Multi-user Power Allocation Games

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What is Game Theory?

Basically, Game Theory is the mathematics of strategy.

Game Theory aims to help us to understand situations in which decision makers interact.

The primary theory is the Minimax Theorem which basically says that if all the players of a game play the best, most rational strategy, the resulting outcome of the game is predictable.
**Static games in Strategic Games**

**Strategic Games**: A strategic game consists of:

- A set of **players**.
- for each player, a set of **actions**.
- for each player, **preferences** over the set of action profiles.

**Example 1**: the players may be firms, the actions prices and the preferences the firm’s profit.

**Example 2**: the players may be candidates for political office, the actions campaign expenditures and the preferences a reflection of the candidates’s probabilities of winning.

We consider for the moment only static games (the players have only one move as a strategy).
Players: Let $P$ be the set of players. The subscript $-i$ designates all the players belonging to $P$ except $i$ himself.

Remark: These players are often designated as being opponents of $i$. In a two player games, player $i$ has one opponent referred as $j$. 
Utility (or payoff function): Let $u_i(s)$ be the benefit of player $i$ given the strategy profile $s$. In the two players case, we have $U = \{u_1(s), u_2(s)\}$.

Remark: The payoff function represents a decision maker’s preferences in the sense that if he prefers $a \in S$ to $b \in S$ then $u(a) > u(b)$. 
Let us start: the prisoner’s dilemma

**Context:** Two suspects in a major crime are held in separate cells.

- If they both stay quiet, each will be convicted of the minor offense and spend one year in prison.
- If one and only one of them finks, he will be freed and used as a witness against the other who will spend four years in prison.
- If they both fink, each will spend three years in prison.
Modeling the game

Players: The two suspects

Actions: Each player’s set of actions is \(\{Quite, Fink\}\)

Preferences: We need a function \(u_1\) such as:

\[
 u_1(\text{Fink, Quiet}) > u_1(\text{Quiet, Quiet}) > u_1(\text{Fink, Fink}) > u_1(\text{Quite, Fink})
\]

For example,

- \(u_1(\text{Fink, Quiet}) = 3\).
- \(u_1(\text{Quiet, Quiet}) = 2\).
- \(u_1(\text{Fink, Fink}) = 1\).
- \(u_1(\text{Quite, Fink}) = 0\).
Games and matrices

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- There are gains from cooperation (each player prefers that both players choose Quiet than they both choose Fink).
- However, each player has an incentive to “free ride” (choose fink?) whatever the other player does.

**Question:** How to solve it? In other words, how to predict the strategy of each player, considering the information the game offers and assuming that the players are rational?
Methods to solve static games

- Iterated dominance.
- Nash Equilibrium.
- Mixed Strategies.

In all these techniques, we assume non-cooperative games. Cooperative games require agreements between the decision makers and might be more difficult to realize (additional signalization).
Best response framework

- If player 1 is quiet, then the best response of player 2 is to fink.
- If player 1 finks, then 2 is better off finking also.

**Definition** Denote $b_i(s_{-i})$ the best response of player $i$ to the opponent’s strategy vector $s_{-i}$. The best response $b_i(s_{-i})$ of player $i$ to the profile of strategies $s_{-i}$ is a strategy $s_i$ such that:

$$b_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

One can see that if two strategies are mutual best responses to each other, then no player would have a reason to deviate from the given strategy profile.

This is the start of the Nash equilibrium
**Definition** The pure strategy profile \( s^* \) constitutes a Nash equilibrium if, for each player \( i \),

\[
u_i(s_i^*, s^*_{-i}) \geq u_i(s_i, s^*_{-i}), \quad \forall s_i \in S_i
\]

Expressed differently, a Nash equilibrium embodies a steady stable "social norm": if everyone else adheres to it, no individual wishes to deviate from it.
Cooperative, non-cooperative and coordination

We only went through one type of equilibria in the case of non-cooperative games.

Nash equilibrium may be very efficient and cooperation is sometimes better off (Pareto equilibrium).

However, we don’t necessarily need cooperation. Coordination is enough (Correlated equilibrium).

Example: (Aumann and Schelling, Nobel prize): People can coordinate rather well without communicating.


- Consider the game where two people are asked to select a positive integer.
- If they choose the same integer, both get an award, otherwise no award is given.
- In such a setting, the majority tends to select the number 1. This number is distinctive, since it is the smallest integer.
Game theory and power allocation

Basic example: flat fading case

We consider the uplink of a single cell network.

$K$ users are simultaneously communicating with a base station in the flat fading case.

At each time instant, the carrier of each user $k$ is characterized by a fading realization $h_k$.

As a consequence, the received signal at the base station is given by:

$$ y = \sum_k h_k x_k + n $$

where $n$ is a zero mean gaussian noise with variance $\sigma^2$. 
Basic example: flat fading case

At the base station, the SINR of user $k$

$$\text{SINR}_k = \frac{P_k h_k}{\sigma^2 + \sum_{j=1, j \neq k}^{K} P_j h_j}.$$ 

The corresponding capacity of user $k$ on a given carrier:

$$C_k = \log_2 \left( 1 + \frac{P_k h_k}{\sigma^2 + \sum_{j=1, j \neq k}^{K} P_j h_j} \right)$$
Utility function

In order to place ourselves in a game theoretic setting, we have to define a utility for the users.

**Utility** measures the gain of a user as a result of the strategy this user plays.

It is natural to define utility as the ratio of the throughput to the transmit power.

This is a relevant performance measure, as each mobile wants to use its (limited) battery power to transmit the maximum possible amount of information.
Utility function

Therefore, the utility of user $k$ can be written

$$u_k = \frac{C_k}{P_k}.$$  \hfill (1)

This utility is expressed in bits per joule.

In the non-cooperative game setting, each user wants to selfishly maximize his utility.

A Nash equilibrium is obtained when no user can benefit by unilaterally deviating from his strategy.
Utility function

From now on, we denote $\text{SINR}_k = \beta_k$, whichever filter is actually used.

The throughput is given by $C_k = C(\beta_k)$.

To obtain the maximum utility achievable by user $k$, we derive $u_k$ with respect to the power $P_k$ and equate to 0. We obtain

$$P_k \frac{\partial \beta_k}{\partial P_k} C'(\beta_k) - C(\beta_k) = 0.$$

However, in our case (and CDMA case using asymptotic random matrix theory), we have that $P_k \frac{\partial \beta_k}{\partial P_k} = \beta_k$.

Thus, the problem reduces to finding $\beta^*$ that satisfies

$$\beta_k C'(\beta_k) - C(\beta_k) = 0.$$
Utility function

\[ \beta_k C'(\beta_k) - C'(\beta_k) = 0. \]  \hspace{1cm} (2)

The existence of a solution is guaranteed as long as the utility is a quasiconcave function of the SINR.

In addition, we assume that the function \( C'(\cdot) \) takes value \( C(0) = 0 \), so that users cannot achieve an infinite utility by not transmitting.

The uniqueness of the solution \( \beta^* \) is due to the fact that the SINR of each user is a strictly increasing function of its transmit power.
Define $C_k = R_k \cdot f(\beta_k)$ where $R_k$ is the transmission rate and $f(\beta_k)$ is the efficiency function.

The efficiency depends on the modulation, coding and packet size but should be increasing with the following constraints: $f(0) = 0$ and $f(\infty) = 1$.

Example: For a packet which contains $M$ bits, $f(\beta_k) = (1 - e^{-\beta_k})^M$.

In this case, with $M = 100$, $\beta^* = 6.48 = 8.1 dB$. 
Solution

From the definition of the SINR, we have:

\[
p_1 = \frac{1}{|h_1|^2} \frac{\sigma^2 \beta^* (1 + \beta^*)}{1 - (\beta^*)^2}
\]

\[
p_2 = \frac{1}{|h_2|^2} \frac{\sigma^2 \beta^* (1 + \beta^*)}{1 - (\beta^*)^2}
\]
Utility function with delay constraints

Let $X$ be the number of retransmissions required for a packet to be received without any errors.

Any constraint on the transmission delay can be expressed on the number of retransmissions.

$$Pr(X = m) = f(\beta)(1 - f(\beta))^{m-1}$$

The delay requirements of a particular user is given by the pair $(D, q)$ such as:

$$Pr(X \leq D) \geq q$$

which translates into $\sum_{m=1}^{D} f(\beta)(1 - f(\beta))^{m-1} \geq q$. 
Utility function with delay constraints

\[
\sum_{m=1}^{D} f(\beta)(1 - f(\beta))^{m-1} \geq q
\]

The delay constraint translates into a lower bound on the SINR \( \beta \).

Hence, the lower bound is given by:

\[
\beta_{k}^{\text{lower}} = f^{-1}(1 - (1 - q_k)^{\frac{1}{D_k}})
\]
Nash equilibrium of the game with delay constraints

Each user will seek to maximize its utility while satisfying its SINR requirement.

This can be captured by defining a delay-constrained utility for user $k$ as

- $u_{k}^{\text{delay}} = u_k$ if $\beta_k \leq \beta_k^{\text{lower}}$
- $u_{k}^{\text{delay}} = 0$ if $\beta_k > \beta_k^{\text{lower}}$

Remember that $\beta^*$ is solution of $f(\beta) = \beta f'(\beta)$.

It can be shown that for some type of efficiency functions (the one we exhibited for example) that $u_k$ is an increasing function of $p_k$ if $\beta_k < \beta^*$ and for all $\beta_k > \beta^*$, $u_k$ is a decreasing function of $p_k$

$u_{k}^{\text{delay}}$ is maximized when the user transmits at a power level such as:

$$\beta_{k}^{\text{delay}*} = \max(\beta_k^{\text{lower}}, \beta^*)$$
Nash equilibrium of the game with delay constraints

Result. The Nash equilibrium for the non-cooperative game is given by

\[ p_k^{\text{delay}*} = \min(p_k^*, P_{\text{max}}) \]

where \( p_k^* \) is the transmit power that results in the SINR equal to \( \beta_k^{\text{delay}*} \), solution of

\[ \beta_k^{\text{delay}*} = \max(\beta_k^{\text{lower}}, \beta^*) \]
Proof. \( \beta_k^{\text{delay}*} \) is maximized when the transmit power is such as
\[
\beta_k^{\text{delay}*} = \max(\beta_k^{\text{lower}}, \beta^*).
\]
If \( \beta_k^{\text{lower}} \) can not be achieved, the user must transmit at maximum power level to maximize his utility.

Let \( p_k^{\text{lower}} \) be the power level for which the SINR is equal to \( \beta_k^{\text{lower}} \).

We have in any case \( p_k^{\text{lower}} \leq P_{\text{max}} \) (otherwise, the user is not admit it in the network) and because \( u_k^{\text{delay}} = 0 \) for \( p_k < p_k^{\text{lower}} \), there is no incentive for the user to transmit at a power level smaller than \( p_k^{\text{lower}} \).

Hence, the set of strategies are restricted to \([p_k^{\text{lower}}, P_{\text{max}}]\), an interval in which the utility is continuous and quasiconcave and thus has a Nash equilibrium.
Proof. As a result, $\beta^*$ which is the solution of

$$f(\beta) = \beta f'(\beta)$$

is unique and as a result $\beta^*_k$ is unique.
What is the users do not behave selfishly?

On the contrary, in a social optimum scenario, the purpose for the users is to maximize the total utility of the system, i.e., the total throughput divided by the total power needed to attain this throughput:

$$u_{\text{coop}} = \frac{\sum_{k=1}^{K} C_k}{\sum_{k=1}^{K} P_k}.$$  \hspace{1cm} (3)

$C_k$, as previously, will depend on the kind of receiver considered.
Given the definition of $P^*(k)$, for all $k$, $\frac{C(\beta^k)}{P_k} \leq \frac{C(\beta^*)}{P^*(k)}$. Since $P^*(k_0) = \min_k P^*(k)$, we further have $\frac{C(\beta^k)}{P_k} \leq \frac{C(\beta^*)}{P^*(k_0)}$. Thus we obtain

$$\frac{C(\beta_k)}{C(\beta^*)} \leq \frac{P_k}{P^*(k_0)}.$$ 

Summing over $k = 1 \ldots K$, we finally obtain

$$\frac{\sum_{k=1}^{K} C(\beta_k)}{\sum_{k=1}^{K} P_k} \leq \frac{C(\beta^*)}{P^*(k_0)}.$$ 

This completes the proof.

Only the user with the best channel transmits at a time. Hence, the optimal policy is time sharing.
Conclusion

We have only touched upon the basic applications of game theory

- Wardrop Equilibrium (large system analysis)

- Stackelberg equilibrium
THANK YOU!