Game Theory
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Lecture 2

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Fall 2016
Lecture 1 recap

• Defined games in normal form
• Defined dominance notion
  – Iterative deletion
  – Does not always give a solution
• Defined best response and Nash equilibrium
  – Computed Nash equilibrium in some examples

→ Are some Nash equilibria better than others?
→ Can we always find a Nash equilibrium?
Outline

1. Coordination games and Pareto optimality
2. Games with continuous action sets
   - Equilibrium computation and existence theorem
   - Example: Cournot duopoly
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The Investment Game

• The players: you
• The strategies: each of you chooses between investing nothing in a class project ($0) or investing ($10)
• Payoffs:
  – If you don’t invest your payoff is $0
  – If you invest you make a net profit of $5 (gross profit = $15; investment $10) if more than 90% of the class chooses to invest. Otherwise, you lose $10

• Choose your action (no communication!)
Nash equilibrium

• What are the Nash equilibria?

• Remark: to find Nash equilibria, we used a “guess and check method”
  – Checking is easy, guessing can be hard
The Investment Game again

• Recall that:
  – Players: you
  – Strategies: invest $0 or invest $10
  – Payoffs:
    • If no invest $0 → $0
    • If invest $10 $5 net profit if ≥ 90% invest
    – $10 net profit if < 90% invest

• Let’s play again! (no communication)

• We are heading toward an equilibrium
  ➔ There are certain cases in which playing converges in a natural sense to an equilibrium
Pareto domination

• Is one equilibrium better than the other?

Definition: Pareto domination

A strategy profile s Pareto dominates strategy profile s’ iif for all i, $u_i(s) \geq u_i(s')$ and there exists j such that $u_j(s) > u_j(s')$; i.e., all players have at least as high payoffs and at least one player has strictly higher payoff.

• In the investment game?
Convergence to equilibrium in the Investment Game

• Why did we converge to the wrong NE?
• Remember when we started playing
  – We were more or less 50% investing
• The starting point was already bad for the people who invested for them to lose confidence
• Then we just tumbled down

• What would have happened if we started with 95% of the class investing?
Coordination game

• This is a *coordination game*  
  – We’d like everyone to coordinate their actions and *invest*  
• Many other examples of coordination games  
  – Party in a Villa  
  – On-line Web Sites  
  – Establishment of technological monopolies (Microsoft, HDTV)  
  – Bank runs  
• Unlike in prisoner’s dilemma, *communication helps* in coordination games → *scope for leadership*  
  – In prisoner’s dilemma, a trusted third party (TTP) would need to impose players to adopt a strictly dominated strategy  
  – In coordination games, a TTP just leads the crowd towards a better NE point (there is no dominated strategy)
# Battle of the sexes

<table>
<thead>
<tr>
<th></th>
<th>Player 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Opera</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Soccer</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
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- Find the NE
- Is there a NE better than the other(s)?
Coordination Games

• Pure coordination games: there is no conflict whether one NE is better than the other
  – E.g.: in the investment game, we all agreed that the NE with everyone investing was a “better” NE

• General coordination games: there is a source of conflict as players would agree to coordinate, but one NE is “better” for a player and not for the other
  – E.g.: Battle of the Sexes

➔ Communication might fail in this case
Pareto optimality

**Definition: Pareto optimality**

A strategy profile $s$ is Pareto optimal if there does not exist a strategy profile $s'$ that Pareto dominates $s$.

- Battle of the sexes?
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The partnership game (see exercise sheet 2)

• Two partners choose effort $s_i$ in $S_i=[0, 4]$ 

• Share revenue and have quadratic costs 

$$u_1(s_1, s_2) = \frac{1}{2} \left[ 4 (s_1 + s_2 + b s_1 s_2) \right] - s_1^2$$

$$u_2(s_1, s_2) = \frac{1}{2} \left[ 4 (s_1 + s_2 + b s_1 s_2) \right] - s_2^2$$

• Best responses:

$$\hat{s}_1 = 1 + b s_2 = BR_1(s_2)$$

$$\hat{s}_2 = 1 + b s_1 = BR_2(s_1)$$
Finding the best response (with twice continuously differentiable utilities)

\[
\frac{\partial u_1(s_1, s_2)}{\partial s_1} = 0
\]

• First order condition (FOC)

\[
\frac{\partial^2 u_1(s_1, s_2)}{\partial^2 s_1} \leq 0
\]

• Second order condition (SOC)

• Remark: the SOC is automatically satisfied if \( u_i(s_i, s_{-i}) \) is concave in \( s_i \) for all \( s_{-i} \) (very standard assumption)

• Remark 2: be careful with the borders!
  – Example \( u_1(s_1, s_2) = 10-(s_1+s_2)^2 \)
  – \( S_1=[0, 4] \), what is the BR to \( s_2=2 \)?
  – Solving the FOC, what do we get?
    – When the FOC solution is outside \( S_i \), the BR is at the border
Nash equilibrium graphically

- NE is fixed point of \((s_1, s_2) \rightarrow (\text{BR}(s_2), \text{BR}(s_1))\)
Best response correspondence

• Definition: \( \hat{s}_i \) is a BR to \( s_{-i} \) if \( \hat{s}_i \) solves \( \max u_i(s_i, s_{-i}) \)
• The BR to \( s_{-i} \) may not be unique!
• \( BR(s_{-i}) \): set of \( s_i \) that solve \( \max u_i(s_i, s_{-i}) \)
• The definition can be written:
  \( \hat{s}_i \) is a BR to \( s_{-i} \) if \( \hat{s}_i \in BR_i(s_{-i}) = \arg\max_{s_i} u_i(s_i, s_{-i}) \)

• Best response correspondence of \( i \): \( s_{-i} \rightarrow BR_i(s_{-i}) \)
• (Correspondence = set-valued function)
Nash equilibrium as a fixed point

• Game \( \left( N, (S_i)_{i \in N}, (u_i)_{i \in N} \right) \)

• Let’s define \( S = \times_{i \in N} S_i \) (set of strategy profiles) and the correspondence

\[ B : S \rightarrow S \]

\[ s \mapsto B(s) = \times_{i \in N} BR_i(s_{-i}) \]

• For a given \( s \), \( B(s) \) is the set of strategy profiles \( s' \) such that \( s'_i \) is a BR to \( s_{-i} \) for all \( i \).

• A strategy profile \( s^* \) is a Nash eq. iif \( s^* \in B(s^*) \) (just a re-writing of the definition)
Kakutani’s fixed point theorem

Theorem: Kakutani’s fixed point theorem

Let $X$ be a compact convex subset of $\mathbb{R}^n$ and let $f : X \to X$ be a set-valued function for which:

- for all $x \in X$, the set $f(x)$ is nonempty convex;
- the graph of $f$ is closed.

Then there exists $x^* \in X$ such that $x^* \in f(x^*)$. 
Closed graph (upper hemicontinuity)

- Definition: f has closed graph if for all sequences \((x_n)\) and \((y_n)\) such that \(y_n\) is in \(f(x_n)\) for all \(n\), \(x_n \rightarrow x\) and \(y_n \rightarrow y\), \(y\) is in \(f(x)\)
- Alternative definition: f has closed graph if for all \(x\) we have the following property: for any open neighborhood \(V\) of \(f(x)\), there exists a neighborhood \(U\) of \(x\) such that for all \(x\) in \(U\), \(f(x)\) is a subset of \(V\).
- Examples:
Existence of (pure strategy) Nash equilibrium

**Theorem: Existence of pure strategy NE**

Suppose that the game \( (N, (S_i)_{i \in N}, (u_i)_{i \in N}) \) satisfies:

- The action set \( S_i \) of each player is a nonempty compact convex subset of \( \mathbb{R}^n \)
- The utility \( u_i \) of each player is continuous in \( S \) (on \( S \)) and concave in \( S_i \) (on \( S_i \))

Then, there exists a (pure strategy) Nash equilibrium.

- Remark: the concave assumption can be relaxed
Proof

• Define B as before. B satisfies the assumptions of Kakutani’s fixed point theorem.
• Therefore B has a fixed point which by definition is a Nash equilibrium!

• Now, we need to actually verify that B satisfies the assumptions of Kakutani’s fixed point theorem!
Example: the partnership game

- $N = \{1, 2\}$
- $S = [0,4] \times [0,4]$ compact convex
- Utilities are continuous and concave
  \[
  u_1(s_1, s_2) = \frac{1}{2} [4 (s_1 + s_2 + b s_1 s_2)] - s_1^2
  \]
  \[
  u_2(s_1, s_2) = \frac{1}{2} [4 (s_1 + s_2 + b s_1 s_2)] - s_2^2
  \]
- Conclusion: there exists a NE!

- Ok, for this game, we already knew it!
- But the theorem is much more general and applies to games where finding the equilibrium is much more difficult
One more word on the partnership game before we move on

• We have found (see exercises) that
  – At Nash equilibrium:
    \[ s^*_1 = s^*_2 = \frac{1}{1-b} \]
  – To maximize the sum of utilities:
    \[ s^{W}_1 = s^{W}_2 = \frac{1}{1/2-b} > s^*_1 \]

• Sum of utilities called social welfare
• Both partners would be better off if they worked \( s^{W}_1 \) (with social planner, contract)
• Why do they work less than efficient?
Externality

• At the margin, I bear the cost for the extra unit of effort I contribute, but I’m only reaping half of the induced profits, because of profit sharing

• This is known as an “externality”

→ When I’m figuring out the effort I have to put I don’t take into account that other half of profit that goes to my partner

→ In other words, my effort benefits my partner, not just me

• Externalities are omnipresent: public good problems, free riding, etc. (see more in the netecon course)
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Cournot Duopoly

• Example of application of games with continuous action set
• This game lies between two extreme cases in economics, in situations where firms (e.g. two companies) are competing on the same market
  – Perfect competition
  – Monopoly
• We’re interested in understanding what happens in the middle
  – The game analysis will give us interesting economic insights on the duopoly market
Cournot Duopoly: the game

- The players: 2 Firms, e.g., Coke and Pepsi

- Strategies: quantities players produce of **identical** products: \( q_i, q_{-i} \)
  - Products are **perfect substitutes**

- Cost of production: \( c \times q \)
  - Simple model of **constant marginal cost**

- Prices: \( p = a - b (q_1 + q_2) = a - bQ \)
  - Market-clearing price
Price in the Cournot duopoly

Slope: \(-b\)

Demand curve

Tells the quantity demanded for a given price
Cournot Duopoly: payoffs

• The payoffs: firms aim to maximize profit

\[ u_1(q_1,q_2) = p * q_1 - c * q_1 \]
\[ p = a - b (q_1 + q_2) \]

\[ u_1(q_1,q_2) = a * q_1 - b * q_1^2 - b * q_1 q_2 - c * q_1 \]

• The game is symmetric

\[ u_2(q_1,q_2) = a * q_2 - b * q_2^2 - b * q_1 q_2 - c * q_2 \]
Cournot Duopoly: best responses

- First order condition
  \[ a - 2bq_1 - bq_2 - c = 0 \]

- Second order condition
  \[-2b < 0\]

  [make sure it's a max]

\[ \begin{align*}
\hat{q}_1 &= BR_1(q_2) = \frac{a - c}{2b} - \frac{q_2}{2} \\
\hat{q}_2 &= BR_2(q_1) = \frac{a - c}{2b} - \frac{q_1}{2}
\end{align*} \]
Cournot Duopoly: best response diagram and Nash equilibrium

\[ q_{\text{Cournot}} = \frac{a - c}{3b} \]

\[ q_1 = \frac{a - c}{2b} \]

\[ q_2 = \frac{a - c}{b} \]
Best response at $q_2=0$

- $BR_1(q_2=0) = (a-c)/(2b)$
- Interpretation: monopoly quantity
  ➢ marginal revenue = marginal cost
When is $BR_1(q_2) = 0$?

- $BR_1(q_2=(a-c)/b) = 0$
- Perfect competition quantity

➤ Demand = marginal cost

If Firm 1 would produce more, the selling price would not cover her costs
Cournot Duopoly: best response diagram and Nash equilibrium

\[ q_{\text{Cournot}} = \frac{a - c}{3b} \]

Monopoly

\[ q_{\text{Monopoly}} = \frac{a - c}{b} \]

Perfect competition

\[ q_{\text{Perfect}} = \frac{a - c}{2b} \]
Strategic substitutes/complements

• In Cournot duopoly: the more the other player does, the less I would do
  ➔ This is a game of *strategic substitutes*
    – Note: of course the goods were substitutes
    – We’re talking about strategies here

• In the partnership game, it was the opposite: the more the other player would the more I would do
  ➔ This is a game of *strategic complements*
Cournot duopoly: Market perspective

- Total industry profit maximized for monopoly

Industry profits are maximized

\[ q_{\text{Cournot}} = \frac{a - c}{3b} \]

\[ \frac{a - c}{b} \]

\[ \frac{a - c}{2b} \]
Cartel, agreement

- How could the firms set an agreement to increase profit?
- What can the problems be with this agreement?

Both firms produce half of the monopoly quantity
Cournot Duopoly: last observations

• How do quantities and prices we’ve encountered so far compare?

Perfect Competition

\[ \frac{a - c}{b} \]

Cournot Quantity

\[ \frac{2(a - c)}{3b} \]

Monopoly

\[ \frac{a - c}{2b} \]
Summary

• Coordination games
  – Pareto optimal NE sometimes exist
  – Scope for communication / leadership

• Games with continuous action sets (pure strategies)
  – Compute equilibrium with FOC, SOC
  – Equilibrium exists under concavity and continuity conditions
  – Cournot duopoly