Efficient Space-Time Codes from Cyclic Division Algebras

(Invited Paper)

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Abstract—An overview of space-time code construction based on cyclic division algebras (CDA) is presented. Applications of such space-time codes to the construction of codes optimal under the diversity-multiplexing gain (D-MG) tradeoff, to the construction of the so-called perfect space-time codes, to the construction of optimal space-time codes for the ARQ channel as well as to the construction of codes optimal for the cooperative relay network channel are discussed. We also present a construction of optimal codes based on CDA for a class of Orthogonal Amplify and Forward (OAF) protocols for the cooperative relay network.

I. INTRODUCTION

Consider the quasi-static, Rayleigh fading, space-time (ST) MIMO channel with quasi-static interval $T$, $n_t$ transmit and $n_r$ receive antennas. The $(n_r \times T)$ received signal matrix $Y$ is given by

$$Y = \theta H X + W$$

where $X$ is a $(n_t \times T)$ code matrix drawn from a ST code $\mathcal{X}$, $H$ the $(n_r \times n_t)$ channel matrix and $W$ represents additive noise. The entries of $H$ and $W$ are assumed to be i.i.d., circularly symmetric, complex Gaussian $CN(0,1)$ random variables. Many of the results presented here, however, carry over to the case of a channel matrix $H$ having an arbitrary statistical description [9]. The real scalar $\theta$ ensures that the energy constraint

$$\theta^2 \|X\|^2_F \leq T \text{SNR}, \quad \forall X \in \mathcal{X},$$

is met.

Multiple transmit and receive antennas have the potential of increasing reliability of communication as well as permitting communication at higher rates, quantified by the diversity and spatial multiplexing gains respectively. Zheng and Tse [5] showed that there is a fundamental tradeoff explained below, between diversity and multiplexing gain, referred to as the diversity-multiplexing gain (D-MG) tradeoff.

The space-time code $\mathcal{X}$ transmits

$$R = \frac{1}{T} \log(|\mathcal{X}|)$$

bits per channel use. Let $r$ be the normalized rate given by $R = r \log\text{SNR}$. Following [5], we will refer to $r$ as the (spatial) multiplexing gain. Let the diversity gain $d(r)$ corresponding to transmission at normalized rate $r$ be defined by

$$d(r) = -\lim_{\text{SNR} \to \infty} \frac{\log(P_e)}{\log\text{SNR}},$$

where $P_e$ denotes the probability of codeword error. We will follow the exponential equality notation of [5] under which this relationship can equivalently be expressed by

$$P_e \approx \text{SNR}^{-d(r)}.$$

A principal result in [5] is the proof that for a fixed integer multiplexing gain $r$, and $T \geq n_t + n_r - 1$, the maximum achievable diversity gain $d(r)$ is governed by

$$d(r) = \begin{cases} (n_t - r)(n_r - r) & 0 \leq r \leq \min\{n_t, n_r\} \\ 0 & \text{else} \end{cases}$$

Fig. 1. Upper and Lower Bounds on the D-MG Tradeoff in the case of 4 transmit and 4 receive antennas.

The value of $d(r)$ for non-integer values of $r$ is obtained through straight-line interpolation. For $T < n_t + n_r - 1$ upper and lower bounds on the maximum possible $d(r)$ are provided in [5]. The corresponding bounds for $n_t = n_r = T = 4$ are shown in Fig. 1.

Zheng and Tse[5] establish that the D-MG tradeoff curve coincides with the plot of outage probability and that random
Gaussian codes achieve the D-MG tradeoff provided $T \geq n_t + n_r - 1$.

We shall refer to a ST code that achieves the upper-bound on the D-MG tradeoff as being a D-MG optimal ST code, or more simply, an optimal code.

a) **Optimal 2 Antenna Codes:** Yao and Wornell [2], [3] were the first to exhibit an optimal code and did so for the case $n_t = T = 2$. The diagonal and anti-diagonal threads of their $(2 \times 2)$ code matrix are unitary transformations of a pair of $(2 \times 1)$ vectors with QAM components. An appropriate choice of the unitary transformation caused the determinant of the difference code matrix to be bounded below irrespective of SNR and was shown to lead to D-MG optimality. Subsequent constructions by Dayal and Varanasi [4], Belfiore and Rekaya [12], Liao et al. [11] and Oggier et al. [14] also possess this property and hence are also D-MG optimal.

b) **LAST Codes:** In [7], El Gamal et al. consider a lattice-based construction of space-time block codes termed LAST codes. Here, a code matrix $X$ in the space-time code $C$ is identified with a vector in $\mathbb{C}^{n_r T}$. The construction calls for a lattice $\Lambda_c$ and a sublattice $\Lambda_s$. Message symbols are mapped onto coset representatives $\{z\}$ of the subgroup $\Lambda_s$ of $\Lambda_c$ that lie within the fundamental region $V_s$ of the sublattice $\Lambda_s$. Thus the fundamental region of the sublattice serves as a shaping region for the lattice. The transmitted vector $x$ is then given by

$$x = c - u \pmod{\Lambda_s},$$

where $u$ is a pseudorandom “dither” vector known to the receiver and chosen with uniform probability from $V_s$. The lattice pair $\Lambda_s, \Lambda_c$ is drawn from an ensemble of lattices having good “covering” properties. It is shown that this ensemble of lattices contains a lattice such that the resultant space-time code, when suitably decoded using generalized minimum Euclidean distance lattice decoding, achieves the D-MG tradeoff for all $T \geq n_t + n_r - 1$. In actual code construction, a lattice drawn at random from the ensemble of lattices is used.

c) **ST Codes from Division Algebras:** Space-time code construction from division algebras was first proposed by Sethuraman and Rajan [17], [19], [16], [18] and independently by Belfiore and Rekaya [12]. In [18], the authors consider the construction of ST codes from field extensions as well as division algebras (DA). It is shown how Alamouti’s code arises as a special instance of the DA construction. A method of constructing cyclic division algebras (CDA) using transcendental elements is given and the capacity of the corresponding STBCs studied. A second construction of CDA due to Brauer is also applied to construct ST codes.

d) **Non-Vanishing Determinant:** In [12], the notion of a non-vanishing determinant (NVD) is introduced by Belfiore and Rekaya. The coding gain of a space-time code as determined by pairwise error probability considerations, is a function of the determinant of the difference code matrix. It is therefore of interest to maximize the value of this determinant. The authors of [12] note that while many constructions of space-time codes have the property of having a non-zero determinant, this determinant often vanishes as the SNR increases and the size of the signal constellation is accordingly increased. In [12], the authors describe an approach for constructing CDA-based square ST codes with the NVD property for $n_t = T = 2^k$ and $n_r = T = 2 \cdot 2^k$.

e) **Constructions with NVD:** In [15], square ST codes with the NVD property are constructed by Kiran and Rajan for $n_t = T = 2^k, 3 \cdot 2^k, 2 \cdot 3^k$ or $n_r = T = q^k(q - 1)/2$, where $q = 4k + 3$ is a prime. Also contained in this paper, is a Lemma that simplifies CDA-based NVD code construction.

f) **Perfect Codes:** In [14], Oggier et al. define a square $(n \times n)$ STBC to be a perfect code if

- the code is a full-rank, linear-dispersion code using $n^2$ information symbols drawn from either a QAM or HEX constellation,
- the minimum determinant of the code is bounded away from zero even as $M \to \infty$,
- the $2M^2$-dimensional real lattice generated by the vectorized codewords, is either $\mathbb{Z}^{2M^2}$ or $A_2^{M}$ ($A_2$ is the hexagonal lattice), and
- each symbol is drawn in the code matrix has the same value of average energy.

Perfect codes have been shown through simulation, to have excellent performance as judged by codeword error probability. The authors of [14] show the existence of perfect CDA-based space-time codes for dimensions $n = 2, 3, 4, 6$. The Golden code is an example of a perfect code in 2 dimensions. More recently, Elia et al. [20] show how the perfect code construction can be generalized to yield perfect codes for any value of the integer $n$.

g) **Approximate Universality:** In [9], Tavildar and Viswanath consider the general case where the channel matrix $H$ has an arbitrary statistical description. They show the existence of a class of codes called permutation codes that achieve the D-MG tradeoff for the parallel (diagonal $H$) channel. A sufficient condition for a ST code to be approximately universal, i.e., be D-MG optimal for every correlated MIMO fading channel is provided in terms of the product of the squared-singular values of the difference code matrix. This is used to show approximate universality of the Yao-Wornell codes as well as the codes of Elia et al. [1].

II. BACKGROUND ON DIVISION ALGEBRAS

Division algebras are rings with identity element and inverse, i.e., each nonzero element has a multiplicative inverse. As commutative division algebras are fields, it is the non-commutativity of a division algebra that gives it its unique identity.

**Example 1 (Quaternion Division Algebra):** The canonical example of a division algebra is the ring $D = \mathbb{R}(e, i, j, k)$ of quaternions over the real numbers $\mathbb{R}$ where $e$ is the identity element with $e^2 = 1$ and $i, j, k$ are elements satisfying

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$ (4)
Division algebras are in retrospect, a natural algebraic object from which to construct full-rank space-time codes i.e., space-time codes where the difference between any space-time code matrices has full rank. The construction makes use of the representation as square matrices, of elements in a division algebra. Such a representation arises when one considers an element \( d \) in a division algebra as a linear transformation corresponding to multiplication of elements in the division algebra by \( d \). We illustrate by considering the field of complex numbers as a (commutative) division algebra over the reals.

Example 2 (Matrix Representation of a Complex Number): Consider the linear operator \( A: \mathbb{C} \to \mathbb{C} \), given by
\[
A(u + \bar{v}) = (a + ib)(u + \bar{v}) .
\]
By regarding \( \mathbb{C} \) as a vector space over \( \mathbb{R} \), we arrive at the matrix representation
\[
A = \begin{bmatrix}
a & -b \\
b & a
\end{bmatrix} .
\]
Since the mapping:
\[
(a + ib) \to A = \begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}
\]
is a ring homomorphism, as long as \( (a + ib) \neq 0 \), \( A \) is guaranteed to have an inverse.

A. Cyclic Division Algebras

Cyclic division algebras (CDA) have a particularly simple structure and a general technique for the construction of a CDA can be found in [22], Proposition 11 of [18], or Theorem 1 of [12].

We review this construction procedure here as well as the construction given in [18], for the construction of space-time codes from the CDA. Let \( F, L \) be number fields, with \( L \) a finite, cyclic Galois extension of \( F \) of degree \( n \). Let \( \sigma \) denote the generator of the Galois group \( \text{Gal}(L/F) \). Let \( z \) be some symbol that satisfies the relations
\[
\ell z = z \sigma(\ell) \quad \forall \ell \in L \quad \text{and} \quad z^n = \gamma
\]
for some \( \gamma \in F^* \) having the property that the smallest integer \( t \) for which \( \gamma^t \) is the relative norm \( N_{L/F}(u) \) of some element \( u \) in \( L^* \), is \( n \). In a slight abuse of terminology, we shall refer to an element \( \gamma \) satisfying this property, as a "non-norm" element.

Then a cyclic division algebra \( D(L/F, \sigma, \gamma) \) with center \( F \) and maximal subfield \( L \) can be constructed by setting
\[
D = L \oplus zL \oplus \ldots \oplus z^{n-1}L.
\]
A space-time code \( X \) can be associated to \( D \) by selecting the set of matrices corresponding to the matrix representation of elements of a finite subset of \( D \). This so-called "left-regular" representation is along the lines in which the Alamouti code was derived as the matrix representation of elements of the quaternion algebra. Then each codeword matrix \( X \) in the space-time code can be shown to be of the form
\[
X = \begin{bmatrix}
\ell_0 & \gamma \sigma(\ell_{n-1}) & \ldots & \gamma^{n-1} \sigma(\ell_1) \\
\ell_1 & \sigma(\ell_0) & \ldots & \gamma^{n-1} \sigma(\ell_2) \\
\vdots & \vdots & \ddots & \vdots \\
\ell_{n-1} & \sigma(\ell_{n-2}) & \ldots & \sigma^{n-1} \ell_0
\end{bmatrix}
\]
where \( \ell_i \in L \).

1) Selection of "non-norm" parameter \( \gamma \): The above CDA-based construction of \( (n_t = n, T = n) \) square space-time codes calls for a cyclic (Galois) field extension \( L/F \) of degree \( n \) and a "non-norm" element \( \gamma \). In [18], \( \gamma \) is chosen to be transcendental over \( L \), whereas in [12], [10], [13], the focus is on choosing \( \gamma \) to lie in \( F \). As pointed out in [10], the advantage of choosing \( \gamma \) to lie in \( F \) is that the determinant of the corresponding matrix representation lies in \( F \). A sketch of the proof of this fact is shown in [12] for the case \( n = 2 \) and \( n = 3 \cdot 2^l \). While known much earlier in the mathematical literature [23], a simple self-contained proof valid for all \( n \) is given in [1].

2) Non-vanishing determinant (NVD) property: Further, as pointed out in [12], [10], when \( F = \mathbb{Q}, F = \mathbb{Q}(i) \) or \( F = \mathbb{Q}(\omega_3) \) where \( \omega_m = \exp\left(i \frac{2\pi}{m}\right) \), and the entries of codeword matrices are restricted to the ring of integers \( \mathcal{O}_L \) of \( L \), then the squared magnitude of the determinant of difference matrices will always take on an integer value. This endows the space-time code with the "non-vanishing determinant" property [12], [10], [13]. A space-time code is said to have non-vanishing determinant if the determinant \( \det(\Delta X) \) of the difference \( \Delta X \) of any pair of distinct code matrices in \( X \), is bounded away from 0 even in the limit as the SNR and hence the size of the signal constellation, tends to \( \infty \).

3) Cyclic extensions of some number fields: These observations call for the construction of cyclic extensions \( L \) of \( F \) of degree \( n \) where \( F = \mathbb{Q}, \mathbb{Q}(i) \) or \( \mathbb{Q}(\omega_3) \) and for the identification of a "non-norm" element \( \gamma \). In [12], the authors identify cyclic extensions of \( \mathbb{Q}(i) \) and \( \mathbb{Q}(\omega_3) \) of degree \( 2^t \) and \( 3 \cdot 2^t \) respectively. A non-norm element \( \gamma \) is explicitly identified for the cases \( t = 2, 3, 4, 6 \) in [12], [10], [13]. In [15], the authors provide construction techniques for cyclic extensions of \( \mathbb{Q}(i) \) of degree \( 2^t \) and \( 3 \cdot 2^t \) and cyclic extensions of \( \mathbb{Q}(\omega_3) \) of degree \( 3^t \) and \( 3 \cdot 3^t \). In each case, they also explicitly identify a non-norm element \( \gamma \). If \( q \) is a prime of the form \( q = 4k + 3 \), the ring of integers of \( \mathbb{Q}(\sqrt{-q}) \) is given by \( \mathbb{Z}[\frac{1 + \sqrt{-q}}{2}] \) and then every element in \( \mathbb{Z}[\frac{1 + \sqrt{-q}}{2}] \) also has integer norm. With this in mind, the authors of [15] provide a construction for a cyclic extension of \( \mathbb{Q}(\sqrt{-q}) \) of degree \( \frac{q(q-1)}{2} \) and show how a non-norm element \( \gamma \) can be found in this case as well.
B. D-MG Optimality of CDA-Based Space-Time Codes

In [1], the authors establish that CDA-based space-time codes having the NVD property achieve the D-MG tradeoff. They also show how one can identify cyclic extensions of \( \mathbb{Q}(\gamma) \) of any desired degree \( n \) and identify in every case, a non-norm element \( \gamma \) as desired. This makes it possible to construct a CDA and a CDA-based \( (n_t = T = n) \) space-time code that has the NVD property for any integer \( n \). By suitably modifying the square codes, the authors were able to construct D-MG optimal codes for all \( T \geq n_t \). As a result, the codes constructed in [1] represent an explicit construction of space-time codes that achieve the D-MG tradeoff for any combination of \( T \geq n_t \) and \( n_r \).

The results in [1] also imply the D-MG optimality of some other code constructions as well including the \((2 \times 2)\) Golden code construction of [10], the codes in [13] and the CDA-based codes constructed in [15].

III. D-MG Tradeoff of ARQ Channel

Consider an ARQ scheme in which the receiver is allowed to pass back one bit of information to the transmitter, corresponding to successful reception or otherwise, of the transmission. If the receiver does not deem the transmission to be successful, the transmitter is permitted to send a second code matrix relating to the same underlying data and so on up to a maximum of \( L \) transmissions. In this setting, a principal result derived in [7] is that the optimal D-MG curves \( d_{ARQ}(r) \) for the long-term static ARQ channel and for the non-ARQ channel \( d(r) \) are related by

\[
d_{ARQ}(r) = d\left(\frac{r}{L}\right), \quad 0 \leq r \leq \min\{n_t, n_r\}
\]

The corresponding plots for \( n_t = n_r = 4 \) and \( L = 1, 2, 4, 8, 20 \) are presented in Fig. 2.

For a broad class of channels that includes the Rayleigh-fading channel model, explicit space-time (ST) code constructions based on cyclic division algebras are provided which optimally trade diversity for multiplexing gain in the ARQ setting whenever either \( n_t \mid L \) or \( L \mid n_t \). The codes so constructed, incur minimum possible delay as well.

IV. D-MG Tradeoff for Cooperative Relay Networks

Cyclic Division Algebras can be used to construct D-MG optimal codes for cooperative relay networks. As an example we examine the D-MG tradeoff for a class of Orthogonal Amplify and Forward Protocols (OAF), and provide a CDA-based ST construction that is D-MG optimal.

A. OAF protocol:

Consider a system model in which there are \( n + 1 \) nodes that cooperate in the communication between source node \( S \) and destination node \( D \). The remaining \((n-1)\) nodes thus act as relays.

We assume the following:
- all nodes have a single transmit and single receive antenna
- that the channels are quasi static fading with \( g_j \) denoting the fade coefficients between \( S \) and \( D_j \), \( (g_j)_i \) the fade coefficients between \( S \) and \( j^\text{th} \) relay node \( R_j \), and between \( R_i \) and \( D \) respectively.
- all fade coefficients are i.i.d., circularly symmetric complex gaussian \( \mathcal{CN}(0,1) \) random variables
- the noise at the receivers is assumed to be i.i.d., circularly symmetric complex gaussian \( \mathcal{CN}(0,\sigma^2) \) random variables with variance \( \sigma^2 \).
- we assume half-duplex operation at each node, i.e., at any given instant a node can either transmit or receive but not both.

In this protocol, we assume that the source \( S \) transmits signal to the relays \( \{R_j\} \) and the destination \( D \) for a duration of \( p \) channel uses. All the relays then transmit a linear transformation of the received signal from the source to \( D \) for \( q \) channel uses, while the source remains silent. We allow the time duration \( p \) for which source transmits to the destination to vary with the multiplexing gain \( r \).

Hence we can write the signal model for the protocol, in matrix form as,

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
g_1 f_p \\
\sum_{j=2}^n g_j h_j A_j
\end{bmatrix} x + \begin{bmatrix}
w_1 \\
\sum_{j=2}^n h_j A_i w_j + w_2
\end{bmatrix}
\]

Fig. 2. The D-MG Tradeoff of the ARQ Channel.

Fig. 3. Cooperative Relaying in networks.
where $\mathbf{z}$ is the signal transmitted by the source and $[y_1, y_2]^T$ is the signal received by the destination. The $\{A_j\}$ are $(q \times p)$ matrices that represent the linear transformation taking place at the relay nodes. The vectors $[y_j, y_2]^T$ and $[w_1, w_2]^T$, represent the additive noise seen by the receivers located at the relay nodes and destination respectively. We set the Frobenius norm of the relay matrices $\{A_j\}$ as
\begin{equation}
\|A_j\|_F^2 = \alpha_j^2
\end{equation}
so that we can impose a constraint on the average energy of the signal transmitted by the relays $R_j$ by varying $\alpha_j^2$.

**Theorem 1:** (General OAF Upper Bound) For the class of OAF protocols described above, for any choice of transformation matrices $\{A_j\}$ satisfying (10), the D-MG tradeoff satisfies the upper bounds given below. In the bounds, we have set $m = (p + q)$.

For $(q > p)$ or $(p > q)$ and $\frac{p}{m} \leq \frac{n}{2n-1}$,
\begin{equation}
d(r) \leq \left\{ \begin{array}{ll}
n(1 - \frac{mr}{p}) & 0 \leq r \leq \frac{n}{2n-1} \frac{(n-1)}{p} \\
(1 - r) & \frac{1}{2} \leq r \leq 1
\end{array} \right. \quad (11)
\end{equation}
For $p > q$ and $\frac{p}{m} > \frac{n}{2n-1}$, we have
\begin{equation}
d(r) \leq \left\{ \begin{array}{ll}
n(1 - \frac{(n-1)mr}{2n}) & 0 \leq r \leq \frac{n}{2n-1} \\
(2n-1) & \frac{1}{2} \leq r \leq 1
\end{array} \right. \quad (12)
\end{equation}

**Remark 1:** In deriving these bounds, we have permitted the source to exercise the option of not using cooperative relaying whenever it is advantageous to avoid cooperative relaying.

**Remark 2:** These upper bounds on the D-MG tradeoff were derived by upper bounding the outage probability of the space-time channel resulting from a choice of matrix set $\{A_j\}$.

**Remark 3:** The largest value of upper bound on D-MG tradeoff results when one selects the parameters $p, q, m$ to satisfy $\frac{p}{m} = \frac{n}{2n-1}$ in which case one obtains the upper bound
\begin{equation}
d(r) \leq \left\{ \begin{array}{ll}
n(1 - \frac{(2n-1)r}{n}) & 0 \leq r \leq \frac{1}{2} \\
(1 - r) & \frac{1}{2} \leq r \leq 1
\end{array} \right. \quad (13)
\end{equation}
It turns out that in this specific instance, the right hand-side is precisely the outage probability of the space-time channel that results from the matrix selection
\begin{equation}
[A_j]_{kl} = \left\{ \begin{array}{ll}
\alpha_j & k = j - 1, l = j \\
0 & \text{elsewhere}
\end{array} \right. \quad (14)
\end{equation}
Furthermore, a CDA-based ST code is constructed below, that achieves this largest value of upper bound on D-MG tradeoff.

**B. D-MG Optimal Codes for the OAF Protocol**

In this subsection, we restrict our attention to the parameter set $p = n$, $q = (n - 1)$ so that $m = (2n - 1)$. In our constructions, we select the matrices $\{A_j\}$ as specified by (14). Without loss of generality (insofar as D-MG tradeoff is concerned), for simplicity, we set
\begin{equation}
\alpha_j = 1 \quad \text{all } 2 \leq j \leq n.
\end{equation}

Specification of the ST code will be complete upon identifying the vector $\mathbf{z}$ appearing in (9). We use CDA to construct this vector $\mathbf{z}$.

Consider a CDA having center $\mathbb{F} = \mathbb{Q}(i)$ and maximum subfield $\mathbb{L}$ that is a degree-$n$ cyclic Galois extension $\mathbb{L}/\mathbb{F}$ of $\mathbb{F}$. Let $\sigma$ be the generator of the cyclic Galois group $\text{Gal}(\mathbb{L}/\mathbb{F})$. Let $\mathcal{O}_\mathbb{F}$, $\mathcal{O}_\mathbb{L}$ denote the ring of algebraic integers in $\mathbb{F}$, $\mathbb{L}$ respectively. It is known that $\mathcal{O}_\mathbb{F} = \mathbb{Z}[i]$. Let $\{\beta_1, \ldots, \beta_n\}$ be the integral basis for $\mathcal{O}_\mathbb{L}/\mathcal{O}_\mathbb{F}$.

Let
\begin{equation}
\mathcal{A}_\text{OAF}(\beta_1, \ldots, \beta_n) = \left\{ \sum_i a_i \beta_i \mid a_i \in \mathcal{A}_\text{OAF} \right\}
\end{equation}

where
\begin{equation}
\mathcal{A}_\text{OAF}(\beta_1, \ldots, \beta_n) = \left\{ \sum_i a_i \beta_i \mid a_i \in \mathcal{A}_\text{OAF} \right\}
\end{equation}

With this background, we are now in a position to specify our choice of code vector $\mathbf{z}$. We simply set
\begin{equation}
\mathbf{z} = [\ell_0 \quad \sigma(\ell_0) \quad \ldots \quad \sigma^{n-1}(\ell_0)]
\end{equation}
It can be shown that this choice achieves the largest upper bound on D-MG performance. Fig. 4 plots the best possible performance for the case $(n = 4)$ when three nodes act as relays.

![Fig. 4. Optimal D-MG tradeoff for 3 relay OAF protocol.](image)

**Theorem 2:** The D-MG tradeoff of the code constructed above (with $\{A_j\}$ as in (14) and $\mathbf{z}$ as in (15)) is given by
\begin{equation}
d(r) \leq \left\{ \begin{array}{ll}
n(1 - \frac{(2n-1)r}{n}) & 0 \leq r \leq \frac{1}{2} \\
(1 - r) & \frac{1}{2} \leq r \leq 1
\end{array} \right. \quad (16)
\end{equation}
Thus this code achieves the best possible D-MG performance under the class of OAF protocols discussed here.

As before, we permit the source to exercise the option of not using cooperative relaying whenever it is advantageous to
avoid cooperative relaying and this is the case for $r > \frac{1}{2}$ in the equation above.)

**Remark 4.** This theorem establishes that the performance of the best OAF protocol under the class of protocols considered here is identical to that of the best possible performance under the NAF (non-orthogonal amplify-and-forward) protocol discussed in [27].

**Remark 5.** Furthermore, we show that this optimal OAF performance can be achieved using a minimum-delay CDA-based code possessing a simple structure.

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