Analyzing the Effect of Antenna Type and MAC Protocol on Wireless Ad-Hoc Network Capacity

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Abstract—In this paper, we analyze the capacity of wireless ad-hoc networks, defined as the maximum achievable network throughput, for different antenna types and Media Access Control (MAC) protocols. Specifically, we perform a detailed analysis for omni-directional and directional antennas, under various propagation environments. We provide results for both the exact (constant factor) and asymptotic behavior of the capacity, and illustrate the effect of important antenna parameters and different MAC protocols on it. In order to do so, we use a joint interference & protocol model that identifies how and when the capacity is bounded by the average interference level in the network and when by the MAC protocol used. We argue that if the MAC protocol is the bounding factor for the capacity, then the spatial resources of the network are underutilized, and the MAC protocol can be made to perform more efficiently. To support our argument, we analyze the performance of the DMAC (directional antenna MAC) protocol [6] [7], identify the cases when DMAC becomes the bounding factor, and improve it to maximize capacity. In terms of asymptotic behavior, we prove using a different approach, that the bounds derived by Kumar and Gupta [2] hold, for both omni-directional and finite-gain directional antennas, for path loss exponents n > 2, but need to be scaled by a factor (logN)^-1 for free-space propagation environments. Finally, we analyze the capacity of a heterogeneous network, where a fraction p of nodes uses omni-directional antennas and the remaining 1-p use directional ones. We prove that the performance degradation over the fully directional case is linear on p (ap+b), a result which is encouraging for the gradual deployment of directional antennas.

Index Terms—ad-hoc network, capacity, directional antenna, interference.

I. INTRODUCTION

Wireless ad-hoc networks are multi-hop networks where all nodes cooperatively maintain connectivity. The ability to be set up fast and operate without the need of any wired infrastructure (e.g. base stations, routers, etc.) makes them a promising candidate for military, disaster relief, and law enforcement applications. Furthermore, the rapidly growing interest in sensor network applications has created a need for protocols and algorithms for large scale self-organizing ad-hoc networks, consisting of hundreds or thousands of nodes.

A vast number of protocols and algorithms have been proposed for wireless ad-hoc networks [17]. However, the unique nature of such networks, stemming from probabilistic and highly volatile wireless connectivity, various mobility patterns, the need for every node to act as a relay, etc., renders theoretical analysis of the performance of different architectures and protocols a very challenging task. As a result, there are still a number of fundamental questions, regarding intrinsic characteristics and/or limitations of such networks, which remain unanswered or unclear.

One such important attribute of a wireless ad-hoc network is its capacity. The need for nodes to compete, in order to gain access to the shared wireless media, raises important questions: How many nodes can be allowed to communicate concurrently in the network, without garbling each other transmissions? How is that affected by operation environment, transceiver technology (e.g. radio sensitivity, antenna type, etc.), and access protocols used?

There have been a number of papers trying to answer some of these questions. In their very influential paper [2] Kumar and Gupta performed a capacity analysis for networks consisting of nodes using omni-directional antennas, and derived tight asymptotic bounds for a variety of cases. Their most important result was that in wireless network of N nodes lying in a disk of area A, the capacity available to each node for its own traffic scales Θ(√A/√N), even if nodes’ positions and communication pairs are chosen arbitrarily. On the other hand, if nodes are uniformly distributed and communication pairs are randomly chosen, capacity further decreases to Θ(√A / √N logN).

The anticipated gains of using directional antennas, in terms of energy efficiency [16], increased throughput [15], etc., has led a number of researchers to extend the analysis of Kumar and Gupta, to the cases of directional [5], [10], or even smart antennas [1] [10]. A different approach is taken by the authors of [4], where concepts from network flow theory are used to reevaluate Kumar’s results and extend them to the case of directional antennas.

On a different level, the authors in [3] explore how node mobility can be exploited in order to overcome the fundamental capacity bound established in [2] and achieve a per node capacity of Θ(1). The price paid for this improvement is that no delay guarantees are provided. On the other hand, when there is a maximum allowable delay
requirement $d$, it was shown in [11] that there is a critical value for $d$, below which no significant improvement can be achieved through node mobility, and above which capacity increases as $d^{2/3}$.

A number of these works base their analysis on the assumption that the media access protocol establishes disjoint silence regions around each communicating node (e.g. 802.11 [9]), and compute network capacity using geometric arguments to bound the maximum number of such disjoint regions. Others employ an elementary interference model, where the Signal to Interference and Noise Ratio (SINR) at each receiving node must remain above a radio-specific threshold value. It is common to call the former, the Protocol Model and the latter the Interference or Physical Model [2].

However, it is important to understand that it is the subtle interplay between the interference level requirement and MAC protocol behavior that defines the actual capacity bound for each scenario. For example, in a harsh operating environment, where attenuation of signals is quite large (e.g. path loss exponent $n > 4$), or where nodes are using highly directional antennas, the average interference level at each node is expected to be relatively low. In such environments, it is the protocol that will most probably impose a bound on the capacity, and the SINR requirement. On the other hand, when each transmitting node is contributing a large amount of interference in the surrounding environment, far less concurrent transmissions can occur successfully, than the MAC protocol would allow. It is for this reason that we introduce in this paper the Joint Interference and Protocol Model that accurately models these interactions. An important consequence of it is that, whenever the capacity is bounded by the MAC protocol used, it implies that the spatial resources of the network are underutilized and the protocol can be improved.

In this paper we utilize the Joint Interference and Protocol Model to perform a detailed capacity analysis for the cases of:
- networks of nodes using omni-directional antennas
- networks of nodes using directional antennas
- heterogeneous networks where only a percentage $p$ of nodes are using omni-directional antennas, while the rest are using directional ones.

Doing so, we investigate the effect that antenna parameters, propagation environment, radio sensitivity and MAC protocol have on the maximum capacity. Our results describe both the exact (constant factor), as well as asymptotic behavior of the capacity.

In the next section we describe the joint interference and protocol model in more detail, and present important models and definitions we shall use in our analysis. In section 3 we perform a thorough analysis for the case of nodes using omni-directional antennas. After we justify the importance of a silence region established by the MAC protocol around a communicating node, we derive analytic results for the capacity under different propagation environments. We also re-evaluate the scaling laws of Kumar and Gupta, and prove that they need to be scaled by a factor of $(\log N)^{3}$, when free space propagation is assumed. In section 4 we assume that all nodes are using directional antennas and implement the DMAC protocol, a directional version of 802.11 [6] [7]. We derive analytic capacity results and quantify the achievable performance improvement over the omni-directional case, as a function of important antenna parameters like gain and beamwidth, under different operating environments. We also identify the conditions under which DMAC becomes the bounding factor for the capacity, and propose a modified version, we call $DMAC+$, which improves the capacity under these conditions. The importance of this methodology comes from a network designer’s point of view and is twofold:
- One can choose the right (i.e. most efficient) MAC protocol to use, if antenna type, radio, and propagation environment are known.
- It can be generalized to analytically compare the performance of different MAC protocols and identify conditions under which one will more efficient than the other.

We close this section by proving that the asymptotic behavior for both protocols is the same as that for the omni-directional case, when directional antennas of finitely large gain are used. We conclude our analysis with the case of a heterogeneous network consisting of a fraction $p$ of nodes being equipped with omni-directional antennas, while the remaining $1-p$ are using directional ones, is analyzed in Section 6. We prove that the performance degradation over the fully directional case is linear on $p (ap+b)$, a result which is encouraging for the gradual deployment of directional antennas. Finally, in Section 6, we discuss a few interesting topics that we plan to further pursue, and draw a conclusion.

II. PRELIMINARIES

The setting on which we formulate our problem is similar to that used in many capacity related papers on ad-hoc networks [2] [5] [10], generalized or simplified wherever deemed appropriate.

A. Problem Formulation and Capacity Measures

We formulate our problem as follows:
- There are $N$ nodes uniformly distributed on a planar disk $A$ of radius $k$.
- Time is assumed to be slotted and nodes synchronized. A slot is assumed to be long enough to comprise a complete communication exchange between two nodes, depending on the protocol used (e.g. RTS, CTS, DATA, ACK, etc). The assumption of slotted time is only used to facilitate our analysis. It is not however necessary for our results to hold.
- Without any loss of generality, we assume that a single channel is shared among all nodes. The channel capacity and slot duration can be easily scaled to
achieve same performance as the multiple channel case.

- Any node engaged in communication within a time slot is considered as active during that slot. An active node may receive (or transmit) data throughout the duration of the slot, and thus incur (or generate) interference by (or towards) other active nodes.
- $\rho_{\text{act}}$ denotes the average density of active nodes in the network.
- Network throughput $C_N$ is measured in terms of the number of active nodes that can successfully communicate, during any time slot. If we require a node to be active only if it can successfully communicate during that slot, then $\rho_{\text{act}} = C_N / \|A\|^1$ (active nodes/m$^2$).

**Definition 2.1:** We define the network capacity or capacity $C_{\text{MAX}}$ as the maximum achievable network throughput $C_N$.

**B. Joint Interference and Protocol Model**

Each node is transmitting with the same power $P$. We assume that $P$ may not be high enough to reach every node in the network, and therefore relaying of packets may be required. We assume a log-distance path loss model as follows: the received power $Pr$ of a node at distance $d$ from the transmitter is given by

$$Pr = P * c / d^n$$  \hspace{1cm} (1)

We let the path loss exponent $n$ take different values (roughly ranging from 2 to 6), in order to capture in our analysis the effect of different propagation environments [14]. We ignore any shadowing and small-scale fading effects.

The receiver is characterized by two sensitivity values, namely $Pr_{\text{min}}$ and $\text{SINR}_{\text{min}}$. These two values depend on the receiver technology, coding scheme, modulation scheme, etc.

- $Pr_{\text{min}}$ is the minimum received signal power that can be used to successfully recover the useful data, when there is no interference, and defines the maximum transmission range $R_t$ of a node as follows:

$$R_t \leq \sqrt[n]{c(P / Pr_{\text{min}})}$$  \hspace{1cm} (2)

- $\text{SINR}_{\text{min}}$ is the minimum Signal to Interference and Noise Ratio that allows the receiver to capture (i.e., successfully recover) the intended signal in a noisy environment. Interference from different sources is added incoherently at the receiver. Consequently, the total interference power at the receiver is equal to sum of the power received by each individual interference source. We assume that interferers are uniformly distributed with density $\rho_{\text{int}}$ in the interference region $U(I)$. Then the requirement for successful reception at distance $d$ is given by:

$$\frac{P_{\text{cd}}}{N + \int_{U(I)} d l} \geq \text{SINR}_{\text{min}}$$  \hspace{1cm} (3)

Since we are trying to maximize the number of concurrent transmissions, it is safe to assume that the interference level at any node will be considerably higher than internal noise $N$. The model described up to this point is an integral form typical Interference Model used in the related literature. We shall use this model in the next section, in order to demonstrate the shortcomings of the lack of a silence region.

**Definition 2.2:** In the Joint Interference and Protocol Model, the MAC protocol establishes a silence region $U(NI)$ around any active node $AN$. $U(NI)$ is a no interference zone. The interference zone is defined $U(I) = A - U(NI)$. No overlap between silence regions is allowed.

Any node in $U(NI)$ is forced to refrain from transmission throughout the duration of this slot, and hence cannot act as a source of interference. This could be achieved, for example, by means of RTS and CTS packets, as in the case of 802.11. On the other hand, any node outside the silence region may potentially interfere with AN. Finally, we note that, in a real setting, silence regions may partly overlap. However, we assume that a slack $\Delta$ is included in the establishment of the silence region, so as to guarantee that silence regions are disjoint. This assumption will make our results only slightly more pessimistic.

**C. Antenna Models**

In Section 3 we assume that nodes are using omni-directional antennas that have uniform gain $G = 1$ towards every direction in the azimuth plane, for both transmission and reception.

On the other hand, a directional antenna is characterized by its gain function:

$$G(\phi, \theta) = \frac{EIRP(\phi, \theta)}{P}$$

$\text{EIRP}(\phi, \theta)$ is the effective isotropically radiated power towards direction $(\phi, \theta)$, defined as the amount of power that would need to be transmitted by an ideal isotropical antenna, in order to achieve the same power density at that direction. We assume further that nodes use the same directional antenna for both transmission and reception. Therefore, we know that $G_T(\phi, \theta) = G_R(\phi, \theta)$ (reciprocity theorem). Our log-distance path loss model needs to be modified, in order to include antenna gain as follows:

$$Pr = P_c G_T G_R \frac{d^n}{d^n}$$  \hspace{1cm} (4)

The model used for directional antennas, throughout this paper, is that of an ideal, flat-topped antenna, with an antenna pattern as depicted in Fig.1. This is a useful directional antenna model, frequently used in related literature, which is simple enough to allow any analytic work.
yet captures successfully important antenna characteristics like antenna beam-width and side-lobe gain. Since all nodes are placed on the plane we are only interested in the gain $G(\phi)$. Additionally, we normalize $G(\phi)$ to 1, that is, we assume that the transmitting power $P$ is adjusted so as to achieve the same transmission range $R_s$, as an omni-directional antenna. $R_s$ (and consequently $P$) are dictated by the antenna beam-width, and $G_s$ is the side-lobe gain. Finally, we assume that the flat-topped antenna pattern is electronically steerable towards any desired $\phi_0$.

The gain function $G(\phi)$ for the flat-topped antenna is given by

$$
G(\phi) = \begin{cases} 
1 & |\phi - \phi_0| \leq \frac{\theta}{2} \\
G_s & |\phi - \phi_0| > \frac{\theta}{2}
\end{cases}
$$

(5)

where $\phi_0$ is the direction the antenna is pointing, $\theta$ is the antenna beam-width, and $G_s (<1)$ is the side-lobe gain.

![Figure 1: Normalized flat-topped antenna pattern](image)

**Figure 1: Normalized flat-topped antenna pattern**

### D. Arbitrary vs. Random Networks

The concept of Arbitrary Networks and Random Networks was introduced in [2]. Here, we differentiate between these two types only in regards to the placement of nodes in the disk and the resulting connectivity requirement on the transmission range $R_s$. Furthermore, in contrast to the common assumption of a fixed size disk, we assume that the size of the network grows along with the number of nodes $N$, in order to retain specific density characteristics. Specifically:

- The area $|A|$ of the network grows as $\Theta(N)$. Alternatively, the radius $k$ of the disk grows as $\Theta(\sqrt{N})$.
- Nodes in Random Networks are uniformly distributed on the plane. The transmission range $R_s$ need to be scaled as $\Theta(\sqrt{\log N})$, in order to guarantee connectivity with high probability [12] [13].
- Nodes in Arbitrary Networks can be placed in an arbitrary fashion on the plane. A constant transmission range $R_s = \Theta(1)$, is adequate for connectivity with high probability, provided that nodes are placed appropriately (e.g. in a grid-like manner).

We note that our approach is equivalent to the fixed size disk case, as the authors in [2] also note.

### III. THE OMNI-DIRECTIONAL CASE

#### A. Interference Model - The Need For A “Silence Region”

Our methodology for calculating $C_{MAX}$ when using the Interference Model is the following: The interference level depends on the number of active nodes in the network. The more the active nodes, the higher the interference level at any node. We are therefore looking for the maximum interference level that does not violate Eq.3. The interference $dI$ from an infinitesimal area $dU$ in $U(I)$, at angle $\phi$ and distance $r$ from the receiver is equal to $dI = I(\phi,r) r dU$. Substituting $dI$ in Eq.3 and solving for $\rho_{act}$ we derive the maximum active node density. Multiplying this by the total area $\pi k^2$ of $A$, we get the maximum capacity $C_{MAX}$:

$$
C_{MAX} \leq \frac{Pc d^\alpha \pi k^2}{\text{SINR}_{\text{min}} \int_{U(I)} I(\phi,r) dU}
$$

(6)

Let’s assume that there is no restriction on the minimum distance $d_{\text{min}}$ between two active nodes: $d_{\text{min}} \to 0$. The total number of active nodes is only bounded by Eq. 3.

**Theorem 3.1:** The network capacity $C_{MAX}$ of a network with nodes using omni-directional antennas, when active nodes are allowed to be arbitrarily close is given by:

$$
C_{MAX} \leq \begin{cases} 
\lim_{d_{\text{min}} \to 0} \frac{k^2 R_s^{-n} (n-2)}{2\text{SINR}_{\text{min}} \left( \frac{d_{\text{min}}^{-n+2} - k^{-n+2}}{k^2 R_s^{-n}} \right)}, & n > 2 \\
\lim_{d_{\text{min}} \to 0} \frac{k^2 R_s^{-n} \log(k/d_{\text{min}})}{2\text{SINR}_{\text{min}}}, & n = 2
\end{cases}
$$

(7)

**Proof:** The average number of interferers in a ring of infinitesimal width $dr$ at distance $r$ around the node is equal to $dN = \rho_{act} dU = \rho_{act} 2\pi r dr$. The interference by any node inside this ring is $I(\phi(r)) = I(r) = P c r^{-\alpha}$. We integrate $I(r)dU$ for $U(I)$: $r \in [d_{\text{min}}, k]$ and replace in (3). Taking the limit as $d_{\text{min}} \to 0$, we get:

$$
C_{MAX} \leq \lim_{d_{\text{min}} \to 0} \frac{k^2}{2\text{SINR}_{\text{min}} \int_{d_{\text{min}}}^{k} r^{-n+1} dr}
$$

Calculating the integral in the denominator gives us (7) ■

Theorem 3.1 states that if active nodes are uniformly distributed and allowed to be arbitrary close to each other, then almost no node would be able to communicate successfully, in the average case. The intuitive reason for that is captured in the following Lemma, which is given here without proof, due to lack of space.

2 We use polar coordinates because of the geometry of the problem.
Lemma 3.1: The average distance of the closest interferer to any node, as \( k \to \infty \), approaches zero, when \( d_{\min} \to \infty \).

Lemma 3.1 implies that for large enough \( k \) (i.e. large enough number of uniformly distributed active nodes in disk \( A \)), the closest interferer to any receiving node is going to be within range to be able to prevent successful communication.

The above analysis does not correspond to any practical scenario of interest. Any MAC protocol would resolve simultaneous attempts to communicate by different nodes that are close-by to a single winner, using means of carrier sensing, back-off mechanisms, collision detection, scheduling, etc. Nonetheless, it provides the necessary intuition why MAC protocols that create a silence region around any communicating node perform better, as well as motivates the introduction of the Joint Interference and Protocol Model for our analysis.

B. Joint Interference and Protocol Model

We would like to calculate once more the capacity \( C_{\text{MAX}} \) of the network. We can use the same methodology as in the previous section and maximize the acceptable interference level in Eq.3, to derive a bound on the network. We can use the same methodology as in the previous section and maximize the acceptable interference level in Eq.3, to derive a bound on the network.

Consequently, the MAC protocol can be improved to allow for this higher number.

We can now go on to calculate the two individual bounds, in order to derive \( C_{\text{MAX}} \).

Lemma 3.4: The interference-imposed capacity bound \( C_{\text{MAX}}^{I} \) for a wireless network consisting of nodes using omni-directional antennas is given by:

\[
C_{\text{MAX}}^{I} = \begin{cases} 
\frac{k^2 R_s^n (n-2)}{2\sin R_{\min}^n (R_s^{n+2} - k^{n+2})}, & n > 2 \\
\frac{k^2 R_s^n}{2\sin R_{\min}^n \log(k / R_s)}, & n = 2 
\end{cases}
\]

Proof: The proof is the same as that of Theorem 3.1, where \( d_{\min} \to 0 \) is now replace by \( R_s \).

Lemma 3.5: The protocol-imposed capacity bound \( C_{\text{MAX}}^{P} \) for a wireless network consisting of nodes using omni-directional antennas is given by:

\[
C_{\text{MAX}}^{P} \leq \frac{k^2}{R_s^2}
\]

Proof: The area \( \|U(NI)\| \) of the silence region is equal to \( \pi R_s^2 \), while the area \( \|A\| \) of the planar disk \( A \) is equal to \( \pi k^2 \). Thus at most (\( \|U(NI)\| / \|A\| \)) non-overlapping silence regions can coexist in the network.

Theorem 3.2: The capacity of a wireless ad-hoc network consisting of nodes using omni-directional antennas, whose size is infinitely large (\( k \to \infty \)) is equal to:

\[
C_{\text{MAX}} = \begin{cases} 
C_{\text{MAX}}^{I}, & n = 2 \quad \text{or} \quad \frac{2\sin R_{\min}^n (R_s^{n+2} - k^{n+2})}{n-2} > 1 \\
C_{\text{MAX}}^{P}, & n > 2 \quad \text{and} \quad \frac{2\sin R_{\min}^n}{n-2} < 1 
\end{cases}
\]

Proof: We calculate the ratio \( L = \lim_{k \to \infty} C_{\text{MAX}}^{P} / C_{\text{MAX}}^{I} \).

For \( n > 2 \), using Eq. 8 and 9 we get:

\[
L = \lim_{k \to \infty} \frac{k^2}{R_s^2} = \frac{2\sin R_{\min}^n (R_s^{n+2} - k^{n+2})}{n-2}
\]

For \( n = 2 \), using Eq.8 and 9 we get:

\[
L = \lim_{k \to \infty} \frac{k^2}{R_s^2} = \frac{2\sin R_{\min}^n \log(k / R_s)}{k^2 R_s^n}
\]

However we know that:

\[
- L > 1 \iff \min\{C_{\text{MAX}}^{I}, C_{\text{MAX}}^{P}\} = C_{\text{MAX}}^{I}
\]
\[ \log N \]

between two

\[ (1) \]

\[ \text{SINR}_{\text{min}} = 1 \] implies modern radio

LAN or multi-hop ad-hoc network implementations. On the

other hand, SINR\(_{\text{min}}\) = 10 (10dB) implies practical all for

interference that will limit the number of active nodes

during a slot. The following Corollary of Theorem 3.2

describes this behavior for two different values for SINR\(_{\text{min}}\),

namely 1 and 10. SINR\(_{\text{min}}\) = 10 (10dB), is a typical threshold

value for a number of radio cards frequently used in wireless

technologies and sophisticated coding and/or error correction

schemes.

**Corollary 4.2:** The capacity of a wireless ad-hoc network

consisting of nodes using omni-directional antennas, whose

size is infinitely large \((k \to \infty)\), is:

\[
\begin{align*}
\text{interference-bound, } 2 \leq n < 4 & \quad (\text{SINR}_{\text{min}} = 1) \\
\text{protocol-bound, } n \geq 4 & \quad (\text{SINR}_{\text{min}} = 10)
\end{align*}
\]

**Proof:** The proof follows immediately from (10)

\[ \square \]

C. Asymptotic Capacity Bounds

We now focus our attention into the asymptotic behavior of the capacity of a wireless ad-hoc network. In other words we are interested on the rate of growth of the network throughout, as the number of nodes in the network \(N\) grows to infinity. This behavior is important to evaluate the scalability of the network. The following Lemma defines the capacity of an ad-hoc network, where communication endpoints are chosen to be 1 hop away (def. single-hop capacity).

**Lemma 3.6:** The single-hop capacity of a wireless ad-hoc

network, consisting of nodes using omni-directional antennas, grows asymptotically with the number of nodes \(N\) in the network as:

\[ C_{\text{MAX}}(N) = \Theta\left(\frac{N}{\log N}\right) \]

when \(n = 2\)

\[ C_{\text{MAX}}(N) = O(N) \]

when \(n > 2\)

**Proof:** For \(n = 2\), the network capacity is always

interference bound. Replacing \(k = \sqrt{N}\) into Eq.8 we get:

\[ C_{\text{MAX}}(N) \leq \frac{\text{NR}_{\text{min}}^{-2}}{2\text{SINR}_{\text{min}} \log \left(\sqrt{N}/R_s\right)} \]

For large \(N:\)

\[ C_{\text{MAX}}(N) \leq \frac{R_s^{-2}N}{\text{SINR}_{\text{min}} \log(N) - 2\text{SINR}_{\text{min}} \log(R_s)} = \frac{C N}{\log N} \]

and \(C = (R_s^{-2}/\text{SINR}_{\text{min}})\). Therefore \(C_{\text{MAX}}(N) = O\left(\frac{N}{\log N}\right)\).

For \(n > 2\), and \(N \to \infty\) we get:

\[ C_{\text{MAX}}(N) \leq \frac{\text{NR}_{\text{min}}^{-2}(n-2)}{2\text{SINR}_{\text{min}} R_s^{-n+2} - N^{(-n+2)/2}} = C^1 N, \]

and \(C_{\text{MAX}}(N) \leq \frac{N}{R_s^{-2}} = C^p N, \)

where \(C^1 = \frac{R_s^{-2}(n-2)}{2\text{SINR}_{\text{min}}} \) and \(C^p = R_s^{-2}\).

Consequently, for \(n > 2\): \(C_{\text{MAX}}(N) = O(N) \)

\[ \square \]

We have discussed earlier that the transmission range \(R_s\) of each node is not large enough to cover the whole network, so that relaying of packets from intermediate nodes may be necessary. Consequently, an end-to-end communication exchange between two arbitrary nodes may involve forwarding of packets over a number of intermediate hops. The average number of such hops is given by the following Lemma.

**Lemma 3.7:** The average number of intermediate hops \(\bar{l}\)

in a path between two randomly chosen nodes, in a network

with \(N\) uniformly distributed nodes, is \(\Theta\left(\sqrt{N}/R_s\right)\).

**Proof:** Let us calculate the average distance \(\bar{D}^{(1)}\) between
two random points \(x, y \in [-k, k]\) on a line segment of size 2\(k\).

Each point is uniformly distributed on the line and independently of the other. Consequently, the joint probability density function of the two points is given as

\[ f_{xy}(x, y) = \frac{1}{k^2}, \forall x, y \in [-k, k]. \]

We can now compute \(\bar{D}^{(1)}\) as:

\[ \bar{D}^{(1)} = \frac{1}{k^2} \int_{-k}^{k} \int_{-k}^{k} |x - y|^2 dx dy = \frac{1}{k^2} \int_{-k}^{k} \left[ \int_{-k}^{x} (x - y)^2 dy + \int_{y}^{k} (y - x)^2 dy \right] dx = \frac{2k^3}{3} \]

We can now bind the average distance \(\bar{D}^{(2)}\) between two randomly chosen nodes \(x = (x_1, x_2), y = (y_1, y_2)\), uniformly distributed on a planar disk of radius \(k\), as follows:

\[ \bar{D}^{(1)} \leq \bar{D}^{(2)} \leq E\|x - y\| \leq 2\bar{D}^{(1)} \Rightarrow \frac{2k}{3} \leq \bar{D}^{(2)} \leq \frac{4k}{3} \]

Since the transmission range of each node is equal to \(R_s\), the average number of hops \(\bar{l}\) in the path is given by

\[ \bar{l} = (\bar{D}^{(2)}/R_s) = \frac{2}{3R_s} k \leq \bar{l} \leq \frac{4}{3R_s} k \Rightarrow \bar{l} = \Theta(k/R_s). \]

Finally, we have assumed that the size of the network grows
as \( \Theta(N) \) or equivalently that the radius \( k \) of the disk grows as \( \Theta(\sqrt{N}) \) which implies that \( L \) is \( \Theta(\sqrt{N}/R_s) \).

The need for each node to relay packets neither originating from nor destined to that node, implies that only a fraction of the available capacity is available to each node for its own traffic. The scaling law for the capacity available to each node, as the number \( N \) of nodes in the network grows to infinity, was initially derived by Kumar and Gupta \[2\]. We re-evaluate and generalize this scaling law in the following two important theorems, based on our analysis performed using the more detailed Joint Interference and Protocol Model.

**Theorem 3.3:** In an Arbitrary Network of \( N \) nodes using omni-directional antennas the asymptotic capacity available to each node for its own traffic is

\[
\Theta\left(\frac{1}{\sqrt{N \log N}}\right) \quad \text{for } n = 2
\]

\[
\Theta\left(\frac{1}{\sqrt{N}}\right) \quad \text{for } n > 2
\]

**Proof:** In Lemma 3.6 we derived the asymptotic behavior of the single-hop capacity \( C_{\text{MAX}}(N) \). However, Lemma 3.7 implies that each end-to-end packet will need to be successfully relayed by at least \( L \) intermediate nodes, which view this packet as overhead traffic. Since communication pairs are randomly chosen, the overhead of relaying packets, as \( N \) grows large, will be uniformly distributed among all nodes in the network. Consequently, the capacity available for end-to-end traffic is only a fraction \( C_{\text{MAX}}(N)/L \) of the total single-hop capacity \( C_{\text{MAX}}(N) \). This capacity is shared among all \( N \) nodes. Therefore, the capacity available to each node for its own traffic is \( \frac{C_{\text{MAX}}(N)}{NL} \). We substitute \( C_{\text{MAX}}(N) \) from Lemma 3.6 and \( L \) from Lemma 3.7, and note that in the case of Arbitrary Networks \( R \), does not need to scale with \( N \).  

**Theorem 3.4:** In a Random Network of \( N \) nodes using omni-directional antennas the asymptotic capacity available to each node for its own traffic is

\[
\Theta\left(\frac{R_s^{-1}}{\sqrt{N \log(\sqrt{N}/R_s)}}\right) \quad \text{for } n = 2
\]

\[
\Theta\left(\frac{R_s^{-1}}{\sqrt{N}}\right) \quad \text{for } n > 2
\]

However, in a Random Network, \( R \) needs to scale with \( N \) as \( \Theta(\log N) \), in order to ensure connectivity with a high probability. Therefore,

\[
\Theta\left(\frac{(\log N)^{1/2}}{\sqrt{N \log N}}\right) = \Theta\left(\frac{1}{\sqrt{N (\log N)^{1/2}}}\right) \quad \text{for } n = 2
\]

\[
\Theta\left(\frac{1}{\sqrt{N \log N}}\right) \quad \text{for } n > 2
\]

**IV. THE DIRECTIONAL CASE**

**A. The DMAC Protocol (DMAC)**

In this section we analyze the capacity of wireless ad-hoc networks utilizing flat-topped directional antennas as described in section 2. A directional version of 802.11 MAC protocol (DMAC) is assumed to be implemented by all nodes, in order to access the shared channel using their directional antennas. This protocol was independently proposed in [6] and [7], and has been shown to perform quite efficiently. It is essentially an adaptation of the collision avoidance and virtual carrier sensing mechanisms of the original 802.11, for ad-hoc networks utilizing directional antennas to communicate. For this paper to be self-contained, we will briefly summarize the protocol features that are needed for our analysis. More details can be found in the references:

- All nodes have two modes of operation, directional and omni-directional.
- When nodes are idle, they’re listening to the media omni-directionally. All non-broadcast packets (i.e. RTS, CTS, DATA, and ACK) are transmitted directionally.
- On reception of an RTS packet a node switches to directional mode and points its antenna back to the transmitting node, based on the direction-of-arrival of the RTS packet or knowledge of the location of that sender.
- Directional virtual carrier sensing is implemented as follows. Each node keeps a directional NAV table with a similar use to the NAV value in 802.11. When it overhears an RTS or CTS packet, not destined to itself, it marks the direction-of-arrival in its NAV table as “busy” for the time duration contained inside the packet. This is how the silence region gets established.
- When a node has a packet to transmit, it checks the direction of the intended recipient in its NAV table to see whether there is any ongoing transmission in that direction. If there is, it backs off and tries again later.

The characteristics of the silence and interference regions,
established by the DMAC protocol are described in the following Lemma.

**Lemma 4.1:** The silence region $U(NI)$ and interference region $U(I)$, established by the DMAC protocol, around any active node, whose antenna is pointing at direction $\phi_0$ is given as follows:

$$U(NI) = \begin{cases} r, \phi : 0 \leq r \leq R_s \text{ and } |\phi - \phi_0| \leq \theta/2 \\ r, \phi : 0 \leq r \leq R_{sl} \text{ and } |\phi - \phi_0| > \theta/2 \end{cases}$$

$$U(I) = \begin{cases} r, \phi : R_s < r \leq k \text{ and } |\phi - \phi_0| \leq \theta/2 \\ r, \phi : R_{sl} < r \leq k \text{ and } |\phi - \phi_0| > \theta/2 \end{cases}$$

(12)

$R_{sl} = R_s \sqrt[2/n]{G_s}$

**Proof:** When a node sends an RTS or CTS packet, it is transmitted with gain $G_T = 1$, inside the main beam, and gain $G_T > G_s$ outside the main beam. Furthermore, any node that receives the packet receives it with $G_R = 1$, since nodes in idle mode listen to the media omnidirectionally. Replacing $G_T$ with gain $G_T = 1$, inside the main beam, and gain $G_R$ in Eq.4 and taking $P_r = P_{min}$, we can compute the maximum distance up to which an RTS or CTS packet will be heard correctly: $d_{max} = \frac{\eta \sqrt{c(P/P_{min})}}{\sqrt[2/n]{G_s}}$ for $|\phi - \phi_0| \leq \theta/2$ \r

$d_{max}$ defines the border of the silence region. Finally using Eq.2 we derive $U(I)$ and $U(NI)$ in terms $R_s$.

**Lemma 4.2:** The average antenna gain for an arbitrary interfering signal coming from $U(I)$, when nodes are using flat-topped directional antennas, is given by

$$\overline{G} = \frac{\theta + (2\pi - \theta)}{2\pi}$$

(13)

**Proof:** An active node in the interference region $U(I)$ may be pointing its antenna towards any direction with equal probability. Therefore, $\overline{G}$ is computed easily from Eq.5

We are now ready to proceed with the computation of the network capacity using the Joint Interference and Protocol Model. The interference imposed and protocol imposed capacity bounds $C^I_{MAX}$ and $C^P_{MAX}$, respectively, are given by the following two Lemmas.

**Lemma 4.3:** The interference-imposed capacity bound $C^I_{MAX}$ for a wireless network consisting of nodes using directional antennas and implementing the DMAC protocol in order to communicate is given by:

For $n = 2$,

$$C^I_{MAX} \leq \frac{\text{SINR}_{min}^{-1} \pi k^2 R_s^n}{\theta G \log \left(\frac{k}{R_s}\right) + (2\pi - \theta)G_G^{2n} \log \left(\frac{k}{R_s \sqrt[2/n]{G_s}}\right)}$$

(14)

For $n > 2$,

$$C^I_{MAX} \leq \frac{(n-2)\text{SINR}_{min}^{-1} \pi k^2 R_s^n}{R_s^{n+2}(\theta G + (2\pi - \theta)G_G^{2n}) - k^{n+2}(\theta G + (2\pi - \theta)G_G^{2n})}$$

**Proof:** The reception antenna gain $G_s$ for an arbitrary interfering signal depends on its angle of arrival in respect to the receiving antenna orientation $\phi_0$. Without any loss of generality we assume that $\phi_0 = 0$. Additionally, the transmission antenna gain $G_T$ for any interfering node is given by Lemma 4.2. For this reason we divide the interference area $U(I)$ into $U_{0(I)} = \{r, \phi : R_s < r \leq k \text{ and } |\phi| \leq \theta/2\}$, that lies within the main beam of the receiving antenna and the remaining $U_{2\pi-\theta}(I) = \{r, \phi : R_{sl} < r \leq k \text{ and } |\phi| > \theta/2\}$.

Consequently:

$$\int_I I(\phi(r)dU = \int_{U_{0(I)}} I(\phi(r)dU + \int_{U_{2\pi-\theta}(I)} I(\phi(r)dU$$

The area of an arc in $U_{0(I)}$, that has an angle $\theta$ and infinitesimal width $dr$ at distance $r$ from the node is equal to $dU = \theta dr$. The interference by any node inside this arc is $I(\phi, r) = P_cG_sR^{-n}r^{-n}$. Similarly for an arc in $U_{2\pi-\theta}(I)$ we get:

$$I(\phi(r) = P_cG_sR^{-n}r^{-n}$$

We can now integrate $I(\phi)dU$ within $U(I)$:

$$\int_{U(I)} I(\phi(r)dU = \int_{R_s}^{R_{sl}} \text{PcG}_sR^{-n}r^{-n}dr + (2\pi - \theta) \int_{R_{sl}}^{R_s} \text{PcG}_sR^{-n}r^{-n}dr$$

$$= \text{Pc}[\text{G}_s\log(k/R_s) + (2\pi - \theta)\text{G}_s[k^{n+2} - R_{sl}^{-n+2}]] (n = 2)$$

$$= \frac{\text{Pc}}{2n} [\text{G}_s[k^{n+2} - R_{sl}^{-n+2} + (2\pi - \theta)\text{G}_s[k^{n+2} - R_{sl}^{-n+2}] ] (n > 2)$$

Substituting $R_{sl}$ from Eq.12 and using Eq.3 we get $C^I_{MAX}$

**Lemma 4.4:** The protocol-imposed capacity bound $C^P_{MAX}$ for a wireless network consisting of nodes using directional antennas and implementing the DMAC protocol in order to communicate is given by:

$$C^P_{MAX} = \frac{\pi k^2}{\theta + (\pi - \theta)G_G^{2n}R_s^n}$$

(15)

**Proof:** The area of the silence region is given by:

$$\|U(NI)\| = \frac{\theta}{2}R_s^n + \left[(\pi - \theta)^2R_{sl}^n \right)^{12} \frac{\theta}{2}R_s^n + \left[(\pi - \theta)^2G_G^{2n}R_s^n$$

However, the maximum number of disjoint silence regions
that can fit inside the disk \( A \) is \( \pi k^2 \|U(NI)\| \)

**Theorem 4.1:** The capacity of a wireless ad-hoc network, whose size is infinitely large \( (k \to \infty) \), and consists of nodes using directional antennas and implementing DMAC, is equal to:

\[
C_{MAX} = \begin{cases} 
C_{MAX}^I, & \text{if } n = 2 \text{ or } \frac{2SINR_{min}G}{n-2} > 1 \\
C_{MAX}^P, & \text{if } n > 2 \text{ and } \frac{2SINR_{min}G}{n-2} < 1 
\end{cases}
\]

(16)

**Proof:** We calculate the ratio \( L = \lim_{k \to \infty} \frac{C_{MAX}^P}{C_{MAX}^I} \).

For \( n > 2 \), using Eq.14 and 15 we get:

\[
L = \lim_{k \to \infty} \frac{\pi k^2}{(0/2 + (\pi - 0/2)G_s^{2/n}R_s^2)(n-2)SINR_{min}^{-1} \pi^2 R_s^n}
\]

\[
\times \left[ (\theta_0 G_s + (2 \pi - \theta_0) G_s^{2/n} - k^{-n+2} (\theta_0 G_s + (2 \pi - \theta_0) G_s)) \right]
\]

\[
\Rightarrow L = \frac{SINR_{min} 0 \theta G_s + (2 \pi - \theta_0) G_s^{2/n}}{n-2} \frac{2SINR_{min} G_s}{n-2}
\]

For \( n = 2 \), using Eq.5 we get:

\[
L = \lim_{k \to \infty} \frac{\pi k^2}{0G \log \left( \frac{k}{R_s} \right) + (2 \pi - \theta_0) G_s^{2/n} \log \left( \frac{k}{R_s \sqrt{G_s}} \right)}
\]

Finally, we know that:

\[
\begin{align*}
L &> 1 \iff \min\{C_{MAX}^I, C_{MAX}^P\} = C_{MAX}^I \\
L &< 1 \iff \min\{C_{MAX}^I, C_{MAX}^P\} = C_{MAX}^P
\end{align*}
\]

**Corollary 4.1:** The capacity improvement of using directional antennas and DMAC, over the case of using omni-directional antennas and an 802.11-like protocol is given by:

\[
\begin{align*}
&i) \quad \frac{1}{G} \frac{\|U(NI)_{OMNI}\|}{\|U(NI)_{DMAC}\|}, \quad \text{if } \frac{2SINR_{min}}{n-2} > \frac{1}{G} \\
&ii) \quad \frac{2SINR_{min}}{n-2} \frac{\|U(NI)_{OMNI}\|}{\|U(NI)_{DMAC}\|}, \quad \text{if } 1 < \frac{2SINR_{min}}{n-2} < \frac{1}{G} \\
&iii) \quad \frac{\|U(NI)_{OMNI}\|}{\|U(NI)_{DMAC}\|}, \quad \text{if } \frac{2SINR_{min}}{n-2} < 1
\end{align*}
\]

**Proof:** The full proof is omitted due to lack of space. However, it follows easily from Eq. 8-10 and 14-16, through a few algebraic manipulations.

As explained earlier, when capacity is protocol-bound, it means that DMAC is not effective enough. The reason why this underutilization of available spatial resources occurs is the following: A node AN2 that receives an RTS or CTS by some active node AN1 (i.e. lies within its silence region) will defer from initiating (or accepting) any communication of its own, until the ongoing communication finishes. However, this may not be necessary. It may be the case that the two nodes will not generate enough interference to garble each others communication. One such example is depicted in Fig.3. This is only possible because of the directional nature of the antenna pattern, for both the transmitting and the receiving node. Consequently, AN2 could be allowed to be active in parallel with AN1, despite its being within AN1’s range, i.e. inside the silence region \( U(NI) \) as defined in Lemma 4.1. For this reason we propose a modified version of
DMAC that can efficiently handle situations like the previous one, and improve performance. We call this protocol DMAC+.

Figure 3: Example of spatial underutilization using DMAC

B. Enhanced DMAC Protocol (DMAC+)

The difference between DMAC+ and DMAC is that DMAC+ divides the silence region of DMAC into a smaller silence region U(NI) and a soft (indirect) interference region U(SI). This takes place as follows: any node overhearing an RTS or CTS packet by some active node AN decides whether it is likely to interfere with AN through its side lobe, if it decides to engage itself in communication. If it would interfere, then it refrains from any transmission. Otherwise it can go ahead and transmit “away” from the ongoing transmission. The specific decision rule is given by the following Lemma:

**Lemma 4.5:** Consider a network of nodes using directional antennas, where a node A, implementing DMAC+, receives an RTS or CTS from some node B. Let \( \varphi_b \) denote the angle of arrival (AOA) and \( P_r \) the received power of the incoming signal. Then node A,

i) cannot transmit towards any \( \varphi \), if \( G_s P_r < P_{\min} \) \{U(NI)\}

ii) is allowed to transmit to any direction \( \varphi \notin [\varphi_b - \theta, \varphi_b + \theta] \), if \( G_s P_r < P_{\min} \) \{U(SI)\}

iii) is allowed to transmit towards any \( \varphi \), if \( P_r < P_{\min} \) \{U(I)\}

**Corollary 4.5:** The transmission antenna gain for an arbitrary interfering signal coming from a node AN is

i) \( G_T = 0 \), if AN lies in U(NI)

ii) \( G_T = G_s \), if AN lies in U(SI)

iii) \( G_T = G_s \), if AN lies in U(I)

The following lemma formally defines the boundaries of the regions U(NI), U(SI), and U(I), implied in Lemma 4.5.

**Lemma 4.6:** The silence region U(NI), “soft” interference region U(SI), and interference region U(I), established by the DMAC+ protocol, around any active node, whose antenna is pointing at direction \( \varphi_0 \) are given as

\[
U(\text{NI}) = \begin{cases} 
    \{ r, \varphi : 0 \leq r \leq R_{s1} \text{ and } \varphi - \varphi_0 \leq 0/2 \} \\
    \{ r, \varphi : 0 \leq r \leq R_{s2} \text{ and } \varphi - \varphi_0 > 0/2 \}
\end{cases}
\]

\[
U(\text{SI}) = \begin{cases} 
    \{ r, \varphi : R_{s1} \leq r \leq R_s \text{ and } \varphi - \varphi_0 \leq 0/2 \} \\
    \{ r, \varphi : R_{s2} \leq r \leq R_d \text{ and } \varphi - \varphi_0 > 0/2 \}
\end{cases}
\]

\[
U(\text{I}) = \begin{cases} 
    \{ r, \varphi : R_s < r \leq k \text{ and } \varphi - \varphi_0 \leq 0/2 \} \\
    \{ r, \varphi : R_s < r \leq R_s \text{ and } \varphi - \varphi_0 > 0/2 \}
\end{cases}
\]

\[
R_{s1} = R_s \sqrt{G_s} \cdot R_{s2} = R_s \sqrt{G_s} \quad (17)
\]

**Proof:** Using Lemma 4.5 and Eq.8 it is straightforward to derive the boundaries for the three regions.

We calculate the capacity bounds imposed by interference level and the DMAC+ protocol, respectively, in the following two Lemmas.

**Lemma 4.7:** The interference-imposed capacity bound \( C_{\text{MAX}}^I \) for a wireless network consisting of nodes using directional antennas and implementing the DMAC+ protocol in order to communicate is given by:

For \( n = 2 \),

\[
C_{\text{MAX}}^I \leq \frac{\text{SINR}_{\text{min}}^I \pi k^2 R_s^n}{H_I + H_{SI}} \quad (18)
\]

\[
H_I = 0G \log \left( \frac{k}{R_s} \right) + (2\pi - 0)G_s^{2n} \log \left( \frac{k}{R_s \sqrt{G_s}} \right)
\]

\[
H_{SI} = [0 \theta_s + (2\pi - 0)G_s^2] \log (G_s^{0.5})
\]

For \( n > 2 \),

\[
C_{\text{MAX}}^I \leq \frac{(n-2)\text{SINR}_{\text{min}}^I \pi k^2 R_s^n}{M_I + M_{SI}} \quad (19)
\]

\[
M_I = R_s^{n+2} \left( \theta G_s^2 (2\pi - 0)G_s^n - k^{-n+2} (2\pi - 0)G_s^2 - \theta G_s^2 \right)
\]

\[
M_{SI} = R_s^{n-2} \left[ (G_s^{2} - G_s) + (2\pi - 2\theta) (G_s^{2} - G_s^{n+2} \theta) \right]
\]

**Proof:** \( H_I \) and \( M_I \) are the interference-related terms coming from the interference area U(I), for \( n = 2 \) and \( n > 2 \), respectively. The total interference stemming from U(I) is the same for both DMAC and DMAC+. The only difference between these two cases is an extra term in the denominator, for the case of DMAC+, corresponding to the total (indirect) interference coming from the soft interference region U(SI). These terms are \( H_{SI} \) and \( M_{SI} \) and are calculated using a similar methodology to Lemma 4.3, by

\[
\frac{1}{P_c} \iint_{U(SI)} I(\varphi) dU = \left[ \frac{R_s}{R_s} \int_{R_s} G_s \pi r^{n+1} dr + (2\pi - 0) \int_{R_{s2}} G_s^2 \frac{r^{n+1} dr}{R_{s2}} \right]
\]

Substituting \( R_{s1} \) and \( R_{s2} \) from Eq.17 and computing the
integrals for \( n = 2 \) \( (n > 2) \) we derive \( H_{SI}(M_{SI}) \):

\[
H_{SI} = \log(G_s + (2\pi - \theta)G_s^2) \log\left(G_s^{-0.5}\right)
\]

\[
M_{SI} = R_s^{n+2} \left[ \theta \left( (G_s)^n - G_s \right) + (2\pi - \theta) \left( (G_s)^n - (G_s)^{n+2} \right) \right]
\]

**Lemma 4.8:** The protocol-imposed capacity bound \( C_P \) for a wireless network consisting of nodes using directional antennas and implementing the DMAC+ protocol in order to communicate is given by:

\[
C_P^{\text{MAX}} = \frac{\pi k^2}{2G_s^{2/n} + \frac{\pi}{2} G_s^{4/n} R_s^2}
\]  \( (20) \)

Proof: The area of the silence region is given by:

\[
\|U(NI)\| = \frac{\theta}{2} R_s^2 + \left( \frac{\pi}{2} - \frac{\theta}{2} \right) R_s^2 = \frac{\theta}{2} G_s^{2/n} R_s^2 + \left( \frac{\pi}{2} - \frac{\theta}{2} \right) G_s^{4/n} R_s^2
\]

Dividing the total area \( \|A\| \) by \( \|U(NI)\| \) gives us \( (20) \)

**Theorem 4.2:** The capacity of a wireless ad-hoc network, the size of which is infinitely large \( (k \to \infty) \), and consists of nodes using directional antennas and implementing DMAC+, is equal to:

\[
C_{MAX} = \begin{cases} C_{MAX}^I, & \text{if } n = 2 \text{ or } L > 1 \\ C_{MAX}^p, & \text{if } n > 2 \text{ and } L < 1 \\ \end{cases}
\]

\[
L = \frac{2\text{SINR}_{\text{min}}}{n-2} \left( \frac{\theta(G_s^{2/n} - G_s^2) + (2\pi - \theta) \left( G_s^{2/n} + G_s^{4/n} - G_s^{(n+2)/n} \right) }{\theta G_s^{2/n} + (2\pi - \theta) G_s^{4/n}} \right)
\]

Proof: The proof is similar to that of Theorems 3.1 and 4.1, using Eqs.18, 19, and 20

In Fig 4 we depict how the interference-imposed and protocol-imposed bounds on the capacity relate to each other, as a function of antenna parameters and path-loss exponent. We only depict the behavior for \( \text{SINR}_{\text{min}} = 1 \), because we have found that for high values for \( \text{SINR}_{\text{min}} \) (e.g. equal to 10) the capacity is interference-bound in the vast majority of practical cases.

![Figure 4: Effect of antenna parameters and propagation environment on the capacity behavior of DMAC+. Again, each line in the chart represents the ratio \( L = C_{MAX}^I/C_{MAX}^p \) \( (k \to \infty) \) for the specific parameters mentioned. The dividing point corresponds to \( L \) equal to 1.](image)

**C. Capacity comparison between DMAC and DMAC+**

Let us compare the performance of DMAC and DMAC+, by evaluating the ratio \( \frac{C_{MAX}^I}{C_{MAX}^p} \) \( (k \to \infty) \). \( L > 1 \) implies that DMAC+ performs better than DMAC, while \( L < 1 \) the opposite.

**Theorem 4.3:** Consider a network of nodes using directional antennas. Then the capacity improvement (degradation) \( L \), when using DMAC+ instead of DMAC, as \( k \to \infty \), is given by

i) \( L = \frac{1}{G_s^{2/n}} < 1 \), if \( n = 2 \) or \( \frac{2\text{SINR}_{\text{min}}}{n-2} > \frac{1}{G} \)

ii) \( L = \frac{1}{G_s^{2/n}} > 1 \), if \( \frac{2\text{SINR}_{\text{min}}}{n-2} < \frac{1}{M} \)

iii) \( L = \frac{1}{G_s^{2/n}} < L < \frac{1}{G_s^{2/n}} \), if \( \frac{2\text{SINR}_{\text{min}}}{n-2} < \frac{1}{M} \)

\[
L = \left( \frac{\theta(G_s^{2/n} - G_s^2) + (2\pi - \theta) \left( G_s^{2/n} + G_s^{4/n} - G_s^{(n+2)/n} \right) }{\theta G_s^{2/n} + (2\pi - \theta) G_s^{4/n}} \right)
\]

Proof:

i) This is the case when the capacity for both DMAC and DMAC+ is interference bound. It occurs when

\[
\frac{2\text{SINR}_{\text{min}}}{n-2} > \max \left\{ \frac{1}{M}, \frac{1}{G} \right\} \quad \text{(see Theorems 4.1 and 4.2)}
\]

However, \( M > G \) which implies that \( \max \left\{ \frac{1}{M}, \frac{1}{G} \right\} = \frac{1}{G} \).

When DMAC is interference-bound \( (C_{MAX} = C_{MAX}^I) \), using DMAC+ will only increase the average interference level at every node, due to the extra interference from U(SI), thus decreasing the capacity \( C_{MAX}^I \). Therefore \( L \) is going to be
less than one. The exact value of \( L \) is calculated from Lemmas 4.3 and 4.7.

ii) This is the case when the capacity for both DMAC and DMAC+ is protocol bound, which occurs when
\[
\frac{2\text{SINR}_{\text{min}}}{n-2} < \min\left(\frac{1}{M}, \frac{1}{G}\right) = \frac{1}{M}.
\]
It is straightforward that DMAC+ performs better, because of the smaller silence region established by a factor of \( 1/G_{s}^{2/n} \).

iii) This the case when DMAC capacity is protocol-bound, while DMAC+ capacity is interference-bound, and occurs when
\[
\frac{1}{M} < \frac{2\text{SINR}_{\text{min}}}{n-2} < \frac{1}{G}.
\]
Again, using Lemmas 4.3 and 4.7 we compute \( L \) as
\[
\frac{n-2}{2\text{SINR}_{\text{min}}} \frac{1}{G_{s}^{2/n} M},
\]
whose value lies between the extreme values of cases (i) and (ii).

The importance of Theorem 4.3 comes from a designer’s point of view. Specifically, if we know the operating environment, antenna type and radio sensitivity, we can choose the right MAC protocol that can achieve better performance. This argument can be generalized for many interesting cases where the performance of different protocols is compared. In Fig.5 we compare the capacity improvement of DMAC and DMAC+ over the omni-directional case, for different scenarios.

![Performance Comparison of DMAC vs. DMAC+](image)

**Figure 5:** Capacity improvement of using directional antennas [with DMAC, DMAC+] over omni-directional ones [with 802.11].

D. Asymptotic Capacity Bounds

**Theorem 4.4:** The asymptotic capacity behavior, when nodes are using directional antennas of finitely large gain and directionality, is the same as when nodes are using omni-directional ones, for both Arbitrary and Random Networks.

**Proof:** Let us use the subscripts \( D \) and \( O \) to denote quantities relevant to the directional and omni-directional case, respectively. Then,
\[
C_{D_{\text{MAX}}} \leq C_{D_{\text{MAX}}}^{P} = \left\| U(NI)_{O} \right\| C_{O_{\text{MAX}}}^{P} \Rightarrow
\]
\[
C_{D_{\text{MAX}}} = O(C_{O_{\text{MAX}}}^{P}) = O(C_{O_{\text{MAX}}}^{P}), \text{ if } \left\| U(NI)_{D} \right\| > 0.
\]
Finally, \( \left\| U(NI)_{D} \right\| > 0 \Leftrightarrow 0 > \theta > G_{s} > 0 \), which means that the antenna must have finite directionality (finite gain).

V. HETEROGENEOUS NETWORKS

In this section we consider a network consisting of nodes, some of which are using omni-directional antennas and others using directional ones. We shall call the former ones \( O \)-nodes and the latter ones \( D \)-nodes, hereafter. We assume that:

- The fraction of \( O \)-nodes in the network is equal to \( p \).
- The MAC protocol used by \( O \)-nodes (802.11) and the MAC protocol used by \( D \)-nodes (DMAC) are able to interoperate. This is a reasonable assumption to make, if we look into the detailed functional descriptions of the two protocols.
- The percentage of active \( O \)-nodes in the network is equal to \( p \).

This last assumption needs some more discussion. It is anticipated that in a mixed network with both \( O \)-nodes and \( D \)-nodes, \( D \)-nodes might overwhelm \( O \)-nodes and seize more than their fair share (1−p) of the bandwidth, whenever there is a high traffic load. Consequently, there is a need for a MAC access protocol that would be able to distribute the available bandwidth to \( O \)-nodes and \( D \)-nodes in a fair manner. We plan to look into this MAC fairness issue in a heterogeneous network, in a future work. However, here we assume, for our analysis purposes that 802.11 and DMAC allow a fair media access.

**Lemma 5.1:** In a heterogeneous network consisting of a fraction \( p \) of \( O \)-nodes and a fraction \( 1-p \) of \( D \)-nodes, the average transmission antenna gain for an arbitrary interfering signal is given by
\[
G^{*} = p + (1-p) \left( \frac{2\pi - \theta}{2\pi} \right)
\]

**Proof:** An active node in the interference area \( U(I) \) is an \( O \)-node with probability \( p \) and a \( D \)-node with probability \( 1-p \). Furthermore, an \( O \)-node interferes with an average transmission antenna gain of \( 1 \), while a \( D \)-node interferes with an average transmission gain \( G_{s}^{*} \) as given by Lemma 4.2.

**Lemma 5.2:**

i) The capacity of a heterogeneous network, as \( k \to \infty \), is
defined by the SINR requirement of O-nodes.

ii) The average SINR for a D-node will be higher than that for an O-node, and higher than SINR_{\text{min}} in the limit that maximizes capacity.

**Proof:** Using a similar methodology to Lemmas 3.4 and 4.3, we can compute the average SINR, for an O-node and a D-node, respectively, namely SINR_O and SINR_D.

\[
\begin{align*}
\text{SINR}_O &= \frac{P_c R_s^{-n}}{P_c(\rho_{\text{act}} G_s^{-2} + (2\pi -\theta)G_s^{-2})}, \\
\text{SINR}_D &= \frac{P_c R_s^{-n}}{P_c(\rho_{\text{act}} G_s^{-2} + (2\pi -\theta)G_s^{-2})},
\end{align*}
\]

\[
\text{SINR}_O = \frac{\rho_{\text{act}} G_s^{-2} + (2\pi -\theta)G_s^{-2}}{P_c R_s^{-n}}, \\
\text{SINR}_D = \frac{\rho_{\text{act}} G_s^{-2} + (2\pi -\theta)G_s^{-2}}{P_c R_s^{-n}}.
\]

We calculate the ratio \( L = \lim_{k \to \infty} \frac{\text{SINR}_D}{\text{SINR}_O} \):

For \( n > 2 \):

\[
L = \frac{\frac{2\pi}{\theta + (2\pi - \theta)G_s^{-2}}}{\theta + (2\pi - \theta)G_s^{-2}} > 1 \quad (G_s < 1)
\]

For \( n = 2 \):

\[
L = \lim_{k \to \infty} \frac{\theta + (2\pi - \theta)G_s^{-2}}{\theta + (2\pi - \theta)G_s^{-2}} > 1
\]

\[
L = \lim_{k \to \infty} \frac{\theta + (2\pi - \theta)G_s^{-2}}{\theta + (2\pi - \theta)G_s^{-2}} > 1
\]

We can now calculate the interference-imposed and protocol-imposed capacity bounds for the heterogeneous case.

**Lemma 5.3:** The interference-imposed capacity bound \( C^I_{\text{MAX}} \) for a wireless network consisting of a fraction \( p \) of O-nodes and a fraction \( 1-p \) of D-nodes is given by:

\[
C^I_{\text{MAX}} \leq \begin{cases} 
\frac{k^2 R_n^{-n}}{2SINR_{\text{min}} G_s^2 \log(kR_s^{-1})}, & n = 2 \\
\frac{k^2 R_n^{-n}}{2SINR_{\text{min}} G_s^2 (R_s^{-n} - k^{-n+2})}, & n > 2 
\end{cases}
\]

\[
C^P_{\text{MAX}} = \frac{\pi k^2}{p \pi R_s^2 + (1-p)[\frac{\theta}{2} + \frac{(\theta - \theta)G_s^{-2}}{2}]} 
\]

**Proof:** As Lemma 5.2 states, \( C^I_{\text{MAX}} \) is imposed by the average SINR requirement at an active O-node. Therefore, the proof is exactly the same with that of Lemma 3.4, with the only difference being that an arbitrary active node in U(I) interferes with an average transmission antenna gain equal to \( G^* \).

**Lemma 5.4:** The protocol-imposed capacity bound \( C^P_{\text{MAX}} \) for a wireless network consisting of a fraction \( p \) of O-nodes and a fraction \( 1-p \) of D-nodes is given by:

\[
C^P_{\text{MAX}} = \frac{\pi k^2}{p \pi R_s^2 + (1-p)[\frac{\theta}{2} + \frac{(\theta - \theta)G_s^{-2}}{2}]} 
\]

**Proof:** The proof is straightforward. We only need to calculate the average area of a silence region, when there is a fraction \( p \) of active nodes in the network, and then bound the maximum number of such disjoint silence regions.

**Lemma 5.5:** The capacity of a wireless ad-hoc network, the size of which is infinitely large \( (k \to \infty) \), and consists of a fraction \( p \) of O-nodes and a fraction \( 1-p \) of D-nodes is equal to:

\[
C_{\text{MAX}} = \begin{cases} 
C^I_{\text{MAX}}, & \text{if } n = 2 \text{ or } L > 1 \\
C^P_{\text{MAX}}, & \text{if } n > 2 \text{ and } L < 1 
\end{cases}
\]

\[
L = \frac{n - 2}{4\pi SINR_{\text{min}}} \left\{ \frac{2\pi + (1-p)[\theta + (2\pi - \theta)G_s^{-2}]}{p + (1-p)[\theta + (2\pi - \theta)G_s^{-2}]} \right\}
\]

**Proof:** We define \( L \) as \( L = \lim_{k \to \infty} C^P_{\text{MAX}} C^I_{\text{MAX}} \) and substitute \( C^I_{\text{MAX}} \) and \( C^P_{\text{MAX}} \) from Eq.22 and 23.

**Theorem 5.1:** The performance degradation when a fraction \( p \) of nodes is using omni-directional antennas, over the case where all nodes are using directional ones implementing DMAC, is linear and given by:

\[
\frac{C_{\text{MAX}}(\text{DMAC})}{C_{\text{MAX}}(p - \text{mixed})} = \frac{2\pi(1-G)}{G(1-G) + 2\pi G_s^{2n}} \quad \text{and} \quad b = \frac{2\pi}{2\pi + \theta(1-G_s^{2n})}
\]

**Proof:** We omit the proof due to lack of space.

The linearity of Eq.24 is a very promising result from the network designer’s point of view. The notion of a heterogeneous network, as introduced in this section, is very useful to model the situation where existing networks with
nodes using omni-directional antennas, are gradually upgraded to work with directional antennas. Such an upgrade is not expected to take place at once. Instead, one can envision a scenario where some nodes in the network are new and equipped with directional antennas, while the old ones are still using omni-directional ones. Eq.24 implies that even if all nodes have not yet been upgraded, the performance degradation will only be linear on the number of such nodes. On the other hand, looking closely at (24) we see that there exists a performance degradation equal to b, even when p→0, that is, when very few nodes in the network carry omni-directional antennas. This is because we have assumed that the MAC protocol is fair in distributing the bandwidth among D- and O-nodes. Therefore, even if there exists only one O-node in the network, the interference level is required to be low enough (lower than in the purely directional case), to allow the SINR at the O-node to stay above the threshold.

VI. CONCLUSIONS AND FUTURE WORK

In this paper we have conducted a thorough capacity analysis for wireless ad-hoc networks of nodes using omni-directional or directional antennas. We have provided results and intuition for the detailed dependence of the capacity, on important antenna parameters, propagation environment and radio sensitivity. Furthermore, we have re-evaluated known asymptotic capacity results, using a different, more accurate, methodology, and have strengthened their holding, or corrected them where necessary. We have demonstrated the subtle interplay between physical layer (i.e. interference and SINR requirement) and MAC protocol layer (i.e. silence region), in terms of defining the maximum network capacity, and have explained how a MAC protocol that becomes the bounding factor can be improved. As an example of the latter, we have analyzed the capacity of the DMAC protocol for directional antennas and have proposed a modified version of it that can overcome DMAC’s shortcomings and achieve a higher network throughput. Finally, we have analyzed the capacity of a heterogeneous network, where a percentage p of nodes uses omni-directional antennas and the remaining 1-p use directional ones. We have proven that the performance degradation over the fully directional case is linear on p (ap+b), a result which is very encouraging for the gradual deployment of directional antennas.

We are currently looking into scenarios, where nodes are assumed to have power control and directional antennas with infinitely narrow main beams, and infinitely small side-lobe interference. In this situation, a network throughput of Θ(N) could be achieved in concept, since power could be adjusted accordingly to allow direct communication between end-points, with no significant interference. However, the price paid would be a higher amount of energy spent for the same traffic, a phenomenon which gets exacerbated for higher path loss exponents. Consequently, we will need to define a metric that captures both requirements for high throughput and low energy consumption (e.g. number of bits/Joule) and measure capacity using that metric.

REFERENCES