1)$ matrix obtained from $\overline{\boldsymbol{H}}(n)$ by removing the $k$-th column[2];

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[2]About the identity of the subspaces spanned by the two bases in (5) and (6) see [6].

- A subsequent processing of the projections by a filter designed according to an optimality criterion.

Both the projection and the filter design can be performed jointly for all users or individually for each user. This has effects on both the performance and the complexity of the resulting multistage detector. The joint projection can be obtained using the vectors in (6) as basis of $\chi_{M,k,n}(\overline{\boldsymbol{H}}(n))$. In this case the projector consists of a matched filter $\overline{\boldsymbol{H}}^H$ and $M$ stages each of them performing despreading — filtering by $\overline{\boldsymbol{H}}$ — and successive matched filtering. Using the vectors in (5), no joint computation of the projections is possible for $M > 1$ and $K$ different projectors are required. For the basis (6), filtering design can be performed jointly using the same filter coefficients for all users and choosing them, for example, by enforcing the minimization of the mean square error (MSE) averaged over all users [9]. Alternatively, we can design a different filter for each user minimizing the MSE individually. Table 1 shows the possible combinations and states the denominations. Detectors Type I are known as polynomial expansion detectors and

|  | Joint Projection | Individual Projection |
|---|---|---|
| Joint Filtering | TYPE I | ♯ |
| Individual Filtering | TYPE II | TYPE III |

Table 1: Multistage detector classification.

| Detector | One user's detection | All users' detection |
|---|---|---|
| SUMF | $\mathcal{O}(K)$ | $\mathcal{O}(K)$ |
| TYPE I | $\mathcal{O}(K^2)$ | $\mathcal{O}(K)$ |
| TYPE II | $\mathcal{O}(K^2)$ | $\mathcal{O}(K)$ |
| TYPE III[3] | $\mathcal{O}(K^2)$ | $\mathcal{O}(K^2)$ |
| LMMSE | $\mathcal{O}(K^3)$ | $\mathcal{O}(K^2)$ |

Table 2: Complexity order per symbol.

were proposed in [4]. Detectors Type II were considered in [9] and called there individual LMMSE detectors in $\chi_{M,k,n}(\overline{\boldsymbol{H}})$. Detectors Type III are known as multistage Wiener filters and were presented first in [5]. Detectors Type II and Type III adopt the same optimality criterion in the same subspace and differ only in the choice of the subspace basis. Therefore, they have identical performance. However, they need, in general, different weights.

To be subspace methods does not imply that the multistage detectors have lower complexity order than the full rank LMMSE detector. In fact, the filter coefficient design complexity is $\mathcal{O}(K^3)$ as the complexity order of the LMMSE detector. However, by approximating the optimum filter coefficients with the corresponding weights for large systems, i.e. as $K, N \to \infty$ with $\frac{K}{N} \to \beta$, as proposed in [8], the coefficient design complexity becomes negligible with respect to the projection complexity. The complexity order per bit, for detectors with asymptotic filter coefficients, is shown in Table 2. Table 2 distinguishes two cases: a single user is detected, typically in the forward link, and all users are detected at the receiver, typically in the reverse link. Considering the advantages of the Type II detectors in terms of performance with respect to Type I detectors [9] and in terms of complexity with respect to Type III detectors, in the following we focus on Type II detectors and extend them to the asynchronous case.

A straightforward extension of multistage detectors to asynchronous systems would replace the matrix $\overline{\boldsymbol{H}}(n)$ with the finite matrix $\mathcal{H}_{N,T}(n)$. This would sum up the performance degradation of a subspace method with respect to the full rank approach to the loss due to windowing. However, an implementation of multistage detectors with finite

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[3]For Type III detectors with a single stage ($M = 1$) an implementation with complexity order $0(K)$ is possible as all users are detected (reverse link).
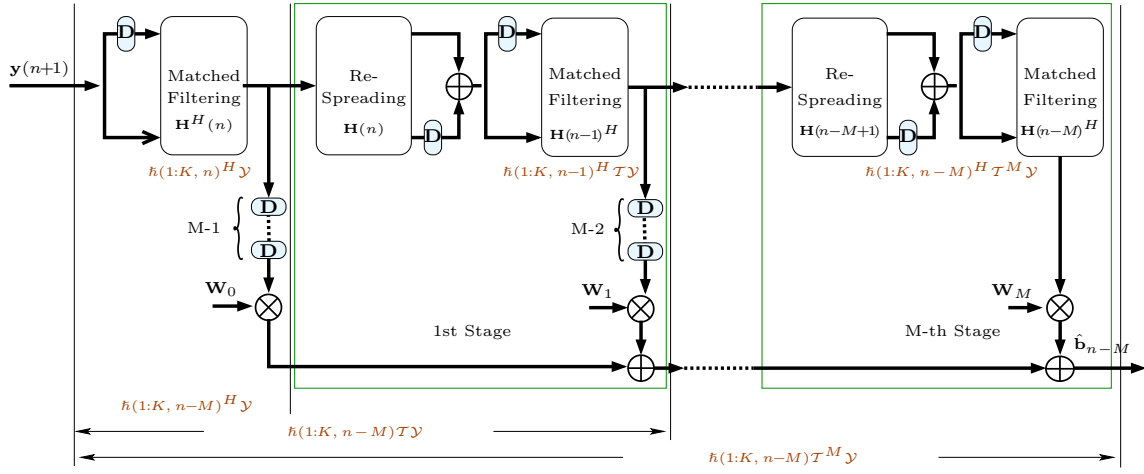
Figure 1: Multistage detector Type II for asynchronous systems.

delay is still possible considering the unlimited system model (1) and using the subspace $\chi_{M,k,n}(\mathcal{H}) = \text{span}\left\{\hbar(k,n)^H \mathcal{T}^m\right\}_{m=0}^M$, where $\hbar(k,n)$ is the column of $\mathcal{H}$ corresponding to user $k$ at time instant $n$ and $\mathcal{T} = \mathcal{H}\mathcal{H}^H$. Because of the bi-diagonal block structure of $\mathcal{H}$, the matrix $\mathcal{T}$ is a tri-diagonal block matrix and its power $\mathcal{T}^m$ is a $(2m+1)$-diagonal matrix. Therefore, the vector $\hbar(k,n)^H \mathcal{T}^m$ has, at most, $(2m+1)N$ nonzero elements and the $M$-stage detector for the unlimited system model can be implemented with a finite delay equal to $MT_s$. Figure 1 shows its structure. The $j$-th stage consists of a re-spreading block that multiplies the input vector by the matrix $\boldsymbol{H}(n-j+1)$ and a filter matched to the transmitted vector at time $n-j$, $\boldsymbol{H}(n-j)^H$. It receives as input vector $\hbar(1\!:\!K,n)\mathcal{T}^j\mathcal{Y}$, where $\hbar(s\!:\!r,n)$ denotes the $s-r+1$ columns of the matrix $\mathcal{H}$ corresponding to the users $r, r+1, \ldots s$ at time instant $n$. The re-spreading block provides two output vectors, the upper part vector $\boldsymbol{H}_u(n-j)\hbar(1\!:\!K,n-j)\mathcal{T}^j\mathcal{Y}$ and $\boldsymbol{H}_d(n-j+1)\hbar(1\!:\!K,n-j)\mathcal{T}^{j-1}\mathcal{Y}$. The input to the following matched filter is given by

$$\begin{bmatrix} \boldsymbol{H}_u(n-j)\hbar(1\!:\!K,n-j-1)^H\mathcal{T}^{j-1}\mathcal{Y} + \boldsymbol{H}_d(n-j-1)\hbar(1\!:\!K,n-j-2)^H\mathcal{T}^{j-1}\mathcal{Y} \\ \boldsymbol{H}_u(n-j+1)\hbar(1\!:\!K,n-j)^H\mathcal{T}^{j-1}\mathcal{Y} + \boldsymbol{H}_d(n-j)\hbar(1\!:\!K,n-j-1)^H\mathcal{T}^{j-1}\mathcal{Y} \end{bmatrix}. \quad (7)$$

The output of the $j$-th stage is delayed by $(M-j)T_s$ before being used as input of the filter to provide the soft estimate of $\boldsymbol{b}(n-M)$.

## 3.2 Design of the Asymptotic Weighting

Let $\mathbf{W}_m(n) = \text{diag}(w_{m1}(n), w_{m2}(n), \ldots, w_{mk}(n))$ and $\mathcal{W}_m = \text{diag}\{\ldots, \mathbf{W}_m(n), \mathbf{W}_m(n{+}1), \ldots\}$. The multistage detector Type II for asynchronous systems is the linear detector $\mathcal{M} = \sum_{m=0}^M \mathcal{W}_m \mathcal{H}^H \mathcal{T}^m$ such that $\text{E}\{\|\mathcal{M}\mathcal{Y} - \mathcal{B}\|^2\}$ is minimum. This is equivalent to the minimization of the mean square error (MSE) for each component $(\boldsymbol{b}(n))_k{}^4$ of $\mathcal{B}$ in the correspondent subspace $\chi_{M,k,n}(\mathcal{H})$. The weight matrices $\mathbf{W}_m(n)$ can be derived by the following equation:

$$\mathbf{w}_k(n) = (\boldsymbol{\Phi}_k(n))^{-1}\varphi_k(n) \quad (8)$$

where $(\mathbf{w}_k(n))_m = (\mathbf{W}_m(n))_{kk}$, $\varphi_k(n)$ is an $(M+1)$-dimensional vector , $\boldsymbol{\Phi}_k(n) \in \mathbb{R}^{(M+1)\times(M+1)}$, $(\varphi_k(n))_m = (\mathcal{R}^{m+1}(n))_{kk}$, $(\boldsymbol{\Phi}_k(n))_{lm} = (\mathcal{R}^{l+m}(n))_{kk} + \sigma^2(\mathcal{R}^{l+m-1}(n))_{kk}$, $\mathcal{R} = \mathcal{H}^H\mathcal{H}$, and $(\mathcal{R}^m(n))_{kk} = \hbar(k,n)^H\mathcal{T}^{m-1}\hbar(k,n)$ denotes the diagonal element of the

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[4]$(.)_m$ denotes the $m$-th component of the vector argument and $(.)_{mn}$ denotes the element $ij$ of the matrix argument.

matrix $\mathcal{R}^m$ corresponding to the user $k$ at time instant $n$. The output SINR of user $k$ is given by [9]

$$\text{SINR}_k(n) = \frac{\varphi_k^T(n)(\Phi_k(n))^{-1}\varphi_k(n)}{1 - \varphi_k^T(n)(\Phi_k(n))^{-1}\varphi_k(n)} \tag{9}$$

In the asymptotic case, as $N, K \to \infty$ with $\frac{K}{N} = \beta$, the expression for Type II detectors requires the existence and the expression of the limits $\lim\limits_{K=\beta N \to \infty}(\mathcal{R}^m(n))_{kk} = \mathcal{R}_{k,\infty}^m(n) \quad k \in [1, K]$ and $1 \leq m \leq 2(M+1)$. It is known [1, 2] that for $K = \beta N$ and $T \to \infty$ the eigenvalue distribution of $\mathcal{R}$ converges to the limit distribution as the eigenvalues of the matrix $\overline{\boldsymbol{R}} = \overline{\boldsymbol{H}}^H \overline{\boldsymbol{H}}$ for synchronous systems. It will be apparent from the discussion in Section 4 that the same property holds also for the diagonal elements $\mathcal{R}_{\infty,k}^m(n)$ and

$$\lim\limits_{K=\beta N \to \infty}(\mathcal{R}^m(n))_{kk} = \overline{\mathbf{R}}_{k,\infty}^m \qquad \forall 1 \leq m \leq 2(M+1). \tag{10}$$

Recursive and closed form expressions for $\overline{\mathbf{R}}_{k,\infty}^m$ can be found in [9]. The extension of these results to detectors of Type I and Type III is straightforward. The asymptotic weights can be found in [9] and [12] respectively.

# 4 Asymptotic Performance of LMMSE Detectors

In this section we propose an algorithm to determine the asymptotic (i.e. $K = \beta N \to \infty$) SINR of linear MMSE detector for asynchronous systems. It is well known that for a finite synchronous system with $K$ users a $K$-stage detector coincides with the linear MMSE detector [4, 6]. Therefore, the SINR of Type II detectors for $M < K$ provides a family of lower bounds for the $\text{SINR}_{\text{LMMSE}}$ of the full rank linear MMSE detector. Additionally, it has been shown that for moderate to heavy loads an 8-stage detector for synchronous system essentially achieves full rank performance [6] independently of the number of users in the system, i.e. the required rank to achieve a fixed level of performance does not scale with the system size. It was established in [7] that the reduced rank multistage filter output SINR converges exponentially in the filter rank toward to the full rank LMMSE filter output SINR. We use this property to determine the performance of a LMMSE detector. We provide a family of lower bounds for the LMMSE detector with finite observation window whose supremum coincides with the $\text{SINR}_{\text{LMMSE}}$. The family of lower bounds consists of the performance of multistage detectors with finite observation windows and varying numbers of stages. We verified numerically that for $M \geq 8$ the lower bounds are so close to the supremum, also for asynchronous systems, to be indistinguishable from it.

Let us consider an asynchronous system with finite observation window $T$ and equal powers. Then, without loss of generality we assume that $\mathbf{A} = \mathbf{I}$ in (3). Making use of (9), the problem of determining the family of lower bounds of $\text{SINR}_{\text{LMMSE}}$ reduces to determining the diagonal elements of the matrix $\mathcal{R}_T(n) = \mathcal{H}_{N,T}^H(n)\mathcal{H}_{N,T}(n)$ as $K = \beta N \to \infty$. A recursive algorithm to determine them is provided by Theorem 1.

**Theorem 1** *Let $\mathcal{H}_{N,T}$ be an $TN \times (T+1)K$ bi-diagonal block matrix with blocks $\boldsymbol{H}(j) = [\boldsymbol{H}_u^T(j), \boldsymbol{H}_d^T(j)]^T \in \mathbb{C}^{2N \times K}$, and $\boldsymbol{H}_u(j), \boldsymbol{H}_d(j) \in \mathbb{C}^{N \times K}$, as follows:*

$$\mathcal{H}_{N,T} = \begin{bmatrix} \boldsymbol{H}_d(1) & \boldsymbol{H}_u(2) & \mathbf{0} & \dots & \dots & \dots \\ \mathbf{0} & \boldsymbol{H}_d(2) & \boldsymbol{H}_u(3) & \mathbf{0} & \dots & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & \dots & \mathbf{0} & \boldsymbol{H}_d(T-1) & \boldsymbol{H}_u(T) & \mathbf{0} \\ \dots & \dots & \dots & \mathbf{0} & \boldsymbol{H}_d(T) & \boldsymbol{H}_u(T+1) \end{bmatrix}. \tag{11}$$

Let $\widetilde{\boldsymbol{H}}(k)$ for $k = 1, \ldots T + 1$ be independent matrices in $\mathbb{C}^{2N \times K}$ with elements $\widetilde{h}_{ij}(k)$, $1 \le i \le N$, $1 \le j \le K$, i.i.d. and such that $\mathrm{E}\{\widetilde{h}_{ij}(k)\} = 0$, $\mathrm{E}\{|\widetilde{h}_{ij}(k)|^2\} = \frac{1}{N}$, and $\widetilde{h}_{ij}(k) \le \frac{\log N}{N}$ and the remaining elements equal to zero. The matrix $\boldsymbol{H}(k)$ is obtained from $\widetilde{\boldsymbol{H}}(k)$ circularly shifting each column by $\tau N$ positions independently of all the others and according to a p.m.f. $P_N(\tau)$, and then, sorting the column vectors by ascending order of $\tau$. The sequence of p.m.f $\{P_N(\tau)\}$ converges to a p.d.f. $p_\tau(\tau)$ with support $[0, \gamma]$ and $\gamma \le 1$, distribution function $F_\tau(\tau)$. We assume the spectrum of the matrix $\mathcal{H}_{N,T}$ be upper bounded for sufficiently large $N$. Define for each $N$ $v_N : [0, T] \times [0, (T+1)\beta] \to \mathbb{R}$ the limiting joint distribution of the variance:

$$v_N(x, y) = N\mathrm{E}\{|h_{ij}|^2\} \qquad \text{for } i, j \text{ satisfying} \tag{12}$$

$$\frac{i}{N} \le x \le \frac{i+1}{N} \qquad \frac{j}{N} \le y \le \frac{j+1}{N}. \tag{13}$$

Then, $v_N(x, y)$ converges uniformly to a limited bounded function $v$ such that $v(x, y) = 1$ in the region whose border is defined by the two curves $r(x)$ and $c(y)$ with

$$r(x) = \begin{cases} \frac{\beta}{\gamma} F_\tau^{-1}\left(\frac{x-i}{\gamma}\right) + i\beta & i \le x \le i + \gamma \\ (i+1)\beta & i + \gamma < x < i + 1 \end{cases} \qquad 0 \le i \le T - 1, \tag{14}$$

$$c(y) = \begin{cases} 0 & 0 \le y \le \beta \\ (i-1) + i\beta F_\tau\left(\frac{\gamma(y - i\beta)}{\beta}\right) & i\beta < y < (i+1)\beta \end{cases} \qquad 1 \le i \le T \tag{15}$$

and $v_N(x, y) = 0$ elsewhere. Moreover, let the function $l(y) \in \mathbb{R}$ be defined as

$$l(y) = \begin{cases} \frac{\beta}{\gamma} F_\tau^{-1}\left(\frac{y}{\gamma}\right) & 0 \le y \le \beta \\ 1 & \beta < y < T\beta \\ \frac{\beta}{\gamma} F_\tau^{-1}\left(\frac{(T+1)\beta - y}{\gamma}\right) + (1 - \gamma) & \beta T \le y \le \beta(T+1) \end{cases} . \tag{16}$$

Then,

$$\lim_{K=\beta N \to \infty} \left(\mathcal{T}_{N,T}^m\right)_{nn} = \lim_{K=\beta N \to \infty} ((\mathcal{H}_{N,T}\mathcal{H}_{N,T}^H)^m)_{nn} \overset{a.s.}{=} \mathcal{T}_T^m(x) \quad \text{and } x = \lim_{N \to \infty} \frac{n(N)}{N} \tag{17}$$

$$\lim_{K=\beta N \to \infty} \left(\mathcal{R}_{N,T}^m\right)_{kk} = \lim_{K=\beta N \to \infty} ((\mathcal{H}_{N,T}^H\mathcal{H}_{N,T})^m)_{kk} \overset{a.s.}{=} \mathcal{R}_T^m(y) \quad \text{and } y = \lim_{N \to \infty} \frac{k(N)}{N} \tag{18}$$

with $\mathcal{R}_T^m(y)$ and $\mathcal{T}_T^m(x)$ determined by the following recursion:

$$\mathcal{T}_T^{n+1}(x) = \beta \sum_{s=0}^n \mathcal{T}_T^s(x) f(\mathcal{R}_T^{n-s}, x) \qquad\qquad 0 \le x \le T \tag{19}$$

$$\mathcal{R}_T^{n+1}(y) = l(y) \sum_{s=0}^n \mathcal{R}_T^s(y) g(\mathcal{T}_T^{n-s}, y) \qquad\qquad 0 \le y \le (T+1)\beta \tag{20}$$

with

$$f(\mathcal{R}_T^n, x) \overset{\triangle}{=} \frac{1}{\beta} \int_{r(x)}^{r(x)+\beta} \mathcal{R}_T^n(y)\, \mathrm{d}y \qquad\qquad 0 \le x \le T \tag{21}$$

$$g(\mathcal{T}_T^n, y) \overset{\triangle}{=} \frac{1}{l(y)} \int_y^{y+l(y)} \mathcal{T}_T^n(x)\, \mathrm{d}x \qquad\qquad 0 \le y \le (T+1)\beta \tag{22}$$

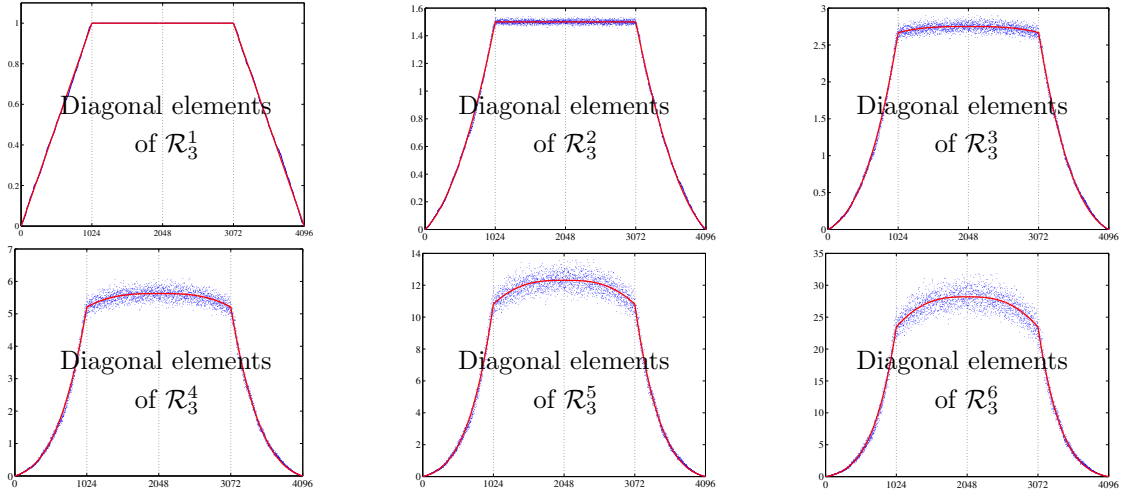and $\mathrm{T}_T^1(x) = \beta$ and $\mathrm{R}_T^1(y) = l(y)$.

Figure 2: Comparison of the theoretical $\mathcal{R}_3^n(y)$ for $n = 1 \ldots 6$ with $\mathcal{R}_{3,kk}^n(2048)$, $\beta = \frac{1}{2}$.

The proof is omitted due to space limitation. The assumption that the spectrum of the matrix $\mathcal{R}_{N,T}$ is upper bounded is of technical nature. Indeed, we conjecture that it follows from the hypotheses on $h_{ij}(k)$. This property is verified for the matrix $\overline{\boldsymbol{H}}$ for synchronous systems. In fact, extensive computer simulations were performed in order to verify it [10] and the property was proven in [11]. However, no analogous result for the matrix $\mathcal{R}_{N,T}$ is known to the authors.

Figure 3 illustrates the meaning of the functions $v(x,y)$, $r(x)$, $c(x)$, and $l(y)$. The following example explains the use of the theorem. Let us assume $T = 3$, $\gamma = 1$ and the delay uniformly distributed in the interval $[0, T_s]$, then $F_\tau(\tau) = \tau$, $r(x) = \beta x \ \forall x \in [0, 3]$,

$$c(y) = \begin{cases} 0 & 0 \leq y \leq \beta \\ \frac{y-\beta}{\beta} & \beta \leq y \leq 4\beta \end{cases} \qquad \text{and} \qquad l(y) = \begin{cases} \frac{y}{\beta} & 0 \leq y \leq \beta \\ 1 & \beta \leq y \leq 3\beta \\ 4\beta - \frac{y}{\beta} & \beta \leq y \leq 4\beta \end{cases} . \qquad (23)$$

Therefore, $\mathcal{T}_T^1(x) = \beta$ and $\mathcal{R}_T^1(y) = l(y)$,

$$f(\mathcal{R}_T^1, x) = \begin{cases} \frac{1}{\beta} \left[ \int\limits_{\beta x}^{\beta} \frac{y}{\beta} \mathrm{d}y + \int\limits_{\beta}^{\beta x + \beta} \mathrm{d}y \right] & 0 \leq x \leq 1 \\ \frac{1}{\beta} \int\limits_{\beta x}^{\beta x + \beta} \mathrm{d}y & 1 \leq x \leq 2 \\ \frac{1}{\beta} \left[ \int\limits_{\beta x}^{3\beta} \mathrm{d}y + \int\limits_{3\beta}^{\beta x + \beta} \left(4\beta - \frac{y}{\beta}\right) \mathrm{d}y \right] & 0 \leq x \leq 1 \end{cases} \qquad (24)$$

and $g(\mathcal{T}_T^1, y) = \beta$, $0 \leq y \leq 4\beta$. We can then apply (19) and (20). In Figure 2 the asymptotic values of $\mathcal{R}_3^n(y)$ for $n = 1 \ldots 6$ are compared to the values $\mathcal{R}_{3,kk}^n(N)$, for $N = 2048$ and $\beta = \frac{1}{2}$, of a single realization. Simulations with various distributions of the elements $h_{ij}$ show that the diagonal elements of finite large matrices match very well the asymptotic values determined by (20).

The difficulty in extending the previous theorem to a system with unbalanced powers ($\mathbf{A} \neq \mathbf{I}$) is due to the difficulty in determining $\mathcal{T}_T^m(x)$. However, for $T \to \infty$ no truncation effects occur and, as for synchronous systems, $\mathcal{T}_T^m(x)$ is independent of $x$ and is equal to the normalized trace of $\mathcal{T}^m$. For $T \to \infty$ it is known [1, 2] that the asymptotic
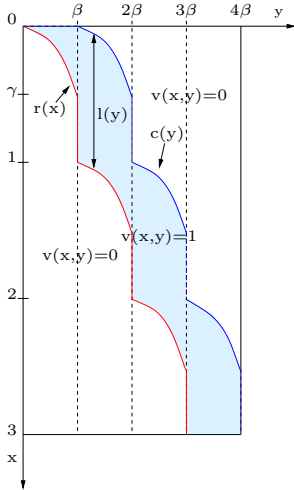
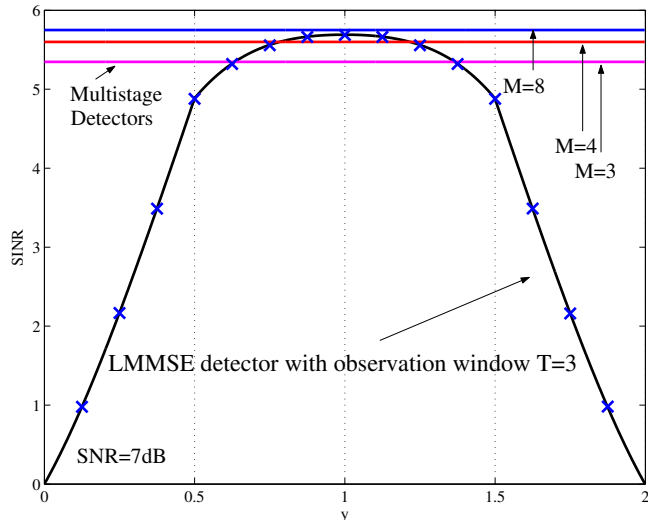Figure 3: Graphical representation of the functions $v(x,y)$, $r(x)$, $c(y)$ and $l(y)$.



Figure 4: Asymptotic $\text{SINR}_{\text{LMMSE}}$ for $T = 3$ and multistage detector SINR for varying $M$ ($\frac{E_b}{N_0} = 7$ dB).

eigenvalue distribution of $\mathcal{T}$ coincides with the eigenvalue distribution for synchronous systems. Hence, with an approach analogous to the one applied to derive Theorem 1, we can derive an equation equivalent to equation (20) for systems with unbalanced powers. This leads to the same results as in the synchronous systems stated in (10).

# 5   Numerical Results

Throughout this section, we consider linear MMSE detectors with observation window $T = 3$. Figure 4 shows the output $\text{SINR}_{\text{LMMSE}}$ for a system with $\beta = \frac{1}{2}$ and $\frac{E_b}{N_0} = 7$ dB. As for the synchronous case, the convergence of lower bounds toward to the supremum is very fast and the lower bound corresponding to $M = 7$ (marks in Figure 4) is undistinguishable from the one obtained for $M = 8$. The SINR reaches its maximum for the transmitted symbol centered in the observation window and decreases smoothly for the transmitted symbols whose spreading is still completely observed ($y \in [\beta, 3\beta]$). The performance degrades rapidly for symbols only partially included in the observation window. In contrast to the synchronous case, in the asynchronous case the multistage detectors with $M$ sufficiently large, can outperform the full rank LMMSE detector with finite observation window $T$. This is due to the fact that both detectors use only a subset of a sufficient statistic, but, with the proposed subspace basis, multistage detectors intrinsically use a wider and wider subset as the number of stages increases, while the full rank LMMSE detector exploits always the same statistic and the use of a wider statistic requires an increment of the observation window size.

# 6   Summary of Results and Conclusion

We proposed a scheme the multistage detectors with linear complexity per bit, which does not suffer from windowing effects in asynchronous systems, in contrast to the full rank LMMSE detector. We also provided an algorithm to determine the performance of the LMMSE detector with finite observation window for all the transmitted symbols that impinge the received signal. In contrast to the synchronous systems, the multistage detector for asynchronous systems can outperform the full rank LMMSE detector with finite window T when choosing a sufficient large rank $M$.

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