How Much Does Transmit Correlation Affect
the Sum-Rate of MIMO Downlink Channels?

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Outline

- Introduction
- Questions of interest in a broadcast scenario
- System model and multiuser scheduling schemes
- Capacity scaling of DPC with channel correlation
- Capacity scaling of beamforming with channel correlation
- Simulations
- Conclusion
Introduction to Broadcast Channels

- Multiple antennas add tremendous value to point to point systems
- Research shifted recently to the role of multiple antennas in multiuser systems
- Broadcast scenarios (point to multi-point) are especially important because downlink scheduling is the major bottleneck for broadband wireless networks
Three Main Questions in a Broadcast Scenario (1)

Q1) Quantify the maximum sum rate possible to all users

A1) Sum-rate is achieved using dirty paper coding (DPC) (Caire and Shamai ’02, Viswanath and Tse ’02, Vishwanath et al. ’02, Yu and Cioffi ’02)

(-) DPC is computationally complex at both Tx and Rx

(-) Requires a great deal of Feedback (CSI for all users at Tx)
Three Main Questions in a Broadcast Scenario (2)

The second question is motivated by the drawbacks of DPC

Q2) Devise computationally efficient algorithms for capturing capacity

A2) Utilize multi-user diversity to achieve performance close to capacity

(+) Opportunist multiple random beamforming coincides asymptotically with DPC (Sharif and Hassibi ’06)

\[ R = M \log \log n + M \log \frac{P}{M} + o(1) \]

(+) Requires simply SINR feedback to Tx
Three Main Questions in a Broadcast Scenario (3)

Q3) With this promising performance, how does opportunistic beam-forming perform under various non-idealities

A3) (i) Time correlation (Kountouris and Gesbert '05)

(ii) Frequency correlation (Fakhereddin, Sharif, and Hassibi '06)

(iii) Channel estimation error (Vikali, Sharif, and Hassibi '06)

(iv) Spatial correlation (D. Park and S Y. Park '05)

Main problem to be addressed:

• For a Gaussian broadcast channel, we would like to quantify the hit that transmit correlation causes to scaling laws of the sum-rate capacity. We consider DPC and various beamforming schemes.
• Base station with $M$ antennas broadcasting to $n$ single-antenna users

• Received signal at each antenna

$$Y_i = \sqrt{P}H_iS + W_i, \quad i = 1, \ldots, n$$

with $E[S^*S] = 1$ and Gaussian noise $W_i \sim CN(0, I)$

• Channel $H_i$ of $i$-th user is $1 \times M$ vector
  
  - Distributed as $CN(0, R)$; $R$ is nonsingular with $\text{tr}(R) = M$
  
  - Known perfectly at receiver
  
  - Follows a bock fading model (with coherence interval $T$)
  
  - $H_i$ is independent from one user to another
Digression: Extreme Value Theory

- Let $x_1, x_2, \ldots, x_n$ be i.i.d random variables with pdf $f(x)$ and CDF $F(x)$. How does $\max_i x_i$ behave?

- Let $z$ denote the limit

$$z = \lim_{x \to \infty} \frac{1 - F(x)}{f(x)}$$

then, for large $n$ we have with high probability

$$\max_i x_i = z \log(n)$$
Scaling of DPC under Correlation

- Sum-rate capacity of DPC

\[ R_{DPC} = E \left\{ \max_{\{p_1, \ldots, p_n, \sum p_i = P\}} \log \det \left( I + \sum_{i=1}^{n} H_i^* P_i H_i \right) \right\} . \]

- Define \( H_i = H_{w_i} R^{\frac{1}{2}} \) and employ the inequality \( \det(A) \leq \left( \frac{\text{tr}(A)}{M} \right)^M \)
  to obtain

\[ R_{DPC} \leq M \log \left( \frac{1}{M} \text{tr}(R^{-1}) + \max_i \|H_{w_i}\|^2 \frac{P}{M} \right) \]

- For large \( n \), \( \max_i \|H_{w_i}\|^2 \) behaves as \( \log n \) with high probability.
Thus,

\[ R_{DPC} \leq M \log \left( \frac{\text{tr}(R^{-1})}{M} + \frac{P}{M} \log n \right) + \log \det R \]

\[ = M \log \log n + M \log \left( \frac{P}{M} \right) + M \log \sqrt[2M]{\det R} \text{ for large } n \]

- This is also a lower bound as it the scaling of deterministic beam forming. So, for large \( n \)

\[ R_{DPC} = M \log \log n + M \log \left( \frac{P}{M} \right) + M \log \sqrt[2M]{\det R} \]

- Compare with rate for spatially uncorrelated channel

\[ R_{DPC} = M \log \log n + M \log \left( \frac{P}{M} \right) \]

Keep in mind that \( \sqrt[2M]{\det R} \leq \frac{\text{Tr}(R)}{M} = 1 \)
What is Random Beam Forming?

- Choose $M$ random orthonormal vectors $\phi_m$, $m = 1, \ldots, M$ (according to an isotropic distribution)

- Construct the signal

$$S(t) = \sum_{m=1}^{M} \phi_m s_m(t), \quad t = 1, \ldots, T$$

where $T$ is less than the coherence interval of the channel.

- After $T$ channel uses we independently choose another isotropic set of orthonormal vectors $\{\phi_m\}$, and so on. So we are transmitting $M$ random beams.

- This is a generalization of the scheme “Opportunistic Beamforming” (Viswanath et al. ’02) in which only one random beam is transmitted and proportional fairness is guaranteed.
Exploit Multi-User Diversity

- Each receiver $i = 1, \ldots, n$ computes the following $M$ SINRs

$$\text{SINR}_{i,m} = \frac{|H_{i\phi_m}|^2}{1/\rho + \sum_{n \neq m} |H_{i\phi_n}|^2}, \quad m = 1, \ldots, M$$

and feeds back the best SINR.

- Rather than randomly assigning the beams, the transmitter assigns signal $s_m$ to the user with the best SINR for that signal. Therefore

$$C = E \sum_{m=1}^{M} \log \left( 1 + \max_{i=1,\ldots,n} \text{SINR}_{i,m} \right)$$

- Due to the symmetry of all the random variables involved:

$$C = ME \log \left( 1 + \max_{i=1,\ldots,n} \text{SINR}_{i,1} \right)$$
Other Beamforming Schemes

- Random Beamforming (RBF)  \( S(t) = \sum_{m=1}^{M} \phi_m s_m(t) \)
- RBF with Channel whitening  
  \[
  S(t) = \sum_{m=1}^{M} \sqrt{\alpha} R^{-1/2} \phi_m s_m(t)
  \]
- RBF with general precoding  
  \[
  S(t) = \sum_{m=1}^{M} \sqrt{\alpha} A \phi_m s_m(t)
  \]
- Deterministic beamforming  
  \[
  S(t) = \sum_{m=1}^{M} \phi_m s_m(t), \quad \phi_m \text{'s are fixed}
  \]
How to Determine Scaling of BF Schemes

1. Sum rate

\[ R_{BF} = E \sum_{m=1}^{M} \log \left( 1 + \max_{i=1,\ldots,n} \text{SINR}_{i,m} \right) \]
\[ = ME \left( 1 + \max_{i=1,\ldots,n} \text{SINR}_{i,m} \right) \]

2. To calculate expectation, condition on beams

\[ R_{BF|\Phi} = ME_{H_{i|\Phi}} \left( 1 + \max_{i=1,\ldots,n} \text{SINR}_{i,m} \right) \]

- \( \text{SINR}_{i,m|\Phi} \) is iid over \( i \)
- Find the distribution of \( \text{SINR}_{i,m|\Phi} \)
- Employ extreme value theory to find \( \max_{i=1,\ldots,n} \text{SINR}_{i,m} \)

3. Average \( R_{BF|\Phi} \) over \( \Phi \)
Statistics of $\text{SINR}_{i,m}$ (White Channel)

- $\text{SINR}_{i,m}$ is defined by

$$\text{SINR}_{i,m} = \frac{|H_i \phi_m|^2}{1/\rho + \sum_{n \neq m} |H_i \phi_n|^2}, \quad m = 1, \ldots, M$$

- Easy to find distribution of $\text{SINR}_{i,m} | \Phi$ when $H_i$ is white

$$f(x) = \frac{e^{-\frac{x}{\rho}}}{(1+x)^M} \left( \frac{1}{\rho}(1+x) + M - 1 \right)$$

$$F(x) = 1 - \frac{e^{-\frac{x}{\rho}}}{(1+x)^M}$$

- Finding these statistics in the correlated case is challenging
Statistics of $\text{SINR}_{i,m}$ Given $\Phi$ (Correlated Case)

- We can show that the CDF of SINR in the correlated case

\[F(x) = 1 - \frac{1}{2\pi^M \det(R)} \lambda_M \prod_{i=1}^{M-1} \frac{\lambda_i \lambda_M}{x(x_i - \lambda_M)} e^{-\frac{1}{\rho} \frac{x}{\lambda_M}}\]

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M$ are the eigenvalues of the matrix

\[A = (1 + x) \Lambda^{1/2} \phi_m \phi_m^* \Lambda^{1/2} - x \Lambda \quad \rho = \frac{P}{M}\]

Note that eigenvalues are a function of $x$.

- pdf is given by

\[f(x) = \frac{1}{2\pi^M \det(R)} e^{-\frac{1}{\rho} \frac{x}{\lambda_M}} \prod_{i=1}^{M-1} \frac{\lambda_i \lambda_M}{x(x_i - \lambda_M)} \times \]

\[\left\{ \frac{1}{\rho} \frac{\|q_M\|^2_C}{\lambda_M} - \|q_M\|^2_B - \sum_{i=1}^{M} \frac{1}{\lambda_i} \frac{\lambda_i^2 M \|q_i\|^2_C - \lambda_i^2 \|q_M\|^2_C}{x(x_i - \lambda_M)} \right\}\]

where $B = \Lambda^{1/2} (\phi_m \phi_m^* - I) \Lambda^{1/2}$ \quad $C = \Lambda^{1/2} \phi_m \phi_m^* \Lambda^{1/2}$
Scaling of Maximum SINR

- Can now show
  \[ \lim_{x \to \infty} \frac{1 - F(x)}{f(x)} = \frac{P}{M} \frac{1}{\|\phi_m\|_\Lambda^{-1}} \]

- Using extreme value theory, we can show that for large \( n \)
  \[ \max_{i=1, \ldots, n} \text{SINR}_{i,m} = \frac{P}{M} \frac{1}{\|\phi_m\|_\Lambda^{-1}} \log n \]

- Conditional sum-rate capacity scales as
  \[ R_{BF|\Phi} = M \log \log n + M \log \frac{P}{M} + M \log \left( \frac{1}{\|\phi_m\|_\Lambda^{-1}} \right) \]

- Sum-rate capacity of random beam-forming
  \[ R_{RBF} = M \log \log n + M \log \frac{P}{M} + ME_{\Phi} \log \left( \frac{1}{\|\phi_m\|_\Lambda^{-1}} \right) \]
Averaging Over the Random Beams

- Need to obtain CDF of $\frac{1}{\|\phi_m\|^2_{\Lambda-1}}$ which is challenging.

- The CDF of $y = \frac{1}{\|\phi\|^2_{\Lambda-1}}$ is given by

$$G(x) = Pr\left(\frac{1}{\|\phi\|^2_{\Lambda-1}} < x\right) = 1 - \sum_i \eta_i \left(\frac{1}{x} - \frac{1}{\lambda_i(\Lambda)}\right)^{M-1} u \left(1 - \frac{x}{\lambda_i(\Lambda)}\right)$$

where $\eta_i = \frac{1}{\prod_{j \neq i} (\frac{1}{x_j(\Lambda)} - \frac{1}{x_i(\Lambda)})}$

- Use CDF to show that

$$R_{RBF} = M \log \log n + M \log \frac{P}{M} +$$

$$\log \lambda_1(\Lambda) + \sum_{i=1}^{M} \eta_i \log \left(\frac{\lambda_i}{\lambda_1}\right) \sum_{k=1}^{M-1} \frac{1}{k+2} \left(\frac{-1}{\lambda_i}\right)^{M-1-k} \frac{1}{y^{k+2}} \lambda_1^{-1}$$
Sum rate of Deterministic Beam Forming

- Sum-rate of deterministic beam forming

\[ R_{BF-D} = M \log \log n + M \log \frac{P}{M} + \sum_{i=1}^{M} \log \left( \frac{1}{\phi_i^* U^* \Lambda^{-1} U \phi_i} \right) \]

\( U^* \Lambda^{-1} U \) is the eigenvalue decomposition of \( R^{-1} \)

- Special case: \( U \phi_i \)'s are the columns of identity matrix

\[ R_{BF-D} = M \log \log n + M \log \frac{P}{M} + M \log M \sqrt{\det(R)} \]

Since \( \text{tr}(R) = M \), the geometric mean satisfies \( \det(R) \leq 1 \)

- Scaling coincides with (D. Park and S Y. Park '05) which focused on the \( M = 2 \) case
**Sum rate of RBF with Channel Whitening**

- For random beam forming with channel whitening,
  \[ S(t) = \sum_{m=1}^{M} \sqrt{\alpha R^{-1/2}} \phi_{m} s_{m}(t) \]

- Set \( \alpha = \frac{\text{tr}(R^{-1})}{M} \) to guarantee \( E[S^*S] \leq 1 \)

- Scaling becomes the same as for white channel case with reduced signal power

\[
R_{BF-W} = M \log \log n + M \log \frac{P}{M} + M \log \frac{M}{\text{tr}(R^{-1})}
\]
Simulations

- Consider a base station with $M = 2$ and $M = 3$ antennas.
- The corresponding correlation matrix is

$$R = \begin{bmatrix}
1 & \alpha \\
\alpha & 1
\end{bmatrix}$$

$$R = \begin{bmatrix}
1 & \alpha & \alpha^2 \\
\alpha & 1 & \alpha \\
\alpha^2 & \alpha & 1
\end{bmatrix}$$
Figure 1: Sum-rate loss versus the correlation factor $\alpha$ for a system with $M = 2$ and $n = 100$. 
Figure 2: Sum-rate versus the correlation factor $\alpha$ for a system with $M = 2$, $P = 10$, and $n = 100$. 
Figure 3: Sum-rate loss versus the correlation factor $\alpha$ for a system with $M = 3$ and $n = 100$. 
Figure 4: Sum-rate versus the number of users in a system with $M = 2$ and $\alpha = 0.5$
Conclusion

- Studied the effect of spatial correlation on various multiuser schemes for MIMO broadcast channels.

- Considered DPC and random, deterministic, and channel whitening schemes.

- All these techniques exhibit the same scaling for iid channels

\[
R_{\text{sum-rate}} = M \log \log n + M \log \frac{P}{M}
\]

- In the presence of correlation between transmit antennas, scaling is

\[
R_{\text{sum-rate}} = M \log \log n + M \log \frac{P}{M} + M \log c
\]

The constant \( 0 < c \leq 1 \) depends on the scheduling scheme and the eigenvalues of the correlation matrix \( R \).
Recent Results:

Scaling Laws of Group Broadcast Channels

- $K$ groups of users
- Each group of users is interested in the same data
- Worst user of each group is the bottleneck
- Worst user is difficult to define in the multi-antenna case
- For $K$ groups with $n$ users each, we show that capacity scales like
  $$C = K \frac{P}{n M}$$
- We show that to have a constant rate, $M$ should grow at least as fast as $\log n$
- This is a joint work with Amir Dana and Babak Hassibi, Cal Tech.
Consider the SINR for the first beam

$$\text{SINR}_{i,1} = \frac{|H_i\phi_1|^2}{1/\rho + \sum_{n=2}^{M} |H_i\phi_n|^2},$$

Define $S$ by

$$S = -\frac{x}{\rho} + H_i^*((1+x)\phi_1\phi_1^* - xI)H_i$$

Then

$$P(\text{SINR}_{i,1} > x) = P(S > 0) = \int_{-\infty}^{\infty} P(H_i)u(S)dH_i$$

$$= \frac{1}{\pi^M \det(R)} \int_{-\infty}^{\infty} e^{-H_i^*R^{-1}H_i}u(S)dH_i$$

To evaluate integral, use the integral representation of unit step

$$u(S) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(j\omega + \beta)S}}{j\omega + \beta} d\omega$$
• Desired probability becomes

\[
P(\text{SINR}_{i,1} > x) = \frac{1}{2\pi^{M+1} \det(R)} \int_{-\infty}^{\infty} d\omega \frac{1}{j\omega + \beta} \int_{-\infty}^{\infty} dH_i e^{(j\omega + \beta)S - H_i^* R^{-1} H_i} \\
= \frac{1}{2\pi^{M+1} \det(R)} \int_{-\infty}^{\infty} d\omega e^{-\frac{(j\omega + \beta)^2}{\rho}} \int_{-\infty}^{\infty} dH_i e^{-H_i^* \tilde{R}^{-1} H_i} \\
= \frac{1}{2\pi^{M+1} \det(R)} \int_{-\infty}^{\infty} d\omega e^{-\frac{(j\omega + \beta)^2}{\rho}} \frac{1}{\det(\tilde{R})}
\]