Vector Precoding in Wireless Communications: A Replica Analysis

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The Problem

Let

\[ E := \frac{1}{K} \min_{x \in \mathcal{X}} x^\dagger J x \]

with \( x \in \mathbb{C}^K \) and \( J \in \mathbb{C}^{K \times K} \).
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\[ \mathcal{X} = \{ x : x^\dagger x = K \} \implies E = \min \lambda(J) \]
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Example 3:

\( \mathcal{X} = \{ x : |x|^2 = 1 \}^K \quad \implies \quad ??? \)
The Gaussian Vector Channel

Let the received vector be given by

\[ y = H t + n \]

where

- \( t \) is the transmitted vector
- \( n \) is uncorrelated (white) Gaussian noise
- \( H \) is a coupling matrix accounting for crosstalk

In many applications, e.g. antenna arrays, code-division multiple-access, the coupling matrix is modelled as a random matrix with independent identically distributed entries (i.i.d. model).

Crosstalk can be processed either at receiver or transmitter.
**Processing at Transmitter**

If the transmitter is a base-station and the receiver is a hand-held device one would prefer to have the complexity at the transmitter.

E.g. let the transmitted vector be

\[ t = H^\dagger (HH^\dagger)^{-1}x \]

where \( x = s \) is the data to be sent.

Then,

\[ y = s + n. \]

No crosstalk anymore due to channel inversion.
Problems of Simple Channel Inversion

Channel inversion implies a significant power amplification, i.e.

\[ x^\dagger (HH^\dagger)^{-1} x > x^\dagger x. \]

In particular, let

- \( \alpha = \frac{K}{N} \leq 1; \)
- the entries of \( H \) are i.i.d. with variance \( 1/N. \)

Then, for fixed aspect ratio \( \alpha \)

\[ \lim_{K \to \infty} \frac{x^\dagger (HH^\dagger)^{-1} x}{x^\dagger x} = \frac{1}{1 - \alpha} \]

with probability 1.
Tomlinson-Harashima Precoding

Tomlinson ’71, Harashima & Miyakawa ’72
**Tomlinson-Harashima Precoding**

*Tomlinson '71, Harashima & Miyakawa '72*
Tomlinson-Harashima Precoding

Tomlinson ’71; Harashima & Miyakawa ’72
Instead of representing the logical "0" by +1, we present it by any element of the set \{\ldots, -7, -3, +1, +5, \ldots \} = 4\mathbb{Z} + 1. Correspondingly, the logical "1" is represented by any element of the set 4\mathbb{Z} - 1.

Choose that representation that gives the smallest transmit power.
Generalized TH Precoding

Let $\mathcal{B}_0$ and $\mathcal{B}_1$ denote the sets presenting 0 and 1, resp.

Let $(s_1, s_2, s_3, \ldots, s_K)$ denote the data to be transmitted.

Then, the transmitted energy per data symbol is given by

$$E = \frac{1}{K} \min_{x \in \mathcal{X}} x^\dagger J x$$

with

$$\mathcal{X} = \mathcal{B}_{s_1} \times \mathcal{B}_{s_2} \times \cdots \times \mathcal{B}_{s_K}$$

and

$$J = (HH^\dagger)^{-1}.$$
Zero Temperature Formulation

Quadratic programming is the problem of finding the zero temperature limit (ground state energy) of a quadratic Hamiltonian.

The transmitted power is written as a zero temperature limit

\[ E = - \lim_{\beta \to \infty} \frac{1}{\beta K} \log \sum_{x \in X} e^{-\beta K \text{Tr}(Jxx^\dagger)} \]

with \( \frac{1}{\beta} \) denoting temperature.
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\]

\[
\quad\quad\quad\quad\rightarrow - \lim_{\beta \to \infty} \lim_{K \to \infty} E \frac{1}{\beta K} \log \sum_{x \in \mathcal{X}} e^{-\beta K \text{Tr}(Jxx^\dagger)}
\]

with \(\frac{1}{\beta}\) denoting temperature.
The Harish-Chandra Integral
(also called the Itzykson-Zuber integral, particular in the physics community)

Let $\tilde{Q}$ be any positive semi-definite matrix of bounded rank $n$, then

$$\lim_{K \to \infty} \frac{1}{K} \log \mathbb{E} e^{-K \text{Tr} J\tilde{Q}} = -\sum_{a=1}^{n} \lambda_a \int_{0}^{\infty} R(-w) dw$$

with $\lambda_a$ denoting the positive eigenvalues of $\tilde{Q}$ (Marinari et al. '94; Guionnet & Maïda '05).

This is a large-deviations result for random matrices.

Recently, it was named the free Fourier transform.
Free Fourier Transform

We want

$$\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta K \text{Tr}(Jxx^\dagger)}.$$

We know (Itzykon & Zuber '80)

$$\lim_{K \to \infty} \frac{1}{K} \log \mathbb{E} e^{-K \text{Tr}(J\mathbf{P})} = -n \sum_{a=1}^{\lambda} \lambda_a \left( \int_0^R J(-w) dw \right).$$
**Free Fourier Transform**

We want

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We know (Itzykon & Zuber '80)

$$\lim_{K \to \infty} \frac{1}{K} \log \mathbf{E} e^{-K \text{Tr} J^P} = - \sum_{a=1}^{n} \lambda_a(P) \int_0^\infty R_J(-w) dw.$$
Free Fourier Transform

We want
\[
\lim_{K \to \infty} \frac{1}{K} \mathbf{E} \log \sum_{x \in \mathcal{X}} e^{-\beta K \operatorname{Tr}(Jxx^\dagger)}.
\]

We know (Itzykon & Zuber '80)
\[
\lim_{K \to \infty} \frac{1}{K} \log \mathbf{E} e^{-K \operatorname{Tr} JP} = -\sum_{a=1}^{n} \lambda_a(P) \int_{0}^{\infty} R_J(-w)dw.
\]

We would like to exchange expectation and logarithm:
\[
\mathbf{E} \log X = \lim_{n \to 0} \frac{1}{n} \log \mathbf{E} X^n.
\]
Replica Continuity

We want

$$\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta K \text{Tr}(J xx^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E}_{J} \left( \sum_{x \in \mathcal{X}} e^{-\beta K \text{Tr}(J xx^\dagger)} \right)^n$$
We want
\[
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\]
\[
= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \prod_{a=1}^{n} \sum_{x_a \in \mathcal{X}} e^{-\beta K \text{Tr}(Jx_a x_a^\dagger)}
\]
We want

\[
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\]

\[
= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \sum_{x_1 \in \mathcal{X}} \cdots \sum_{x_n \in \mathcal{X}} e^{-K \text{Tr}(J \beta \sum_{a=1}^{n} x_a x_a^\dagger)}
\]

with

\[
Q := \sum_{a=1}^{n} x_a x_a^\dagger.
\]
Replica Calculations

\section*{Replica Continuity}

We want

\[
\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta K \text{Tr}(Jx x^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \left( \sum_{x \in \mathcal{X}} e^{-\beta K \text{Tr}(Jx x^\dagger)} \right)^n
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= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \left( \prod_{a=1}^n \sum_{x_a \in \mathcal{X}} e^{-\beta K \text{Tr}(Jx_a x_a^\dagger)} \right)
\]

\[
= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \sum_{x_1 \in \mathcal{X}} \cdots \sum_{x_n \in \mathcal{X}} e^{-K \text{Tr} \left( J \beta \sum_{a=1}^n x_a x_a^\dagger \right)}
\]

\[
= - \lim_{n \to 0} \frac{1}{n} \log \mathbb{E} \exp \left[ \sum_{a=1}^n \beta \lambda_a(Q) \int_0 R_{J}(w) dw \right]
\]

with

\[
Q_{ab} := \frac{1}{K} x_a^\dagger x_b.
\]
**Replica Symmetry**

\[
Q := \begin{bmatrix}
q + \frac{\chi}{\beta} & q & \cdots & q & q \\
q & q + \frac{\chi}{\beta} & \cdots & q & q \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q & q & \cdots & q + \frac{\chi}{\beta} & q \\
q & q & \cdots & q & q + \frac{\chi}{\beta}
\end{bmatrix}
\]

with some macroscopic parameters \( q \) and \( \chi \).

This is the most critical step. In general, the structure of \( Q \) is more complicated. Generalizations are called replica symmetry breaking (RSB).
RS Solution

Let $P(s)$ denote the limit of the empirical distribution of the information symbols $s_1, s_2, \ldots, s_K$ as $K \to \infty$. Let $q$ and $\chi$ be the simultaneous solutions to

$$q = \int \int \text{argmin}_{x \in B_s}^2 \left| z \sqrt{2qR'(-\chi)} - 2xR(-\chi) \right| d z \, d P(s)$$

$$\chi = \frac{1}{\sqrt{2qR'(-\chi)}} \int \int \text{argmin}_{x \in B_s} \left| z \sqrt{2qR'(-\chi)} - 2xR(-\chi) \right| z^* d z \, d P(s)$$

where

$$Dz = \exp(-z^2/2)dz/\sqrt{2\pi}, \quad R(\cdot) \text{ is the R-transform of the limiting eigenvalue spectrum of } J, \quad \text{and } 0 < \chi < \infty.$$  

Then, replica symmetry (RS) implies

$$\frac{1}{K} \min_{x \in \chi} x^\dagger J x \to q \frac{\partial}{\partial \chi} \chi R(-\chi)$$

as $K \to \infty$. 
Some $R$-Transforms

\begin{align*}
\mathbf{I} : & \quad R(w) = 1 \\
HH^\dagger : & \quad R(w) = \frac{1}{1 - \alpha w} \quad \text{Marchenko-Pastur (MP) law} \\
(HH^\dagger)^{-1} : & \quad R(w) = \frac{1 - \alpha - \sqrt{(1 - \alpha)^2 - 4\alpha w}}{2\alpha w} \quad \text{inv. MP} \\
U + U^\dagger : & \quad R(w) = \frac{-1 + \sqrt{1 + 4w^2}}{w}
\end{align*}
Odd Integer Quadrature Lattice
Complex TH Precoding

$E_b := \frac{E}{2} = \frac{4}{3}$ for $L \to \infty$. 

$L = 1, 2, 3, 6, 100$
Complex TH Precoding

![Graph showing E [dB] vs. α]

- RS Solution
- Lower Bound

Vector Precoding in Wireless Communications

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1st Order Replica Symmetry Breaking

\[ Q := \begin{bmatrix}
q + p + \frac{x}{\beta} & q + p & q & \cdots & q & q \\
q + p & q + p + \frac{x}{\beta} & q & \cdots & q & q \\
q & q & q + p + \frac{x}{\beta} & q + p & \cdots & q & q \\
q & q & q + p & q + p + \frac{x}{\beta} & \vdots & \vdots & \\
\vdots & \vdots & \cdots & \ddots & q & q \\
q & q & q & \cdots & q & q + p + \frac{x}{\beta} & q + p \\
q & q & q & \cdots & q & q + p & q + p + \frac{x}{\beta}
\end{bmatrix} \]

with the macroscopic parameters \( q, p \) and \( x \) and the blocksize \( \frac{\mu}{\beta} \).
1st Order Replica Symmetry Breaking

\[ E = \left(q + p + \frac{\chi}{\mu}\right) R(-\chi - \mu p) - \frac{\chi}{\mu} R(-\chi) - q(\mu p + \chi) R'(-\chi - \mu p) \]

The macroscopic parameters \( q, p, \chi \) and \( \mu \) are given by 4 coupled fixed point equations.

Solving those fixed point equations numerically is a very tedious task.
Complex Convex Relaxation

... allows for convex programming.
Energy Penalty Comparison (QPSK)

- RS Solution
- Lower Bound
- 1RSB Solution – L=2
- 1RSB Solution – L=3
- Discrete Lattice (L=2) – Simulation Results: K=27
- Discrete Lattice (L=2) – Simulation Results: K=64
- CR–QPSK – RS Solution
- CR–QPSK – Simulation Results: K=64
Spectral efficiency (per transmit antenna) is given by

\[ C = \frac{1}{N} \sum_{k=1}^{K} I(x_k, y_k) \]
Spectral Efficiency (Optimum $\alpha$)

- DPC ($\alpha=1$)
- DPC ($\alpha\to\infty$)
- Linear ZF – Gaussian Input
- Linear ZF – QPSK Input
- Nonlinear Precoding – QPSK – 2D Discrete Lattice
- Nonlinear Precoding – CR-QPSK

$C$ [bit/sec/Hz/Tx Antenna] vs $E_b/N_0$ [dB]
Inverting Singular Channels

What happens if the MP-law has a mass point at zero \((K > N)\)?

Can we precode without interference?
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The precoder produces

\[
\lim_{\epsilon \to 0} \arg \min_{x \in \mathcal{X}} \frac{x^\dagger (HH^\dagger + \epsilon I)^{-1} x}{K}
\]

The received signal becomes

\[
y = \lim_{\epsilon \to 0} HH^\dagger (HH^\dagger + \epsilon I)^{-1} x + n.
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The received signal becomes

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If the energy is finite, there is no interference.
Overloaded Convex Precoding

There is a high probability that a vector with finite energy can be found

\[
\Pr(E < \infty) = 2^{1-2K} \sum_{\ell=0}^{2N-1} \binom{2K-1}{\ell}
\]

As \( K, N \) to infinity, we get

\[
\Pr(E < \infty) = \begin{cases} 
1 & K < 2N \\
1/2 & K = 2N \\
0 & K > 2N
\end{cases}
\]
Overloaded Convex Precoding

$\frac{1}{1-P(K,N)}$