Vector Precoding in Wireless Communications: A Replica Analysis

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The Problem

Let

$$E := \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^{\dagger} \boldsymbol{J} \boldsymbol{x}$$

with $\boldsymbol{x} \in \mathbb{C}^K$ and $\boldsymbol{J} \in \mathbb{C}^{K \times K}$.

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Example 3:

$$\mathcal{X} = \{x : |x|^2 = 1\}^K \quad \Longrightarrow \quad ???$$

The Gaussian Vector Channel

Let the received vector be given by

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{t} + \boldsymbol{n}$$

where

- $\boldsymbol{t}$ is the transmitted vector
- $\boldsymbol{n}$ is uncorrelated (white) Gaussian noise
- $\boldsymbol{H}$ is a coupling matrix accounting for crosstalk

In many applications, e.g. antenna arrays, code-division multiple-access, the coupling matrix is modelled as a random matrix with independent identically distributed entries (i.i.d. model).

Crosstalk can be processed either at receiver or transmitter

Processing at Transmitter

If the transmitter is a base-station and the receiver is a hand-held device one would prefer to have the complexity at the transmitter.

E.g. let the transmitted vector be

$$\boldsymbol{t} = \boldsymbol{H}^\dagger (\boldsymbol{H}\boldsymbol{H}^\dagger)^{-1} \boldsymbol{x}$$

where $\boldsymbol{x} = \boldsymbol{s}$ is the data to be sent.

Then,

$$\boldsymbol{y} = \boldsymbol{s} + \boldsymbol{n}.$$

No crosstalk anymore due to channel inversion.

Problems of Simple Channel Inversion

Channel inversion implies a significant power amplification, i.e.

$$\boldsymbol{x}^\dagger \left(\boldsymbol{H}\boldsymbol{H}^\dagger\right)^{-1} \boldsymbol{x} > \boldsymbol{x}^\dagger \boldsymbol{x}.$$

In particular, let

- $\alpha = \frac{K}{N} \leq 1$;
- the entries of $\boldsymbol{H}$ are i.i.d. with variance $1/N$.

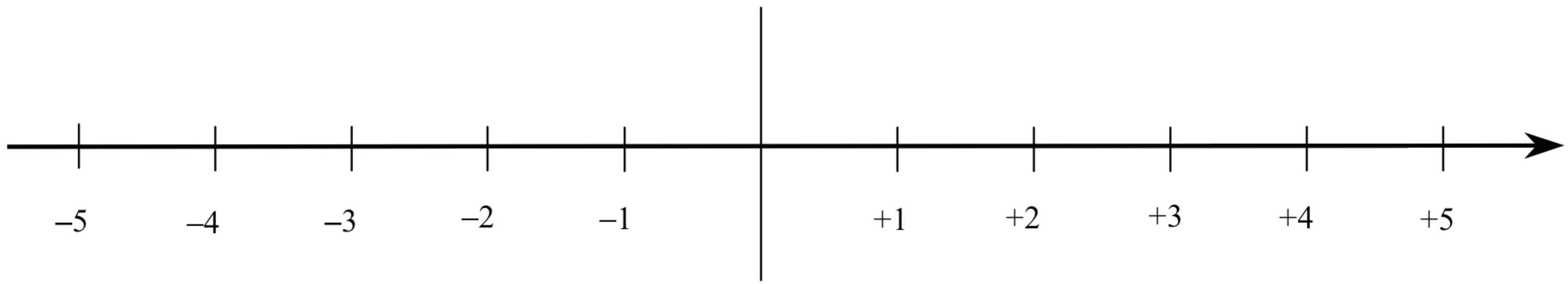
Then, for fixed aspect ratio α

$$\lim_{K\to\infty} \frac{\boldsymbol{x}^\dagger \left(\boldsymbol{H}\boldsymbol{H}^\dagger\right)^{-1} \boldsymbol{x}}{\boldsymbol{x}^\dagger \boldsymbol{x}} = \frac{1}{1-\alpha}$$

with probability 1.

Tomlinson-Harashima Precoding

Tomlinson '71, Harashima & Miyakawa '72



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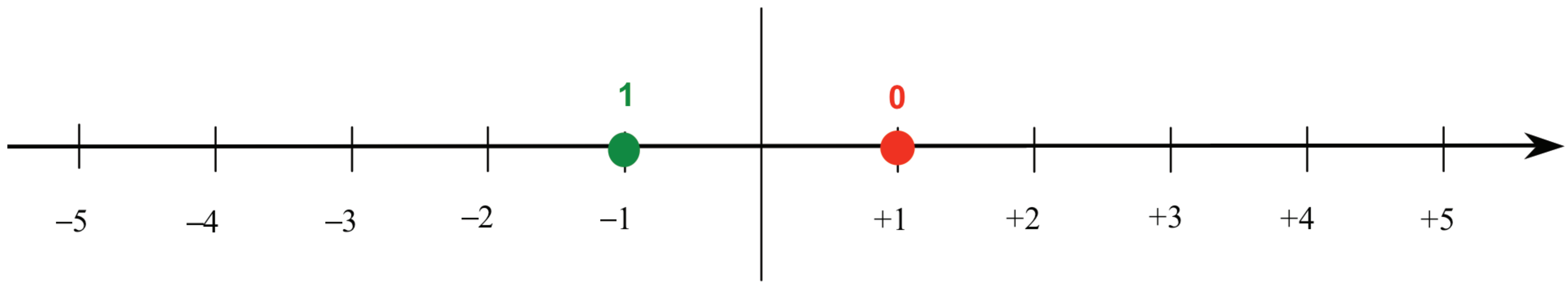

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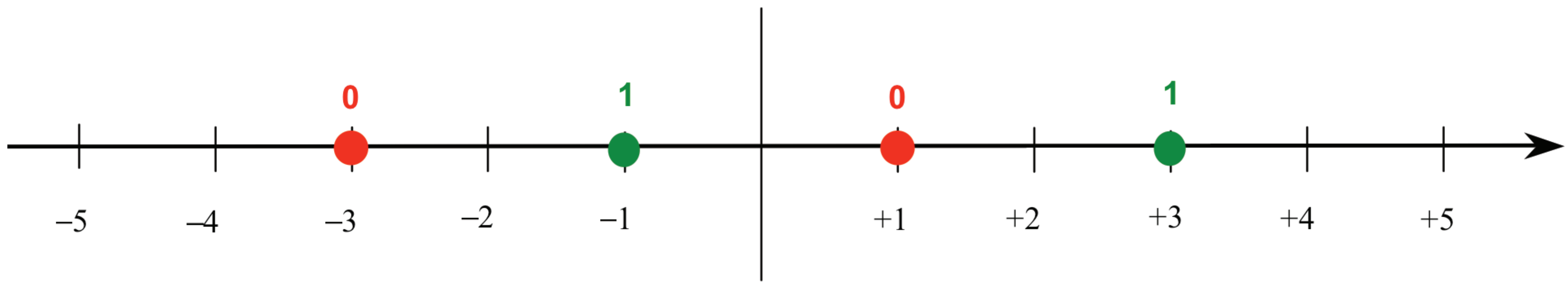

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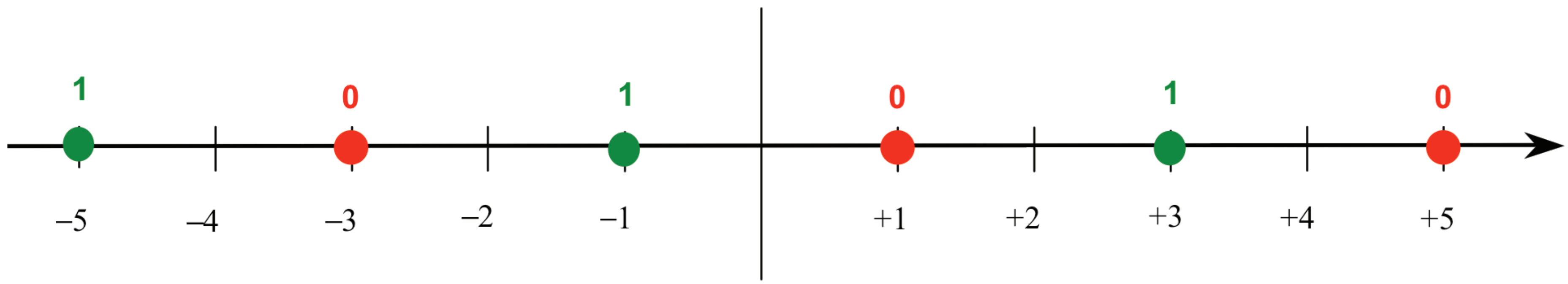

Instead of representing the logical "0" by $+1$, we present it by any element of the set $\{\dots, -7, -3, +1, +5, \dots\} = 4\mathbb{Z} + 1$. Correspondingly, the logical "1" is represented by any element of the set $4\mathbb{Z} - 1$.

Choose that representation that gives the smallest transmit power.

Generalized TH Precoding

Let $\mathcal{B}_0$ and $\mathcal{B}_1$ denote the sets presenting 0 and 1, resp.

Let $(s_1, s_2, s_3, \ldots, s_K)$ denote the data to be transmitted.

Then, the transmitted energy per data symbol is given by

$$E = \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$

with

$$\mathcal{X} = \mathcal{B}_{s_1} \times \mathcal{B}_{s_2} \times \cdots \times \mathcal{B}_{s_K}$$

and

$$\boldsymbol{J} = (\boldsymbol{H}\boldsymbol{H}^\dagger)^{-1}.$$

Zero Temperature Formulation

Quadratic programming is the problem of finding the zero temperature limit (ground state energy) of a quadratic Hamiltonian.

The transmitted power is written as a zero temperature limit

$$E = -\lim_{\beta\to\infty} \frac{1}{\beta K} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)}$$

with $\frac{1}{\beta}$ denoting temperature.

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The Harish-Chandra Integral

(also called the Itzykson-Zuber integral, particular in the physics community)

Let $\tilde{\boldsymbol{Q}}$ be any positive semi-definite matrix of bounded rank n, then

$$\lim_{K\to\infty} \frac{1}{K} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \mathrm{e}^{-K\,\mathrm{Tr}\,\boldsymbol{J}\tilde{\boldsymbol{Q}}} = -\sum_{a=1}^{n} \int_0^{\lambda_a} R(-w)\mathrm{d}w$$

with λ_a denoting the positive eigenvalues of $\tilde{\boldsymbol{Q}}$ (Marinari et al. '94; Guionnet & Maïda '05).

This is a large-deviations result for random matrices.

Recently, it was named the free Fourier transform.

Free Fourier Transform

We want

$$\lim_{K\to\infty} \frac{1}{K} \operatorname*{E}_{\boldsymbol{J}} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \operatorname{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)}.$$

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We know (Itzykon & Zuber '80)

$$\lim_{K\to\infty} \frac{1}{K} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \mathrm{e}^{-K\,\mathrm{Tr}\,\boldsymbol{J}\boldsymbol{P}} = -\sum_{a=1}^{n} \int_0^{\lambda_a(\boldsymbol{P})} R_{\boldsymbol{J}}(-w)\mathrm{d}w.$$

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We would like to exchange expectation and logarithm:

$$\mathop{\mathrm{E}}_{X} \log X = \lim_{n\to 0} \frac{1}{n} \log \mathop{\mathrm{E}}_{X} X^n.$$

Replica Continuity

We want

$$\lim_{K\to\infty} \frac{1}{K} \mathop{\mathrm{E}}_{\boldsymbol{J}} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \operatorname{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} = \lim_{K\to\infty} \lim_{n\to 0} \frac{1}{nK} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \left(\sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \operatorname{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} \right)^n$$

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$$= \lim_{K\to\infty}\lim_{n\to 0}\frac{1}{nK}\log\mathop{\mathrm{E}}_{\boldsymbol{J}}\prod_{a=1}^{n}\sum_{\boldsymbol{x}_a\in\mathcal{X}}\mathrm{e}^{-\beta K\,\mathrm{Tr}\left(\boldsymbol{J}\boldsymbol{x}_a\boldsymbol{x}_a^\dagger\right)}$$

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$$= \lim_{K\to\infty}\lim_{n\to 0}\frac{1}{nK}\log\mathop{\mathrm{E}}_{\boldsymbol{J}}\sum_{\boldsymbol{x}_1\in\mathcal{X}}\cdots\sum_{\boldsymbol{x}_n\in\mathcal{X}}\mathrm{e}^{-K\,\mathrm{Tr}\left(\boldsymbol{J}\beta\sum_{a=1}^{n}\boldsymbol{x}_a\boldsymbol{x}_a^\dagger\right)}$$

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We want

$$\begin{aligned}
\lim_{K\to\infty} \frac{1}{K} \mathop{\mathrm{E}}_{\boldsymbol{J}} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} &= \lim_{K\to\infty} \lim_{n\to 0} \frac{1}{nK} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \left(\sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} \right)^n \\
&= \lim_{K\to\infty} \lim_{n\to 0} \frac{1}{nK} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \prod_{a=1}^{n} \sum_{\boldsymbol{x}_a\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}\left(\boldsymbol{J}\boldsymbol{x}_a\boldsymbol{x}_a^\dagger\right)} \\
&= \lim_{K\to\infty} \lim_{n\to 0} \frac{1}{nK} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \sum_{\boldsymbol{x}_1\in\mathcal{X}} \cdots \sum_{\boldsymbol{x}_n\in\mathcal{X}} \mathrm{e}^{-K \,\mathrm{Tr}\left(\boldsymbol{J}\beta \sum\limits_{a=1}^{n} \boldsymbol{x}_a\boldsymbol{x}_a^\dagger\right)} \\
&= -\lim_{n\to 0} \frac{1}{n} \log \mathop{\mathrm{E}}_{\boldsymbol{Q}} \exp\left[\sum_{a=1}^{n} \int_0^{\beta\lambda_a(\boldsymbol{Q})} R_{\boldsymbol{J}}(-w)\,\mathrm{d}w \right]
\end{aligned}$$

with

$$Q_{ab} := \frac{1}{K} \boldsymbol{x}_a^\dagger \boldsymbol{x}_b.$$

Replica Symmetry

$$\boldsymbol{Q} := \begin{bmatrix} q+\frac{\chi}{\beta} & q & \cdots & q & q \\ q & q+\frac{\chi}{\beta} & \ddots & q & q \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ q & q & \ddots & q+\frac{\chi}{\beta} & q \\ q & q & \cdots & q & q+\frac{\chi}{\beta} \end{bmatrix}$$

with some macroscopic parameters q and χ.

This is the most critical step. In general, the structure of $\boldsymbol{Q}$ is more complicated. Generalizations are called replica symmetry breaking (RSB).

RS Solution

Let $\mathrm{P}(s)$ denote the limit of the empirical distribution of the information symbols $s_1, s_2, \ldots, s_K$ as $K \to \infty$. Let q and χ be the simultaneous solutions to

$$q = \iint \operatorname*{argmin}_{x \in \mathcal{B}_s}{}^2 \left| z\sqrt{2qR'(-\chi)} - 2xR(-\chi) \right| \mathrm{D}z \, \mathrm{dP}(s)$$

$$\chi = \frac{1}{\sqrt{2qR'(-\chi)}} \iint \operatorname*{argmin}_{x \in \mathcal{B}_s} \left| z\sqrt{2qR'(-\chi)} - 2xR(-\chi) \right| z^* \mathrm{D}z \, \mathrm{dP}(s)$$

where $\mathrm{D}z = \exp(-z^2/2)\mathrm{d}z/\sqrt{2\pi}$, $R(\cdot)$ is the R-transform of the limiting eigenvalue spectrum of $\boldsymbol{J}$, and $0 < \chi < \infty$.

Then, replica symmetry (RS) implies

$$\frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x} \to q \frac{\partial}{\partial \chi} \chi R(-\chi)$$

as $K \to \infty$.

Some R-Transforms

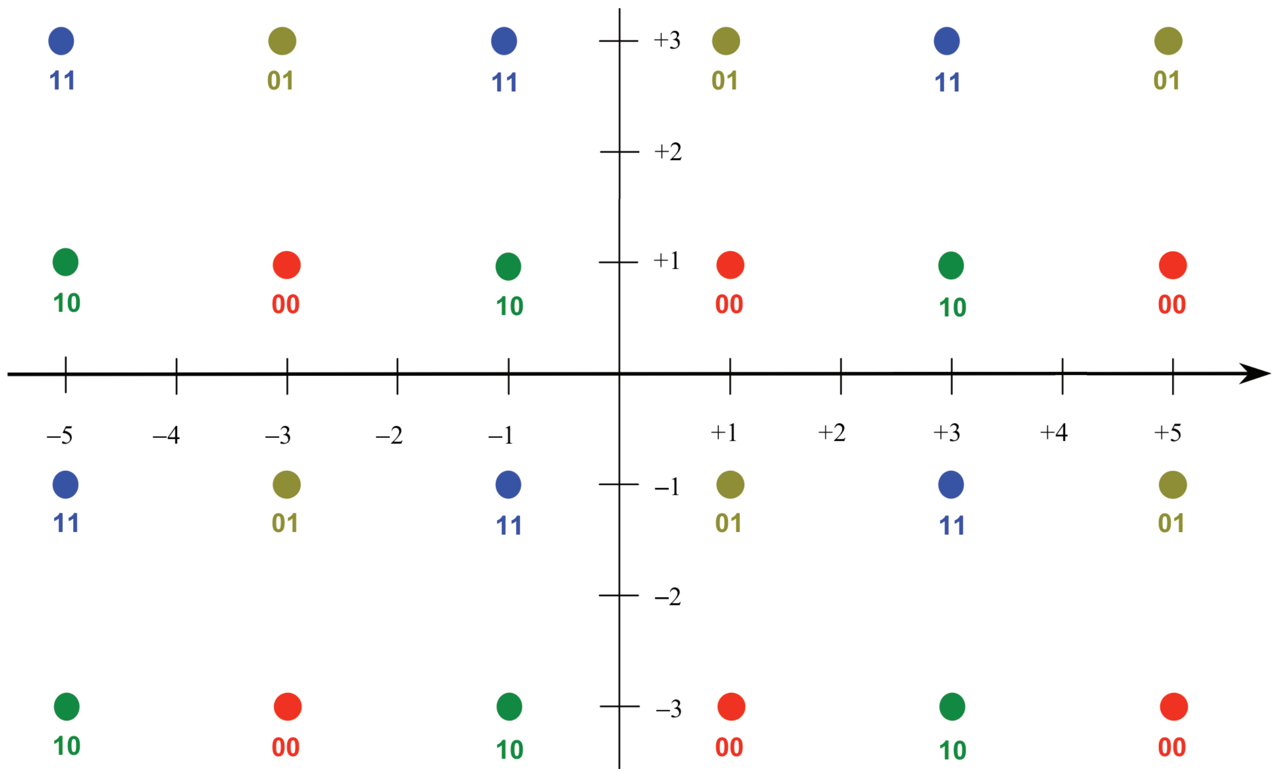
$$\mathbf{I}: \quad R(w) = 1$$

$$\boldsymbol{H}\boldsymbol{H}^{\dagger}: \quad R(w) = \frac{1}{1-\alpha w}$$ Marchenko-Pastur (MP) law

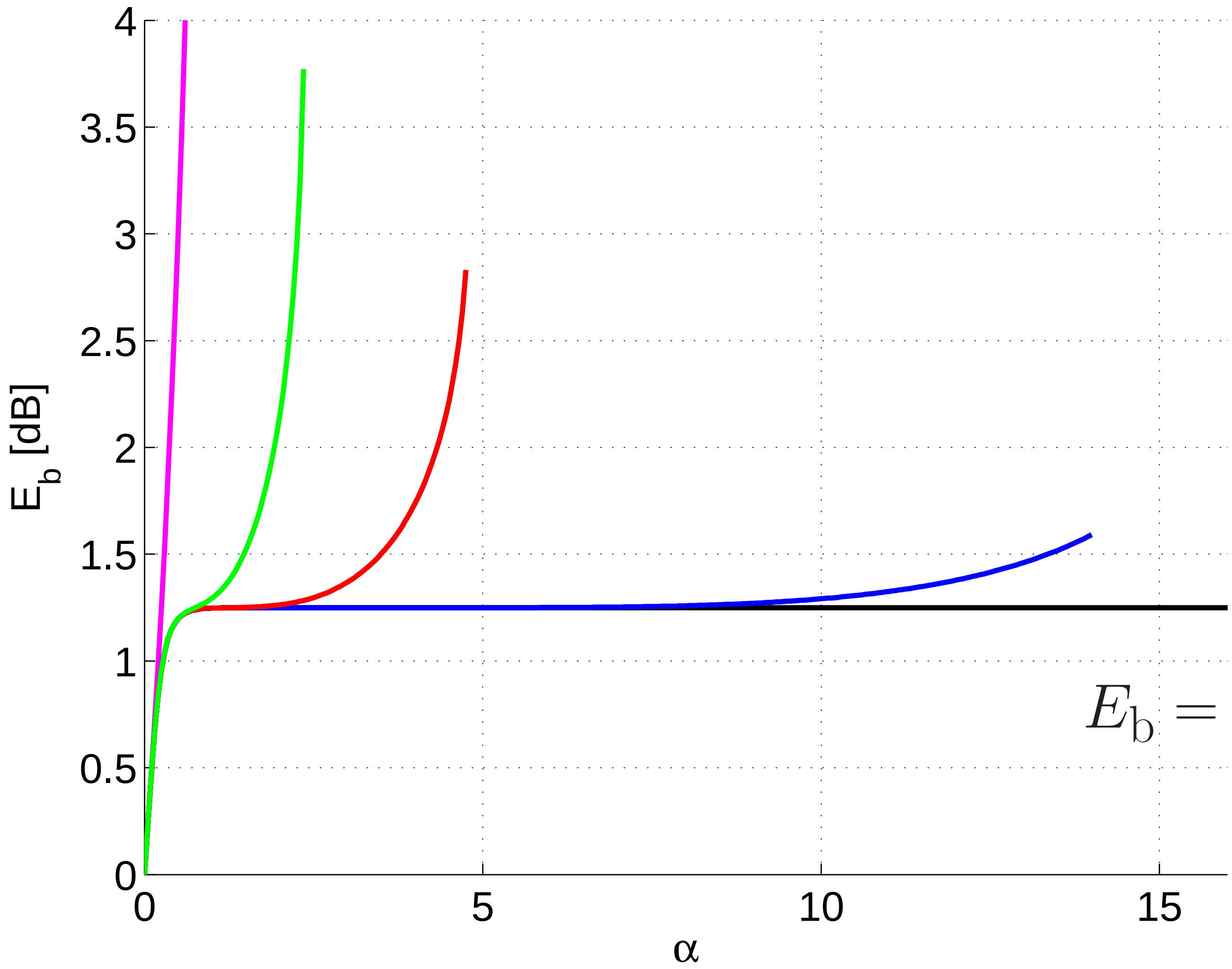
$$(\boldsymbol{H}\boldsymbol{H}^{\dagger})^{-1}: \quad R(w) = \frac{1-\alpha-\sqrt{(1-\alpha)^2-4\alpha w}}{2\alpha w}$$ inv. MP

$$\boldsymbol{U}+\boldsymbol{U}^{\dagger}: \quad R(w) = \frac{-1+\sqrt{1+4w^2}}{w}$$

Odd Integer Quadrature Lattice



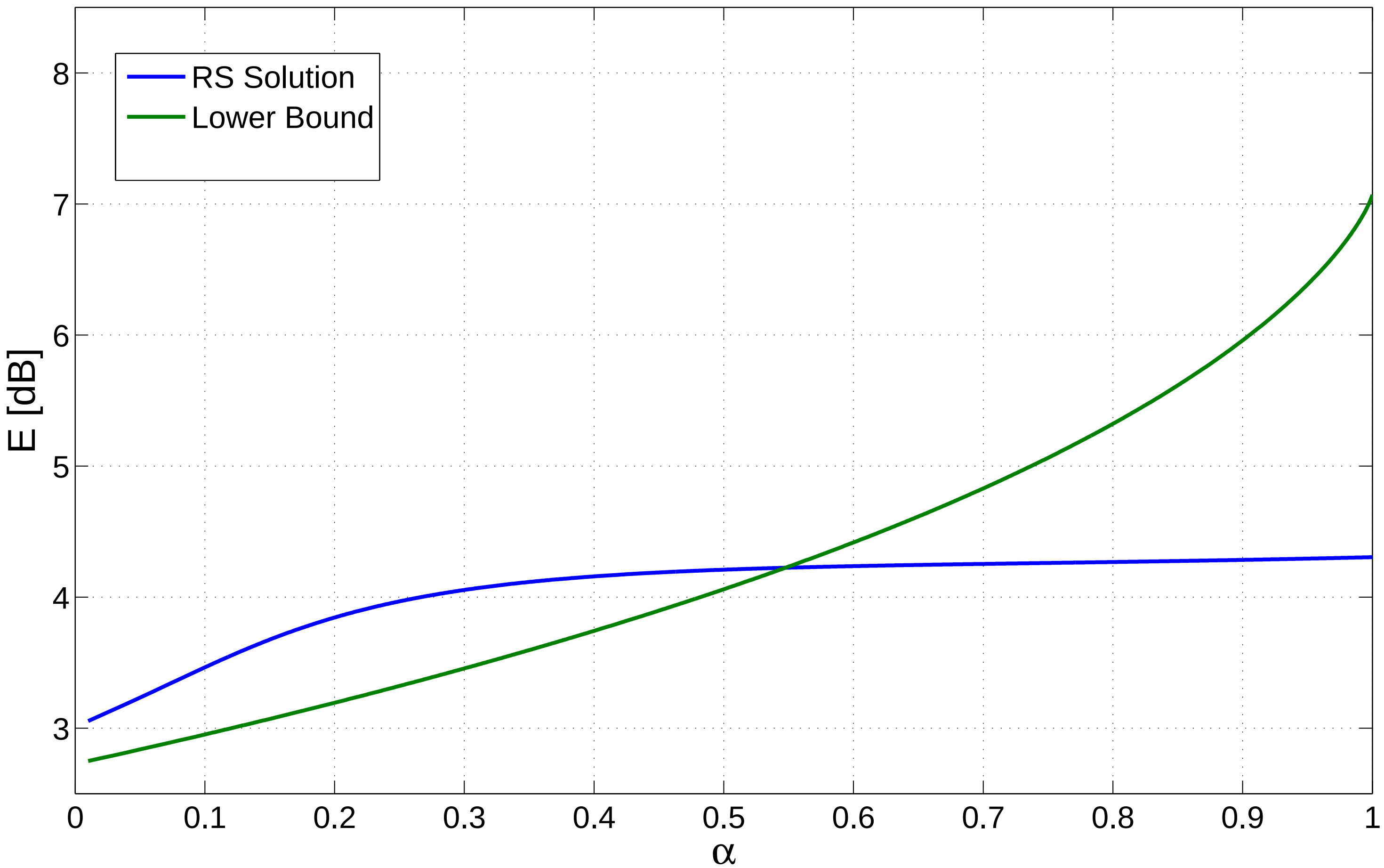
Complex TH Precoding

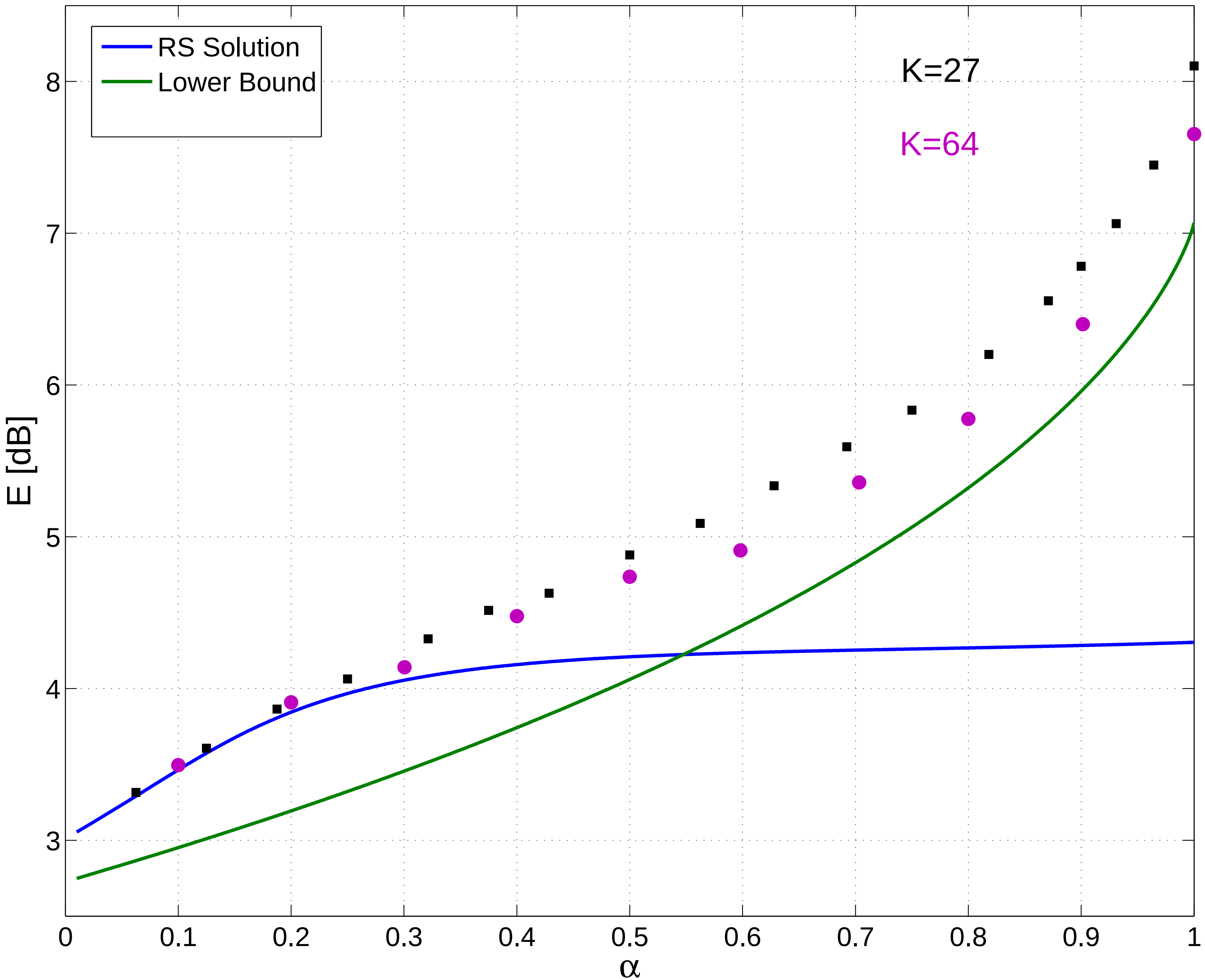


$L = 1, 2, 3, 6, 100$

$E_b = \frac{E}{2} = \frac{4}{3}$ for $L \to \infty$.

Complex TH Precoding





RS Solution
Lower Bound
K=27
K=64
8
7
6
5
4
3
E [dB]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
α

1st Order Replica Symmetry Breaking

$$\boldsymbol{Q} := \overbrace{\qquad\qquad}^{\frac{\mu}{\beta}\text{ columns}} \begin{bmatrix} q+p+\frac{\chi}{\beta} & q+p & q & q & \cdots & q & q \\ q+p & q+p+\frac{\chi}{\beta} & q & q & \cdots & q & q \\ q & q & q+p+\frac{\chi}{\beta} & q+p & \ddots & q & q \\ q & q & q+p & q+p+\frac{\chi}{\beta} & & \vdots & \vdots \\ \vdots & \vdots & \ddots & & \ddots & q & q \\ q & q & q & \cdots & q & q+p+\frac{\chi}{\beta} & q+p \\ q & q & q & \cdots & q & q+p & q+p+\frac{\chi}{\beta} \end{bmatrix}$$

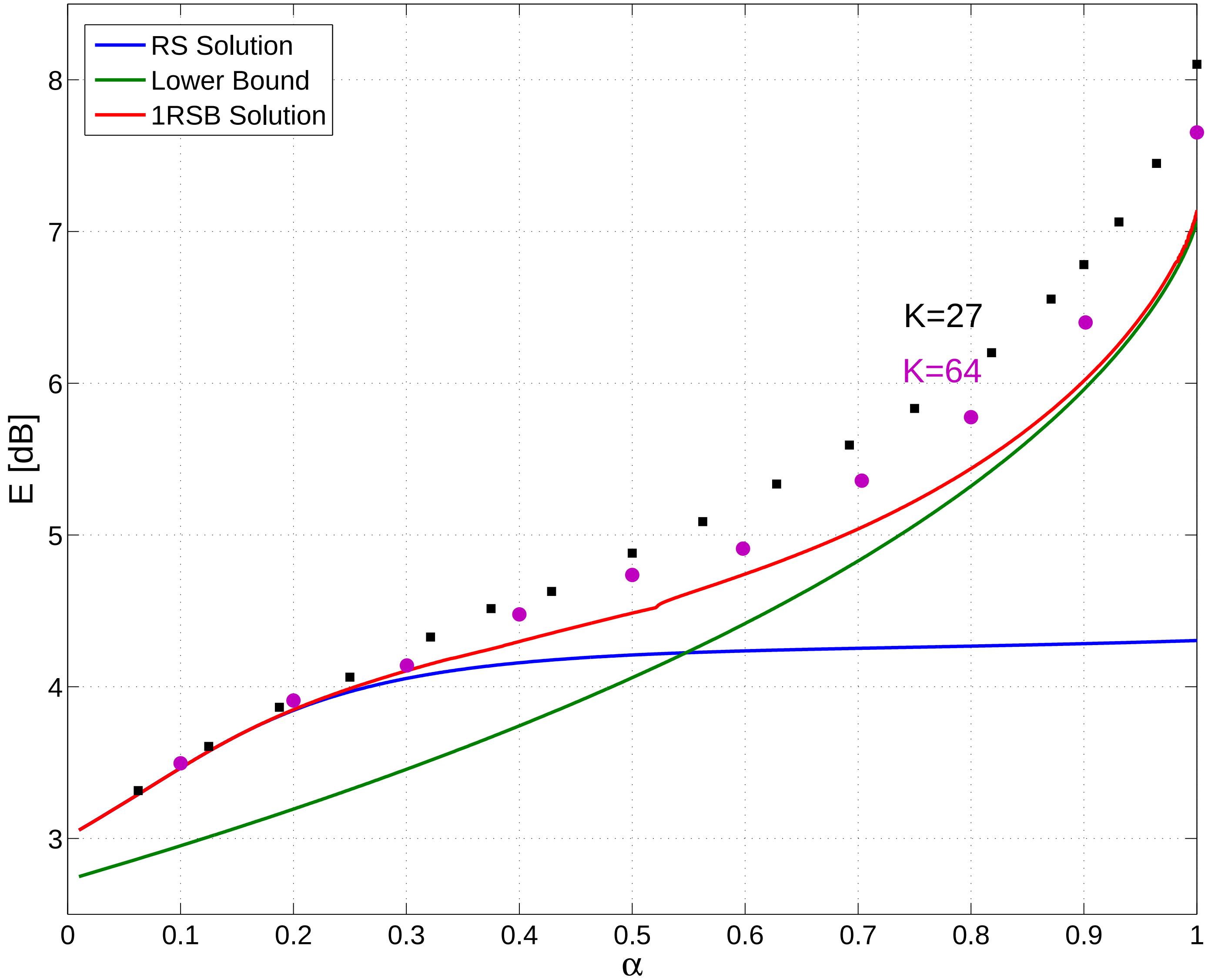
with the macroscopic parameters q, p and χ and the blocksize $\frac{\mu}{\beta}$.

1st Order Replica Symmetry Breaking

$$E = \left(q + p + \frac{\chi}{\mu}\right) R(-\chi - \mu p) - \frac{\chi}{\mu} R(-\chi) - q(\mu p + \chi) R'(-\chi - \mu p)$$

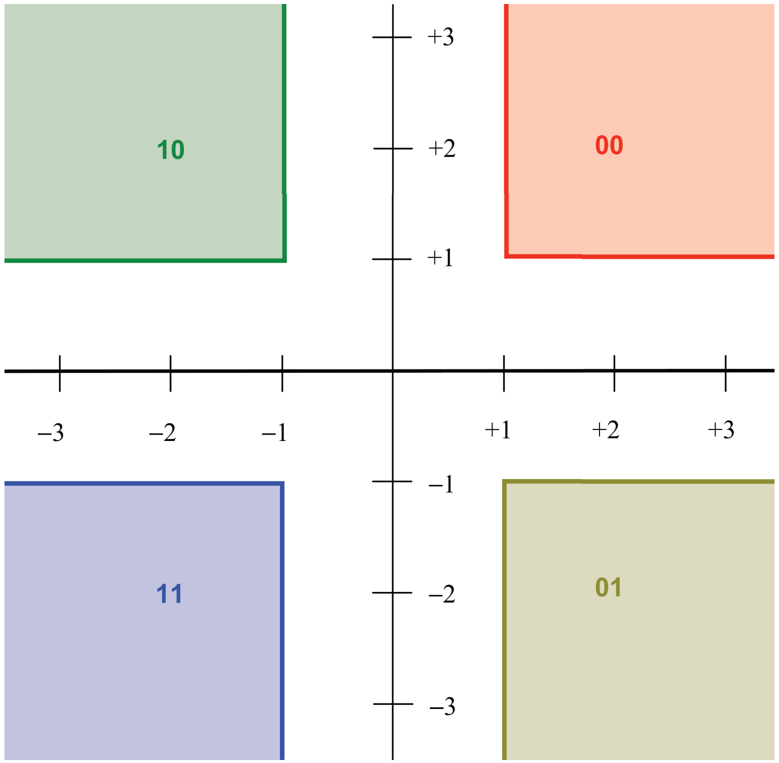
The macroscopic parameters q, p, χ and μ are given by 4 coupled fixed point equations.

Solving those fixed point equations numerically is a very tedious task.

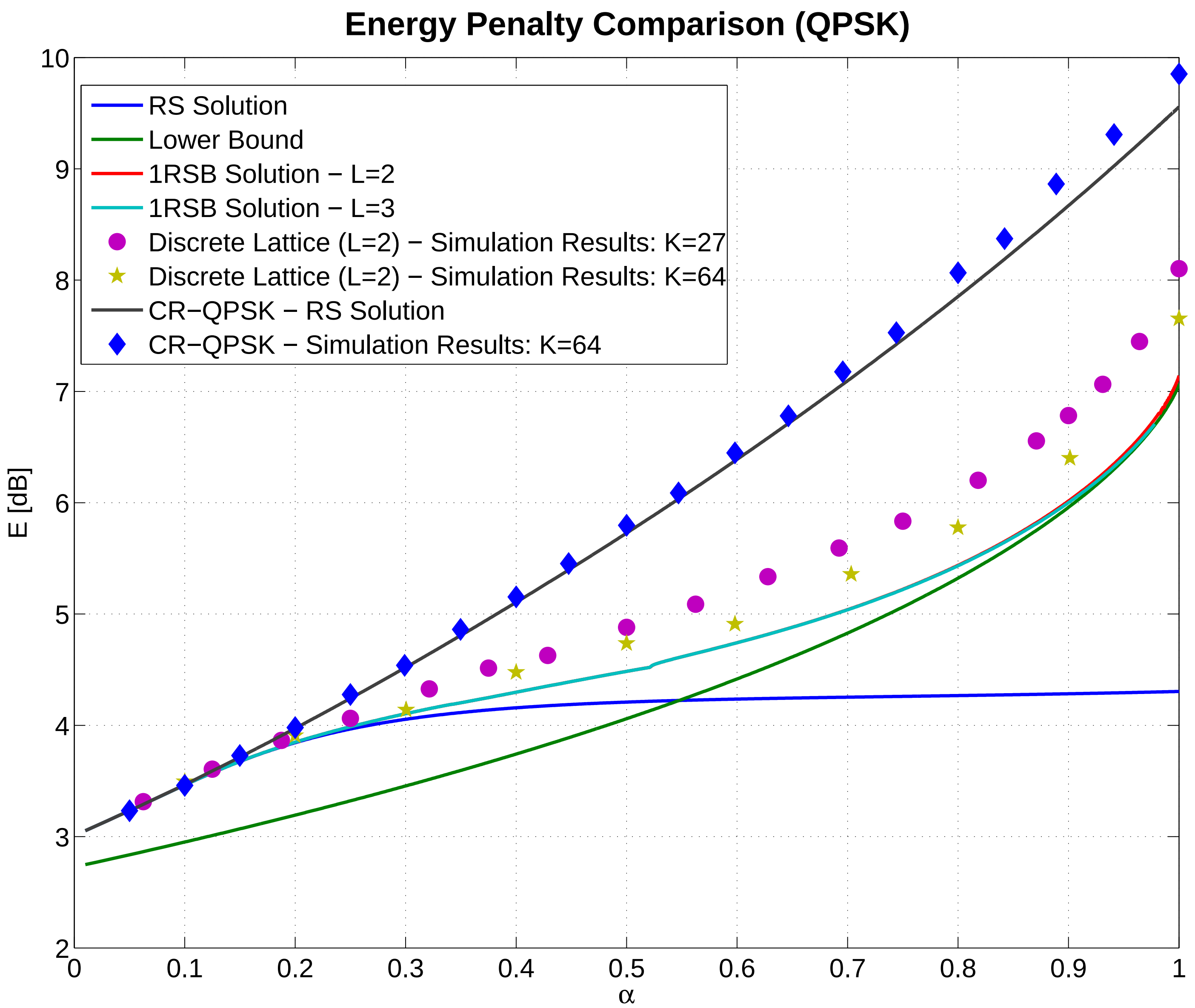


RS Solution
Lower Bound
1RSB Solution
K=27
K=64
E [dB]
α
8
7
6
5
4
3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

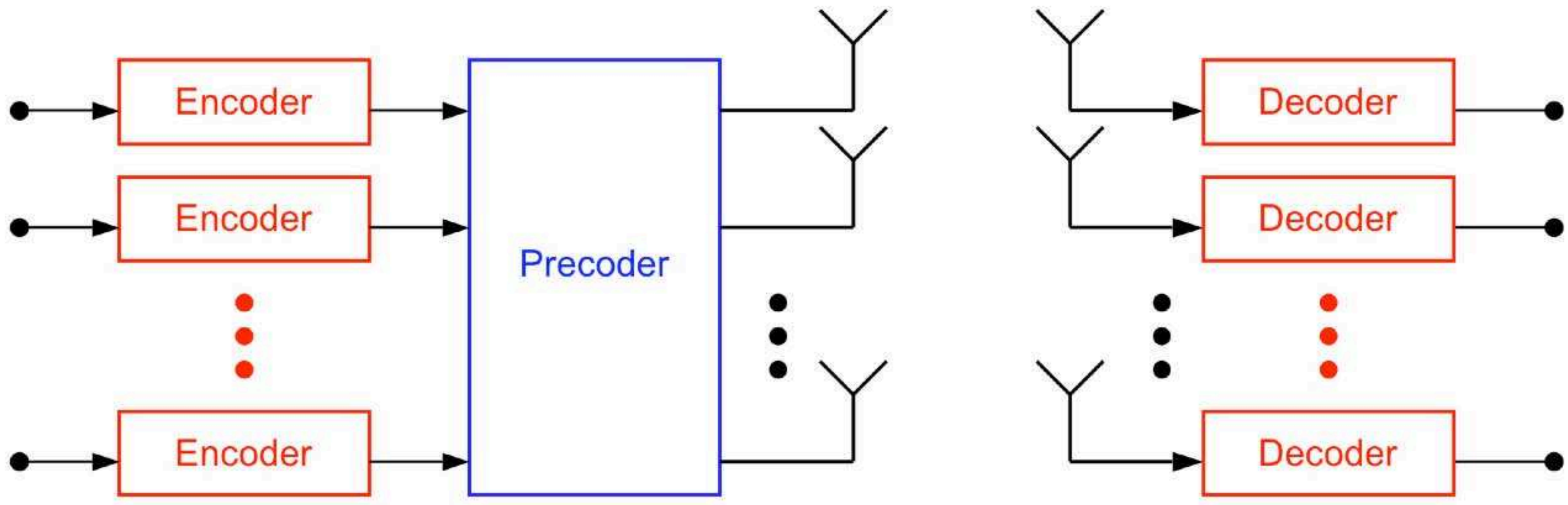
Complex Convex Relaxation



. . . allows for convex programming.

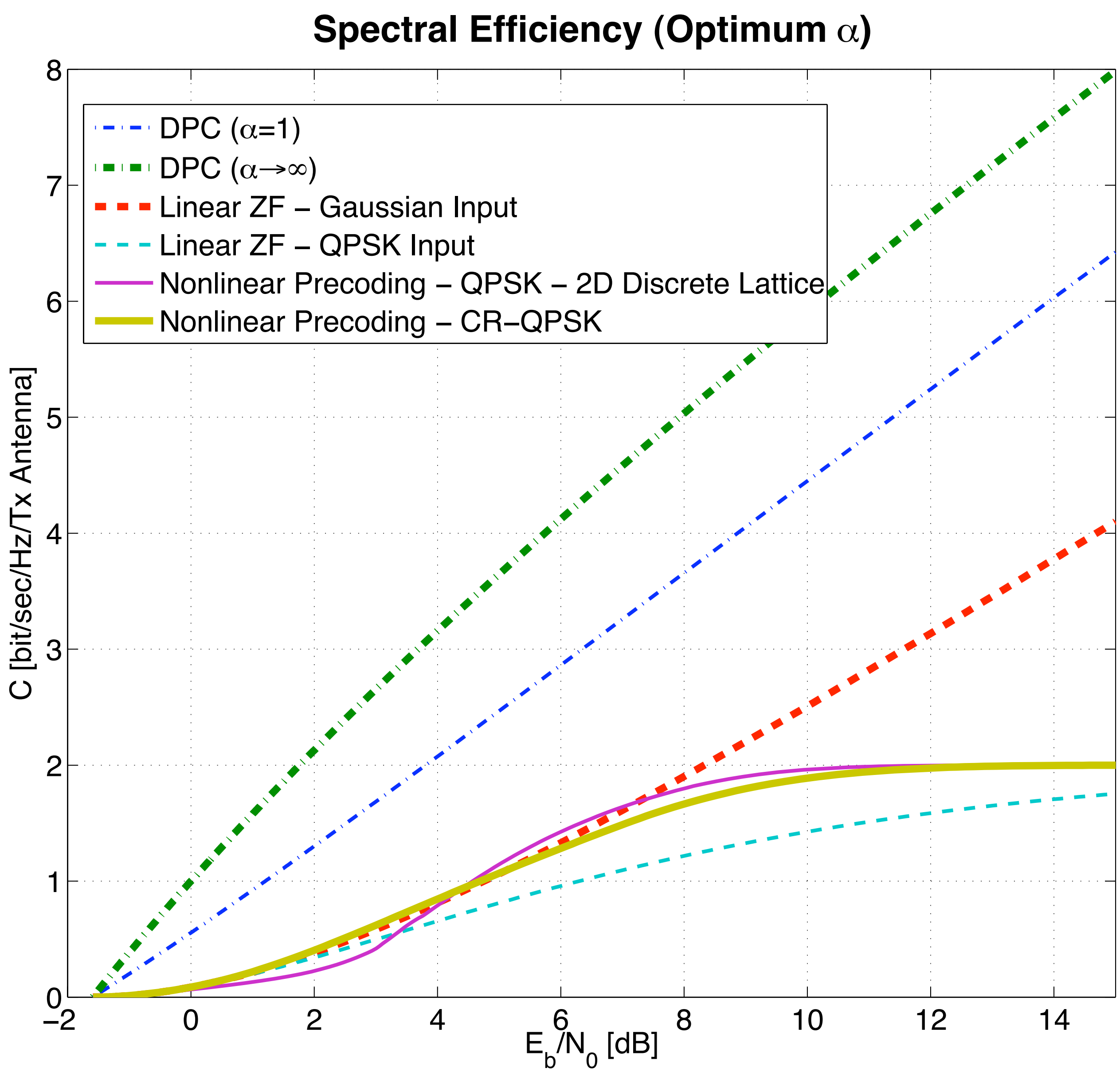

Energy Penalty Comparison (QPSK)
RS Solution
Lower Bound
1RSB Solution − L=2
1RSB Solution − L=3
Discrete Lattice (L=2) − Simulation Results: K=27
Discrete Lattice (L=2) − Simulation Results: K=64
CR−QPSK − RS Solution
CR−QPSK − Simulation Results: K=64
E [dB]
α
10
9
8
7
6
5
4
3
2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Outer Single User Coding



$$\mathrm{I}(x_k, y_k) = \mathrm{h}(y_k) - \mathrm{h}(y_k|x_k)$$

Spectral efficiency (per transmit antenna) is given by $C = \frac{1}{N}\sum_{k=1}^{K} \mathrm{I}(x_k, y_k)$


Spectral Efficiency (Optimum α)
DPC (α=1)
DPC (α→∞)
Linear ZF – Gaussian Input
Linear ZF – QPSK Input
Nonlinear Precoding – QPSK – 2D Discrete Lattice
Nonlinear Precoding – CR–QPSK
C [bit/sec/Hz/Tx Antenna]
E_b/N_0 [dB]
8
7
6
5
4
3
2
1
0
–2
0
2
4
6
8
10
12
14

Inverting Singular Channels

What happens if the MP-law has a mass point at zero $(K > N)$?

Can we precode without interference?

Inverting Singular Channels

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Can we precode without interference?

The precoder produces

$$\lim_{\epsilon \to 0} \operatorname*{argmin}_{\boldsymbol{x} \in \mathcal{X}} \frac{\boldsymbol{x}^\dagger (\boldsymbol{H}\boldsymbol{H}^\dagger + \epsilon \mathbf{I})^{-1} \boldsymbol{x}}{K}$$

The received signal becomes

$$\boldsymbol{y} = \lim_{\epsilon \to 0} \boldsymbol{H}\boldsymbol{H}^\dagger (\boldsymbol{H}\boldsymbol{H}^\dagger + \epsilon \mathbf{I})^{-1} \boldsymbol{x} + \boldsymbol{n}.$$

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If the energy is finite, there is no interference.

Overloaded Convex Precoding

There is a high probability that a vector with finite energy can be found

$$\Pr(E < \infty) = 2^{1-2K} \sum_{\ell=0}^{2N-1} \binom{2K-1}{\ell}$$

As K, N to infinity, we get

$$\Pr(E < \infty) = \begin{cases} 1 & K < 2N \\ 1/2 & K = 2N \\ 0 & K > 2N \end{cases}$$

Overloaded Convex Precoding