On the Performance of Incremental Redundancy Schemes with Turbo Codes

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Abstract

A semi-analytical bound on the frame error performance of rate compatible punctured Turbo (RCPT) codes on the block-fading Gaussian collision channel is presented. Based on this bound, it is concluded that incremental redundancy schemes with RCPT codes have the potential to yield significant throughput improvements compared to similar schemes based on convolutional codes.

1 Introduction

Incremental redundancy techniques, are subject to an increasing interest in order to obtain transmission with low error probability in packet-oriented high-speed wireless links. This is due to the ability of incremental redundancy schemes to achieve high throughput by combining forward error correction (FEC) and automatic repeat request (ARQ) [1].

Information theoretic considerations in [2] indicate that incremental redundancy protocols have the potential to yield simple and efficient error correction for slotted Gaussian multiple-access channels. In particular it is demonstrated that it is possible to achieve higher throughput with incremental redundancy protocols than slotted ALOHA or packet combining [3] protocols. However, a challenging issue is to find practical codes with high performance and low complexity suitable for incremental redundancy protocols.

In contrast to conventional ARQ techniques, incremental redundancy protocols require channel codes that are rate compatible. Rate compatible punctured convolutional (RCPC) codes with application to ARQ were first considered in [4] and subsequently this concept has been extended to larger classes of convolutional codes [5] as well as Turbo codes [6]. In particular, Type-II Hybrid ARQ schemes [1] based on convolutional codes have been studied in [7, 8]. Rate compatible Turbo codes were first introduced in [9] and a significant amount of work on Turbo codes in Type-II Hybrid ARQ schemes has been performed [10–13]. Most of these investigations have been limited to the additive white Gaussian noise (AWGN) channel. There are, however, indications that Turbo codes are effective also on other channels, such as for instance multiple-access channels [14].

Motivated by the increasing deployment of wide-band packet data transmission in mobile multiple-access systems, where interference may be a significant source of performance degradation, we study the performance of RCPT codes on the block-fading Gaussian collision channel. Initial studies of the performance of RCPC codes on this channel have been performed in [15]. This paper extends these investigations to RCPT codes, focusing on the frame error rate (FER) in an incremental redundancy scheme, given a certain number of (re-)transmissions M.

2 System Description

A slotted multiple-access communication system is considered. There are in total N_u users in the system and these users are active according to a Bernoulli distribution with probability p in every time slot. The load of the system is $G = N_u \cdot p$. We consider a static protocol where the transmission probabilities of the individual users are constant in time. The transmission probabilities of individual users are thus not affected by the amount of decoding failures for their previously transmitted packets. A subset of interferers $K(s)$ out of a total of $N_i = N_u - 1$ potential interferers are simultaneously active in time slot s according to the binomial distribution with parameters (p, N_i). The transmitted signal of a user, as well as the interfering signals, are subject to block-independent Rayleigh fading with complex envelope $c_{k,s}$ for each active user k. In the system model, there is also a noise contribution due to thermal noise. The decoder of a particular user k performs single-user detection and treats this noise as well as the interference from the other users of the same time slot as AWGN and sees a channel according to

$$y_{k,s,l} = c_{k,s} x_{k,s,l} + v_{k,s,l}. \tag{1}$$

The transmitted channel symbols $x_{k,s,l}$ from user k in time slot s at symbol position l are assumed real valued and the analysis is restricted to binary antipodal modulation. The interference-plus-noise component $v_{k,s,l}$ includes Gaussian background noise and the interference caused by the other users active in slot s.

The signal-to-noise-and-interference ratio (SNIR) of the signal in (1) corresponding to user $k = 1$ at transmission slot j can be written as

$$\beta_{1,j} = \frac{c_{1,j}^2 E_1}{N_0 + \sum_{k \in K(j)} c_{k,j}^2 E_k}, \tag{2}$$

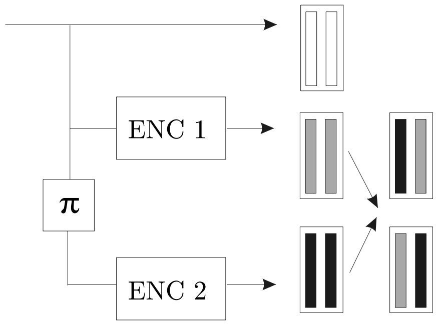

Figure 1: Systematic RCPT encoder.

where E_1 and E_k are the channel symbol energies of the desired user and the interfering users, respectively, and N_0 is the single-sided noise power spectral density. For ease of exposition and with no loss of generality since the system is symmetric, it is sufficient to consider only one particular user $k = 1$. Hence, the user index in the SNIR will in the sequel be omitted letting $\beta_j = \beta_{1,j}$.

The probability density function (PDF) of β_j for this channel model has been derived in [15]. In the system under consideration it is assumed that each transmitter-receiver pair employs an incremental redundancy scheme for transmission over the channel. The transmitters are assumed to have no channel state information. However, perfect channel state knowledge is assumed at the receivers. At the receiver side, maximum-likelihood (ML) detection is assumed available. Although this is not the case for Turbo codes of reasonable size, iterative decoding techniques at hand have been observed to provide near ML-performance [6].

In order to provide a series of incremental-redundancy packets, we consider systematic rate $1/M$ RCPT codes. For the sake of analytical tractability we consider RCPT codes with random interleavers, although improved performance can be achieved with careful interleaver design [16].

The structure of the systematic RCPT codes used in the incremental redundancy system studied is illustrated in Figure 1. The encoder consists of two component encoders with an intermediate interleaver and yields one systematic bit stream and two coded bit streams. These bitstreams are used to compose subpackets of equal length in order to provide incremental redundancy. The systematic bit stream provides the first subpacket. The coded bit streams are punctured in a cyclical manner such that coded bits from both encoders are multiplexed into both the first and the second subpacket of coded bits, *cf.* Figure 1. This enables the decoder to invoke iterative Turbo decoding, already upon reception of the first retransmission. This yields improved performance compared to the approach in [10], where the subpacket of the first retransmission contains explicitly coded bits from the first component encoder which prohibits Turbo decoding until both the subpackets with coded bits have been received.

A higher number of subpackets from RCPT codes can be achieved with different strategies: through 1) the use of low-rate component encoders and appropriate puncturing/multiplexing into subpackets, or 2) the use of multiple component encoders with intermediate interleavers. Due to the expected increase of decoding complexity when using multiple encoders and corresponding multiple decoders the first approach appears to be the most attractive one.

In the Turbo codes under consideration, tailbits are used to force the first encoder back to the zero state in order to terminate the trellis in a known state, which yields a small rate-loss. In our computations and simulations this is reflected in a decrease of the energy of the transmitted symbols.

3 Performance Analysis

A crucial component of the performance analysis is the fact that different subpackets transmitted over the block-fading Gaussian collision channel are subject to different channel states. Therefore, in order to be able to apply distance based bounding techniques derived for the AWGN channel, it is necessary to compute the multivariate distance distribution of the RCPT codes under consideration. For simplicity, the analysis in this paper is restricted to the case of binary antipodal modulation.

The multivariate distance distribution offers the joint multiplicity of Hamming distances between different subpackets belonging to the same sequence of information bits. For a particular mothercode codeword, it is possible compute an equivalent squared Euclidean distance to the all-zeros codeword by computing the scalar product of the Hamming distances and the channel state vector of the different subblocks. By linearity of the codes under consideration, the same set of Euclidean distances to surrounding codewords apply for every codeword and it is therefore sufficient to compute the multivariate Hamming weight distribution.

The multivariate Hamming weight distribution of the punctured component codes of an RCPT can conveniently be computed using the recursive method introduced in [17]. In order to calculate the corresponding weight distribution of an RCPT code one has to take into account the effect of the interleaver. Calculation of the weight distribution resulting from a particular interleaver is today an operation that is far too complex for interleaver lengths of practical interest.

In [18], a tractable method for calculation of the Turbo code weight distribution was presented, based on ensemble averaging over all interleavers of a given length. The method is based on a probabilistic device called a *uniform interleaver*. Using the concept of a uniform interleaver, the expected weight distribution of a Turbo code where the interleaver is chosen as a random permutation can be calculated. Even though the weight distribution of a particular Turbo code does not coincide with the ensemble average weight distribution calculated in this way, the concept of uniform interleaving enables insight into the impact on the weight distribution from different choice of interleaver length and component code parameters such as memory, rate, generator polynomials and puncturing. For instance, improvements of the component codes with respect to minimum distance properties will directly improve the minimum distance distribution of the corresponding, uniformly interleaved, Turbo code.

We will briefly describe how the multivariate Turbo code weight distribution for random choice of interleaver can be enumerated using the concept of uniform interleaving in conjunction with weight-enumerating functions. Define the multivariate weight enumerator of a component code with an input sequence of K bits (information bits and tailbits) as

$$\begin{aligned} & T\left(I, D_1, \ldots, D_{M-1}\right) \qquad (3) \\ \triangleq\ & \sum_{i \geq 0} \sum_{d_1 \geq 0} \cdots \sum_{d_{M-1} \geq 0} t_{i,d_1,\ldots,d_{M-1}} I^i D_1^{d_1} \cdots D_{M-1}^{d_{M-1}}, \end{aligned}$$

where the exponents i and $d_1, \ldots, d_{M-1}$ represent the input sequence weight and the set of output sequence weights. The number of codewords with subpacket weights $d_1, \ldots, d_{M-1}$, generated by input sequences of weight i is contained in $t_{i,d_1,\ldots,d_{M-1}}$. The variables I and $D_1, \ldots, D_{M-1}$ are dummy variables that enable a convenient enumeration of the weight distribution. For instance, multiplication of two weight enumerators corresponds to convolution of the weight distributions. This is a useful property when deriving the multivariate weight distribution for a Turbo code with uniform interleaver, since it corresponds to an M-dimensional convolution of the component code weight distributions. The weight enumerator in (3) is also referred to as the multivariate *input-redundancy weight enumerating function* (IRWEF).

The Turbo codewords are combinations of two component code codewords that both result from the same input-sequence weight i. The *conditional weight enumerating function* (CWEF)

$$\begin{aligned} & T_i\left(D_1, \ldots, D_{M-1}\right) \qquad (4) \\ \triangleq\ & \sum_{d_1 \geq 0} \cdots \sum_{d_{M-1} \geq 0} t_{i,d_1,\ldots,d_{M-1}} D_1^{d_1} \cdots D_{M-1}^{d_{M-1}}, \end{aligned}$$

enumerates the number of codewords of different subpacket weights $d_1, \ldots, d_{M-1}$, corresponding to a certain input-sequence weights $i = 0, 1, \ldots, K$. Given a particular weight of the input sequence, the weight distributions of the two component codes can be represented in terms of the CWEFs as $T_i^{C_1}\left(D_1, \ldots, D_{M-1}\right)$ and $T_i^{C_2}\left(D_1, \ldots, D_{M-1}\right)$.

Using a uniform-interleaver, it is assumed that a certain input-sequence of weight i is permuted with equal probability into all other sequences of weight i. The average CWEF of Turbo code weight distributions, over the ensemble of all interleavers of length K, can be then be calculated as

$$\begin{aligned} & T_i^{TC}\left(D_1, \ldots, D_{M-1}\right) \qquad (5) \\ =\ & \frac{T_i^{C_1}\left(D_1, \ldots, D_{M-1}\right) T_i^{C_2}\left(D_1, \ldots, D_{M-1}\right)}{\binom{K}{i}} \\ =\ & \sum_{d_1 \geq 0} \cdots \sum_{d_{M-1} \geq 0} t_{i,d_1,\ldots,d_{M-1}}^{TC} D_1^{d_1} \cdots D_{M-1}^{d_{M-1}}, \end{aligned}$$

where $1 \Big/ \binom{K}{i}$ is the probability that a weight-i sequence is mapped to a specific weight-i sequence by the uniform interleaver for $i = 0, 1, \ldots, K$.

Having computed the multivariate weight distribution of the RCPT under investigation, using the concept of uniform interleaving, it is possible to calculate upper bounds on the average error probability over the ensemble of interleavers after ML-decoding upon the $1, 2, \ldots, M$th received subpacket. The union bound (see for example [19]), the Viterbi bound [20] and the tangential sphere bound [21] have been considered in order to upper bound the ML-decoding performance of Turbo codes.

The tangential sphere bound by Poltyrev [21], yields the tightest upper bound of the three. However, unfortunately, the tangential sphere bound requires significantly more computational effort than the other two. This causes relatively complex computations in the case of the block-fading Gaussian collision channel, since it is necessary to solve an optimization problem for each simultaneous instance of multiple channel states for the M (re-) transmissions, due to the multiplicative nature of the fading. Asymptotically, for high SNRs, the union bound gives a tight upper bound on the error performance, but the union bound is in practice useless below the cutoff rate since it is very loose in the corresponding region of SNRs. Especially for Turbo codes, operating well below the cutoff rate, the tangential sphere bound and the Viterbi bound are more suitable.

Using the coefficients of the multivariate weight enumerator in (5), the tangential sphere bound can be modified to the case of the block-fading Gaussian collision channel according to

$$\begin{aligned} P(e|\beta_1, \ldots \beta_M) \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z_1^2/2} \cdot \Bigg\{ 1 - \bar{\gamma}\Big(\frac{n-1}{2}, \frac{r_{z_1}^2}{2}\Big) + \\ \sum\nolimits_{i,d_1,\ldots,d_{M-1}\,:\; i\beta_1 + d_1\beta_2 + \ldots + d_{M-1}\beta_M < \alpha^2_{i,d_1,\ldots,d_{M-1}}} \\ t_{i,d_1,\ldots,d_{M-1}}^{TC} \cdot \left(Q(\lambda_{i,d_1,\ldots,d_{M-1}}(z_1)) - Q(r_{z_1})\right) \cdot \\ \cdot \bar{\gamma}\Big(\frac{n-1}{2}, \frac{r_{z_1}^2 - \lambda^2_{i,d_1,\ldots,d_{M-1}}(z_1)}{2}\Big) \Bigg\} \cdot dz_1, \end{aligned}$$

with

$$r_{z_1} = \Big(1 - \frac{z_1}{\sqrt{n_1\beta_1 + \ldots + n_M\beta_M}}\Big) r,$$

$$\lambda_{i,d_1,\ldots,d_{M-1}}(z_1) = r_{z_1}/r \cdot \frac{\sqrt{(n_1\beta_1 + \ldots + n_M\beta_M)(i\beta_1 + \ldots + d_{M-1}\beta_M)}}{\sqrt{(n_1 - i)\beta_1 + \ldots + (n_M - d_{M-1})\beta_M}},$$

$$\alpha_{i,d_1,\ldots,d_{M-1}}(z_1) = r\sqrt{1 - \frac{i\beta_1 + \ldots + d_{M-1}\beta_M}{n_1\beta_1 + \ldots + n_M\beta_M}},$$

$$\bar{\gamma}(a, x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt, \quad a, x > 0 \quad \text{and}$$

$$n = n_1 + \ldots + n_M,$$

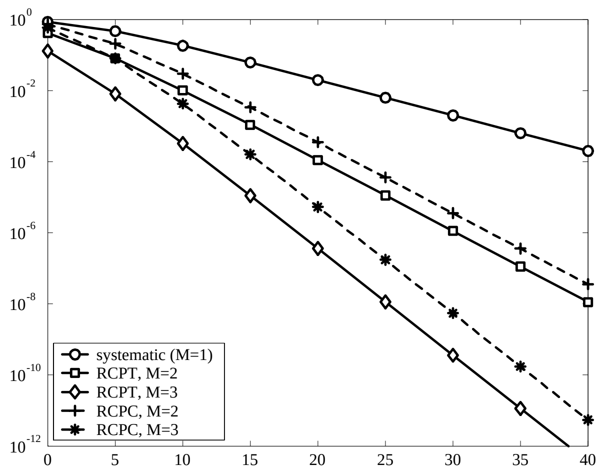

Figure 2: FER performance of systematic RCPT codes vs. systematic RCPC codes with $M = 1, 2$ and 3 (re-)transmissions, on the block-fading channel.

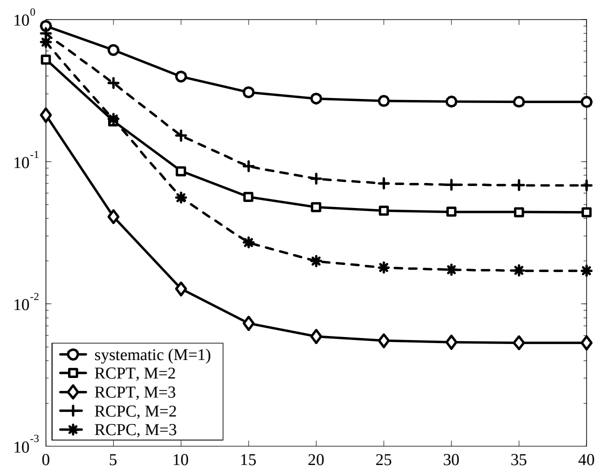

Figure 3: FER performance of systematic RCPT codes vs. systematic RCPC codes with $M = 1, 2$ and 3 (re-)transmissions, on the block-fading Gaussian collision channel with $N_u = 10$ users and transmission probability $p = 0.05$ ($G = 0.5$).

where n_1 is the maximum Hamming weight of the systematic subblock and $n_2, \ldots, n_M$ are the maximum Hamming weights of the remaining $M-1$ redundancy subblocks.

In order to achieve a tight bound there is an associated optimization problem to solve, *i.e.* finding the optimal cone radius r. This is time consuming since it includes finding the root to a relatively complicated integral equation with high requirement of precision in order to achieve a tight bound. The optimization problem can be stated in the following form

$$\frac{\sqrt{\pi}\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})} = \sum\nolimits_{i,d_1,\ldots,d_{M-1}\,:\,i\beta_1+\ldots+d_{M-1}\beta_M<\alpha_k^2} t^{TC}_{i,d_1,\ldots,d_{M-1}} \cdot \int_0^{\theta_{i,d_1,\ldots,d_{M-1}}} \sin^{n-3}\phi d\phi,$$

where

$$\theta_{i,d_1,\ldots,d_{M-1}} = \cos^{-1}\left(\lambda_{i,d_1,\ldots,d_{M-1}}(z_1)/r_{z_1}\right).$$

An upper bound on the FER of a particular coding scheme on the block-fading Gaussian collision channel with M (re-)transmissions can be obtained from the modified tangential sphere bound above and the joint PDF of the channel SNIRs $f(\beta_1, \ldots, \beta_M)$. Since the channel states of different transmissions are independent, the joint PDF can be factorized as

$$f(\beta_1, \ldots, \beta_M) = f(\beta_1) \cdot \cdots \cdot f(\beta_M),$$

with [15]

$$f(\beta_j) = e^{-\beta_j a} \cdot \left(\frac{1+\beta_j - p\beta_j}{1+\beta_j}\right)^{N_i-1} \cdot \frac{p \cdot N_i + a \cdot (1+\beta_j)\left(1+\beta_j - p\beta_j\right)}{(1+\beta_j)^2},$$

where $a = 1 \Big/ \frac{E_k}{N_0}$ is the inverse channel SNR. The upper bound on the FER can now be written as

$$P(e) \le \int_0^\infty \cdots \int_0^\infty P(e|\beta_1, \ldots, \beta_M) \cdot f(\beta_1, \ldots, \beta_M) \cdot d\beta_1 \cdot \ldots \cdot d\beta_M. \quad (6)$$

The integrations in (6) are performed by Gaussian numerical quadrature (see for example [22]), where the semi-infinite integrals intervals are reduced to finite integrals using the bounding technique proposed in [7].

In Figure 2, the tangential sphere bound on the FER performance of a systematic RCPT code and a systematic RCPC code on the block-fading channel (no interference) is compared. The information sequence correponds to 102 bits (including tail bits). The systematic RCPT code uses component codes with feedforward generator 5_8 and feedback generator 7_8, *cf.* Figure 1. The systematic RCPC code uses feedforward generators 7_8 and 5_8. This yields in total $M = 3$ subpackets available for incremental redundancy transmission for each information sequence.

Figure 3 shows the same performance comparison for the case of the block-fading Gaussian collision channel with channel load $G = 0.5$.

4 Conclusions

This paper has presented a performance evaluation technique for rate compatible punctured Turbo (RCPT) codes in incremental redundancy schemes over the block-fading Gaussian collision channel. A conditional upper bound on the frame error rate (FER), given the channel states of different packet (re-)transmissions over this channel, was established by modifying the tangential sphere bound by Poltyrev to apply for multivariate weight distributions. The expectation of this upper bound was then, by knowledge of the channel probability density functions, computed by multidimensional numeric integration.

It was demonstrated that RCPT codes have the potential to yield improved FER performance over the block-fading Gaussian collision channel, compared to rate compatible punctured convolutional (RCPC) codes. This improvement in FER performance translates into increased throughput of incremental redundancy schemes based on RCPT codes relative to RCPC codes.

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