On the Broadcast Phase of Decode-and-Forward Bidirectional Relaying

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Outline

Motivation

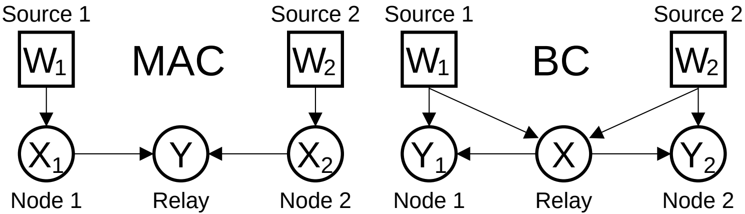
- Motivation: Future wireless systems should provide high data rates reliably in a certain area
  - Challenging task especially in scenarios where the direct link does not have the desired quality
  - ➠ **Cooperative protocols, relay communication**

- Relay cannot transmit & receive at the same time & frequency
- Problem: Most unidirectional relaying protocols allocate additional resources
  - ➠ Inherent loss in spectral efficiency

- Solution: (Two phase decode-and-forward) **bidirectional relaying** based on
  - **Superposition encoding** with interference cancellation at the receiver [Rankov ’05], [Oechtering ’06]
  - **XOR coding** with an XOR operation on decoded data stream (network coding) [Wu ’05], [Yeung ’05], [Fragouli ’06]

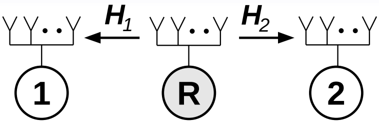
Bidirectional Relaying

- A priori separation of the communication in two phases:
  - **No cooperation** between encoders at node 1 and 2!



- MAC phase: Classical multiple access channel
  - Capacity region and optimal coding strategy are well known [Ahlswede '71], [Liao '72]
- BC phase: Relay node has decoded messages w_1 and w_2
  - Broadcast channel, where w_1 is known at node 1 and relay node and w_2 is known at node 2 and relay node!

Bidirectional Broadcast Channel



- **Multiple transmit and receive antennas** at all nodes
- Input-output relation between the relay node and node k

$$\boldsymbol{y}_k = \boldsymbol{H}_k \boldsymbol{x} + \boldsymbol{n}_k, \quad k = 1, 2$$

with $\boldsymbol{y}_k \in \mathbb{C}^{N_k \times 1}$ the output, $\boldsymbol{H}_k \in \mathbb{C}^{N_k \times N_R}$ the channel matrix, $\boldsymbol{x} \in \mathbb{C}^{N_R \times 1}$ the input, and $\boldsymbol{n}_k \sim \mathcal{CN}(\boldsymbol{0}, \sigma^2 \boldsymbol{I})$ the additive white Gaussian noise

- **Average transmit power constraint** P, i.e., any sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ must satisfy

$$\frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i^H \boldsymbol{x}_i \leq P$$

Motivations for the Broadcast Strategy

- *"...it is in general not optimal to consider the information to be multicast [...] as "fluid" which can simply be routed or replicated at the intermediate nodes. Rather, by employing coding at the nodes, which we refer to as network coding, bandwidth can in general be saved."* [Ahlswede '00]

- **Superposition coding** using interference cancellation techniques regards information as "fluid"
- **Network coding** is originally a multi-terminal source coding problem

➡ Follow the idea of network coding under channel coding aspects

➡ Convey as much information to the receiving nodes which allows them to conclude on the intended message using their own side information

Definitions

- A **Gaussian MIMO bidirectional broadcast channel** consists of two channels between the relay node and nodes 1 and 2 with $\boldsymbol{x} \in \mathcal{X} \subset \mathbb{C}^{N_R \times 1}$ where $\mathcal{X}$ is the set of all input sequences of length n which satisfy the power constraint, i.e., $\mathcal{X}^n := \{(\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n) \in \mathbb{C}^{N_R \times n} : \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i^H \boldsymbol{x}_i \leq P\}$
- A $\boldsymbol{(M_1^{(n)}, M_2^{(n)}, n)}$**-code** for the Gaussian MIMO bidirectional broadcast channel consists of one encoder at the relay node

$$f : \mathcal{V} := \mathcal{W}_1 \times \mathcal{W}_2 \to \mathcal{X}^n$$

and decoders at nodes 1 and 2 with erasure symbol 0

$$g_1 : \mathbb{C}^{N_1 \times n} \times \mathcal{W}_1 \to \mathcal{W}_2 \cup \{0\}$$
$$g_2 : \mathbb{C}^{N_2 \times n} \times \mathcal{W}_2 \to \mathcal{W}_1 \cup \{0\}$$

Definitions (2)

- Probabilities of **decoding error** with $v = (w_1, w_2)$
  - at node one: $\lambda_1(v) := \mathbb{P}[g_1(\boldsymbol{y}_1^n, w_1) \neq w_2 | f(v)$ has been sent$]$
  - at node two: $\lambda_2(v) := \mathbb{P}[g_2(\boldsymbol{y}_2^n, w_2) \neq w_1 | f(v)$ has been sent$]$
- **Average probability of error**
  - at node k: $\mu_k^{(n)} := \frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} \lambda_k(v)$, $k = 1, 2$
  - violating the power constraint: $\lambda_0 := \frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} \mathbb{P}[\frac{1}{n} \|\boldsymbol{X}^n(v)\|^2 > P]$
- A rate pair $[R_1, R_2]$ is said to be **achievable** if for any $\delta > 0$ there is an $n(\delta) \in \mathbb{N}$ and a sequence of $(M_1^{(n)}, M_2^{(n)}, n)$-codes such that for all $n \geq n(\delta)$ we have

$$\frac{1}{n} \log M_1^{(n)} \geq R_2 - \delta \quad \text{and} \quad \frac{1}{n} \log M_2^{(n)} \geq R_1 - \delta$$

  while $\mu_1^{(n)}, \mu_2^{(n)} \to 0$ as $n \to \infty$.
- The **capacity region** $\mathcal{C}_{\text{BC}}$ is the set of all achievable rate pairs

Capacity Region

Capacity of Bidirectional Broadcast Channel [Wyrembelski '08]

The capacity region of the Gaussian MIMO bidirectional broadcast channel with average power constraint P is given by

$$\mathcal{C}_{\mathrm{BC}} := \bigcup_{\mathrm{tr}(\boldsymbol{Q}) \leq P,\ \boldsymbol{Q} \geq 0} \mathrm{dpch}\Big([R_1(\boldsymbol{Q}), R_2(\boldsymbol{Q})] \Big)$$

with

$$R_k(\boldsymbol{Q}) := \log\det\left(\boldsymbol{I}_{N_k} + \tfrac{1}{\sigma^2} \boldsymbol{H}_k \boldsymbol{Q} \boldsymbol{H}_k^H \right), \quad k = 1, 2,$$

where $\mathrm{dpch}(\cdot)$ is the downward comprehensive hull defined for the vector $\boldsymbol{x} \in \mathbb{R}_+^2$ by the set $\mathrm{dpch}(\boldsymbol{x}) := \{ \boldsymbol{y} \in \mathbb{R}_+^2 : y_k \leq x_k, k = 1, 2 \}$.

- Note that both rates depend on the same input

Outline of the Proof of Achievability

- We follow [Oechtering '08] and adapt the random coding proof for broadcast channels with degraded components [Bergmans '73] using [Ash '65]
  - ⇒ Second decoding step, where one message is already successfully decoded, applies here!

- We show that for given covariance matrix $\boldsymbol{Q}$ with $\mathrm{tr}(\boldsymbol{Q}) \leq P$ satisfying the power constraint all rate pairs $[R_1(\boldsymbol{Q}), R_2(\boldsymbol{Q})]$ with

$$R_k(\boldsymbol{Q}) \leq \log\det(\boldsymbol{I}_{N_k} + \frac{1}{\sigma^2}\boldsymbol{H}_k\boldsymbol{Q}\boldsymbol{H}_k^H), \quad k = 1,2$$

  are achievable. Therefore we have to
  - generate independent Gaussian distributed codewords
  - show that the average probability of error, averaged over all codebooks and codewords, gets arbitrary small
  - show that the probability of violating the power constraint gets arbitrary small

Key Ideas of the Proof

- **Codebook Generation:** Define the covariance matrix $\hat{Q} = \frac{\hat{P}}{P} Q$ with $\hat{P} = P - \epsilon_P$, $\epsilon_P \in (0, P]$ and generate each entry of the codewords independently according to $\mathcal{CN}(\mathbf{0}, \hat{Q})$
  - ➠ first bound the error probability with respect to the codebook which might violate the power constraint

- **Decoding:** The receiving nodes will use typical set decoding with $i(\boldsymbol{x}^n; \boldsymbol{y}_k^n) = \frac{1}{n} \log \frac{p(\boldsymbol{y}_k^n | \boldsymbol{x}^n)}{p(\boldsymbol{y}_k^n)}$ and $I(\boldsymbol{X}; \boldsymbol{Y}_k) = \mathbb{E}_{\boldsymbol{X}^n, \boldsymbol{Y}^n}[i(\boldsymbol{X}^n; \boldsymbol{Y}_k^n)]$:

$$\mathcal{S}(\boldsymbol{y}_k^n) = \left\{ \boldsymbol{x}^n \in \mathcal{X}^n : i(\boldsymbol{x}^n; \boldsymbol{y}_k^n) > \frac{R_k + I(\boldsymbol{X}; \boldsymbol{Y}_k)}{2} \right\}$$

  - ➠ node 1 decides on the codeword $\mathcal{S}(\boldsymbol{y}_1^n) \cap \{f(w_1, w_2) : w_2 \in \mathcal{W}_2\}$
  - ➠ node 2 decides on the codeword $\mathcal{S}(\boldsymbol{y}_2^n) \cap \{f(w_1, w_2) : w_1 \in \mathcal{W}_1\}$

➠ **The decoding maps $g_1(\cdot, w_1)$ and $g_2(\cdot, w_2)$ induce at each node for each w_1 and w_2 an individual set partition!**

Key Ideas of the Proof (2)

- **Error Event:** When $f(w_1, w_2)$ has been sent, and $\boldsymbol{y}_1^n$ and $\boldsymbol{y}_2^n$ have been received, and the decoders decide $g_1(\boldsymbol{y}_1^n, w_1) = \hat{w}_2$ and $g_2(\boldsymbol{y}_2^n, w_2) = \hat{w}_1$, then a decoder makes an error if
    - $f(w_1, w_2)$ is not in $\mathcal{S}(\boldsymbol{y}_k^n)$, or
    - at node 1: $f(w_1, \hat{w}_2)$ is in $\mathcal{S}(\boldsymbol{y}_1^n)$ with $\hat{w}_2 \neq w_2$ or
      at node 2: $f(\hat{w}_1, w_2)$ is in $\mathcal{S}(\boldsymbol{y}_2^n)$ with $\hat{w}_1 \neq w_1$

➠ Vanishing average probability of error even if codewords violate the power constraint

- **Satisfying the power constraint:** We have $\mathbb{E}[\|X\|^2] = \hat{P} < P$ so that $\frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} \mathbb{P}[\frac{1}{n}\|\boldsymbol{X}^n(v)\|^2 > P] \to 0$ as $n \to \infty$

➠ Average probability of error $\mu_k^{(n)} \to 0$ as $n \to \infty$

- **Convexity of the region:** Follows from the optimality of Gaussian input and the concavity of the $\log\det$ function

Outline of the Proof of Weak Converse

- For any sequence of $(M_1^{(n)}, M_2^{(n)}, n)$-codes with $\mu_1^{(n)}, \mu_2^{(n)} \to 0$ there exists a covariance matrix $\boldsymbol{Q}$ with $\mathrm{tr}(\boldsymbol{Q}) \le P$ such that

$$R_1 := \liminf_{n\to\infty} \frac{1}{n} \log M_2^{(n)} \le \log\det(\boldsymbol{I}_{N_1} + \frac{1}{\sigma^2}\boldsymbol{H}_1\boldsymbol{Q}\boldsymbol{H}_1^H) + o(n^0)$$

$$R_2 := \liminf_{n\to\infty} \frac{1}{n} \log M_1^{(n)} \le \log\det(\boldsymbol{I}_{N_2} + \frac{1}{\sigma^2}\boldsymbol{H}_2\boldsymbol{Q}\boldsymbol{H}_2^H) + o(n^0)$$

- The proof follows standard arguments with a slight adaptation of Fano's Inequality:

Fano's Inequality

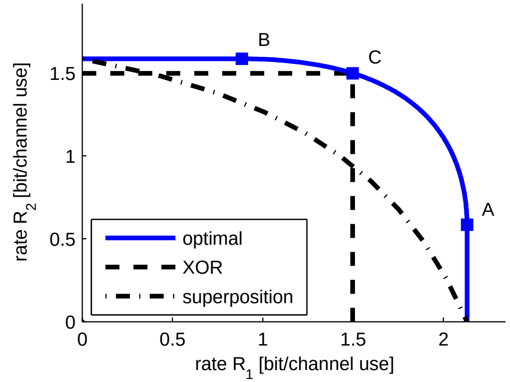
For our context we have Fano's inequality

$$H(W_2|\boldsymbol{Y}_1^n, W_1) \le nR_1\mu_1^{(n)} + 1 = n\epsilon_1^{(n)}$$

$$H(W_1|\boldsymbol{Y}_2^n, W_2) \le nR_2\mu_2^{(n)} + 1 = n\epsilon_2^{(n)}$$

with $\epsilon_k^{(n)} = R_k\mu_k^{(n)} + \frac{1}{n} \to 0$ for $n \to \infty$ as $\mu_k^{(n)} \to 0, k = 1, 2$

Rate Regions



- Both rates depend on the same input
  - Gaussian input will not maximize both links simultaneously
- A and B characterize the **single-user optimal** rate pairs
- C characterizes the **max-min optimal** rate pair

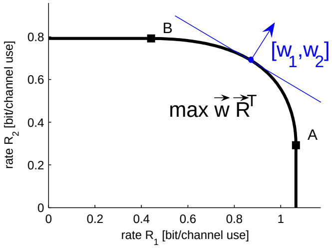
Optimization Problem

- Boundary of the capacity region corresponds to the weighted rate sum optimal rate pairs
- Optimal covariance matrix for weight vector $w = [w_1, w_2]$ given by

$$Q_{\text{opt}}(w) = \underset{\text{tr}(Q)\leq P,\, Q\succeq 0}{\arg\max}\; w_1 \log\left(1 + \frac{1}{\sigma^2}h_1^H Q h_1\right) + w_2 \log\left(1 + \frac{1}{\sigma^2}h_2^H Q h_2\right)$$

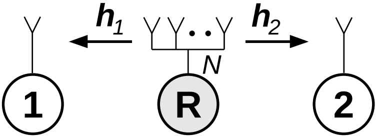
- $Q_{\text{opt}}(w)$ obtained by solving a convex optimization problem!

➠ Efficiently possible using interior point method



➠ **Study Q in detail to learn more about the optimal strategy!**

MISO Bidirectional Broadcast Channel



- **Multiple transmit antennas** at the relay nodes
- Input-output relation between the relay node and node k

$$y_k = \boldsymbol{x}^T \boldsymbol{h}_k + n_k, \quad k = 1, 2$$

with $y_k \in \mathbb{C}$ the output, $\boldsymbol{h}_k \in \mathbb{C}^{N_R \times 1}$ the channel vector, $\boldsymbol{x} \in \mathbb{C}^{N_R \times 1}$ the input, and $n_k \sim \mathcal{CN}(0, \sigma^2)$ the additive white Gaussian noise

- **Channel direction** $\boldsymbol{u}_k = \frac{\boldsymbol{h}_k}{\|\boldsymbol{h}_k\|}$, $k = 1, 2$
- **Correlation** $\rho = \frac{\boldsymbol{h}_1^H \boldsymbol{h}_2}{\|\boldsymbol{h}_1\|\|\boldsymbol{h}_2\|} = \boldsymbol{u}_1^H \boldsymbol{u}_2 = |\rho| e^{i\varphi}$, $0 \leq |\rho| \leq 1$

Single-User Optimal Rate Pairs

For the single-user optimal rate pairs the optimal transmit strategy is known from the single-user MISO channel

MRT-beamforming in the direction of the user's channel

- Optimal strategy for user 1 is $Q^{(1)}_{\text{opt}} = P\,\boldsymbol{u}_1\boldsymbol{u}_1^H$ with $\boldsymbol{u}_1 = \frac{\boldsymbol{h}_1}{\|\boldsymbol{h}_1\|}$
- User 1 can achieve its maximal unidirectional rate but this strategy need not be optimal for user 2.

$$R^{(1)}_{1,\text{opt}} := \log(1 + \tfrac{P}{\sigma^2}\|\boldsymbol{h}_1\|^2), \quad R^{(1)}_{2,\text{opt}} := \log(1 + \tfrac{P}{\sigma^2}|\rho|^2\|\boldsymbol{h}_2\|^2)$$

with $\rho = \boldsymbol{u}_1^H\boldsymbol{u}_2$, $|\rho| \leq 1$ the correlation between the channels

Wanted: Optimal transmit strategy for all rate pairs on the boundary!

First Observation

Proposition: **Subspace optimality**

Let $\boldsymbol{P}$ be the projection matrix for the subspace $\mathcal{W} = \text{span}\{\boldsymbol{h}_1, \boldsymbol{h}_2\}$. Then for any transmit strategy $\boldsymbol{Q}_n$ with rank n and transmit power P_n, the transmit strategy $\boldsymbol{Q} := \boldsymbol{P}^H \boldsymbol{Q}_n \boldsymbol{P}$ achieves the same rate pair with $\text{tr}(\boldsymbol{Q}) = P \leq P_n$ and equality iff we have $\boldsymbol{Q} = \boldsymbol{Q}_n$.

- An optimal transmit strategy transmits only into the subspace spanned by the channels, otherwise transmit power can be reduced while achieving the same rates
- Optimal transmit strategy $\boldsymbol{Q}$ has always $\text{rank}(\boldsymbol{Q}) \leq 2$

Orthogonal Channels

Proposition: Orthogonal channels

Let $\boldsymbol{h}_1, \boldsymbol{h}_2$ be orthogonal channels and $\boldsymbol{P}$ the projection matrix for $\mathcal{W} = \mathrm{span}\{\boldsymbol{h}_1, \boldsymbol{h}_2\}$. Then for any transmit strategy $\boldsymbol{Q}_2$ with rank two, $\boldsymbol{P}^H \boldsymbol{Q}_2 \boldsymbol{P} = \boldsymbol{Q}_2$, and $\mathrm{tr}(\boldsymbol{Q}_2) = P$ we find an equivalent single-beam strategy $\boldsymbol{Q}_1 := \boldsymbol{q}_1 \boldsymbol{q}_1^H$ with $\boldsymbol{q}_1 := \sqrt{\epsilon_1}\boldsymbol{u}_1 + \sqrt{\epsilon_2}\boldsymbol{u}_2$ and $\mathrm{tr}(\boldsymbol{Q}_1) = P$ which achieves the same rate pair and vice versa.

- For orthogonal channels any rate pair can be achieved with a single-beam as well as with a two-beam strategy
- For orthogonal channels the capacity region coincides with the achievable rate region for superposition encoding

Optimality of the Single-Beam Strategy

Theorem: *Theorem:* Single-beam optimality [Oechtering '08]

For arbitrary channels h_1, h_2 the single-beam (rank one) strategy is always an optimal transmit strategy for the MISO bidirectional broadcast channel.

Proof idea: For orthogonal channels optimality follows immediately from the proposition. For non-orthogonal channels show optimality by contradiction using the Karush-Kuhn-Tucker conditions.

➠ **Optimal strategy is to perform a single beam into the subspace spanned by the channels!**

Consequences from the Network Coding Idea

- Classical MISO broadcast (with no common message): Multi-beam strategy is optimal
  - Relay transmits to each user via an individual beam!
- MISO bidirectional broadcast: Single-beam is always optimal
  - Relay broadcasts only a single data stream following the idea of network coding!

Philosophy of network coding affects the signal processing at the relay node!

Consequences from the Single-Beam Optimality

- Coding theorem: Generate a Gaussian code according to the covariance matrix
  - ➠ Codebook with the dimension equal to the number of transmit antennas at the relay node
- Single-beam optimality: any rate pair can be achieved with a covariance matrix of rank one

It is sufficient to use an one-dimensional codebook and apply the optimal single-beam transmit strategy!

➠ Coding complexity reduces significantly!

Parametrization

Theorem: Beamforming vector

Let $\rho = |\rho| e^{i\varphi} = \boldsymbol{u}_1^H \boldsymbol{u}_2$ be the channel correlation and $\boldsymbol{Q} := \boldsymbol{q}\boldsymbol{q}^H$ be an optimal single-beam transmit strategy with $\mathrm{tr}(\boldsymbol{Q}) = P$, then the beamforming vector $\boldsymbol{q}$ can be expressed as $\boldsymbol{q} = a_1 \boldsymbol{u}_1 + a_2 \boldsymbol{u}_2$ with $a_1, a_2 \in \mathbb{C}$ and $\arg(a_1) - arg(a_2) = \varphi$.

- Difference of the phase is fixed
  - Normalized beamforming vector can be expressed as

$$\boldsymbol{q}(t) = \frac{t\boldsymbol{u}_1 + (1-t)e^{-j\varphi}\boldsymbol{u}_2}{||t\boldsymbol{u}_1 + (1-t)e^{-j\varphi}\boldsymbol{u}_2||}, \qquad t \in [0,1]$$

- We can achieve $R_k(t) := \log(1 + \frac{P}{\sigma^2}|\boldsymbol{h}_k^H \boldsymbol{q}(t)|^2)$, $k = 1, 2$
  - Provides a parametrization of the curved section of the boundary!

Egalitarian Solution

- Max-min optimal rate pair denote the egalitarian solution and is characterized by

$$R_1 = \log(1 + \tfrac{P}{\sigma^2}|\boldsymbol{h}_1^H \boldsymbol{q}|^2) = \log(1 + \tfrac{P}{\sigma^2}|\boldsymbol{h}_2^H \boldsymbol{q}|^2) = R_2$$

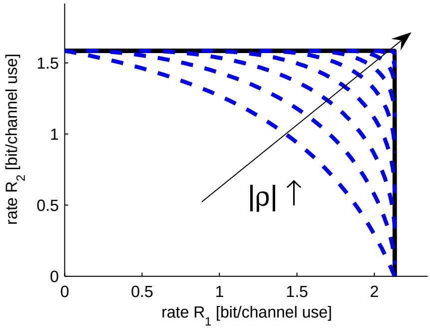
Proposition: Egalitarian solution

If the max-min optimal rate pair is on the curved section of the boundary, the optimal transmit covariance matrix is given by $Q_{\text{eq}} = P\,\boldsymbol{q}(t_{\text{eq}})\boldsymbol{q}^H(t_{\text{eq}})$ with

$$t_{\text{eq}} = \frac{||\boldsymbol{h}_2||(1-|\rho|)}{||\boldsymbol{h}_1||(1-|\rho|) + ||\boldsymbol{h}_2||(1-|\rho|)}.$$

- Characterizes the rate region achievable using XOR coding

Impact of Channel Correlation

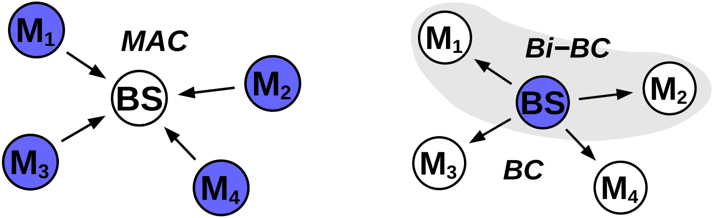


- With increasing correlation the capacity region increases
  - Full correlation: capacity region is a rectangle
  - No correlation: capacity region is equal to the achievable rate region using superposition encoding

➠ Opposite behavior to the classical broadcast channel

Integration into a Cellular Network

- Two users within one cell want to communicate with each other



Bidirectional relaying strategies can be integrated in the communication protocol to improve the performance!

Conclusion

- One input used by both receiving nodes achieves the capacity of the bidirectional broadcast channel
  - Follows the idea of network coding
- There exists always an optimal single-beam transmit strategy for the MISO bidirectional broadcast channel
  - Reflects the single stream processing
  - Manifests the paradigm shift to consider information flows not as ”fluids”
- As a consequence coding with an one-dimensional codebook is sufficient if the relay node applies the optimal transmit strategy.
  - Coding complexity reduces significantly

Thank you for your attention!

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