Diversity Aspects of Linear and Decision-Feedback Equalizers for Frequency-Selective Multi-Antenna Channels

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Outline

- SIMO MFB and SINR of LE and DFE
- outage-rate tradeoff
- outage analysis of suboptimal receiver SINR
- Linear Equalization (LE)
  - LE in Single Carrier Cyclic Prefix (SC-CP) systems
  - non-causal infinite length LE
  - FIR LE
- Decision-Feedback Equalization (DFE)
  - DFE with ideal Feedforward and reduced Feedback filters
  - DFE for SC-CP systems
  - FIR DFE
- frequency-selective MIMO tradeoff
**SIMO system**

\[
\begin{align*}
\mathbf{y}_k &= \mathbf{h}[q] a_k + \mathbf{v}_k \\
&= \begin{pmatrix} y_k \\ y_{k-1} \end{pmatrix} = \begin{pmatrix} h \end{pmatrix} \begin{pmatrix} q \end{pmatrix} \begin{pmatrix} a_k \end{pmatrix} + \begin{pmatrix} v_k \end{pmatrix}
\end{align*}
\]

- \( p \) is the number of subchannels
- The noise power spectral density matrix is \( S_{\mathbf{vv}}(z) = \sigma_v^2 I \)
- \( q^{-1} \) is the unit sample delay operator: \( q^{-1} a_k = a_{k-1} \)
- \( h[z] = \sum_{i=0}^{L} h_i z^{-i} \) is the channel transfer function
- \( L = \text{channel delay spread in symbol periods} \)
- In the Fourier domain: \( h(f) = h[e^{j2\pi f}] \)
Let $\mathbf{h} = \begin{bmatrix} h_0 \\ \vdots \\ h_L \end{bmatrix}$ contain all channel elements.

By default $\mathbf{h} \sim \mathcal{CN}(0, \frac{1}{L+1} I_{p(L+1)})$ i.i.d. channel model

so that spatio-temporal diversity of order $p(L + 1)$ is available.

$\rho = \frac{\sigma_v^2}{\sigma^2}$ average per subchannel SNR

full CSIR, no CSIT
introduce $\delta = \begin{cases} 
0 & \text{, MMSE-ZF design,} \\
1 & \text{, MMSE design.} 
\end{cases}$

- $\text{MFB} = \rho \int_{-\frac{1}{2}}^{\frac{1}{2}} \| \mathbf{h}(f) \|^2 \, df = \rho \int_{-\frac{1}{2}}^{\frac{1}{2}} (\| \mathbf{h}(f) \|^2 + \frac{\delta}{\rho}) \, df - \delta$

  arithmetic average

- $\text{SINR}^{\delta}_{DFE} = \rho \exp \left[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \log(\| \mathbf{h}(f) \|^2 + \frac{\delta}{\rho}) \, df \right] - \delta$

  geometric average

- $\text{SINR}^{\delta}_{LE} = \rho \left[ \int_{-\frac{1}{2}}^{\frac{1}{2}} (\| \mathbf{h}(f) \|^2 + \frac{\delta}{\rho})^{-1} \, df \right]^{-1} - \delta$

  harmonic average

$\text{SINR}^{\delta}_{LE} \leq \text{SINR}^{\delta}_{DFE} \leq \text{MFB} \, , \, \text{SINR}^{0} \leq \text{SINR}^{1}$
Outage-Rate Tradeoff

- normalized SINR $\gamma$, $\text{SINR} = \rho \gamma$
- dominating term in cdf of $\gamma$: $\text{Prob}\{\gamma \leq \epsilon\} = c \epsilon^k$ for small $\epsilon > 0$
- outage probability $\text{Prob}\{\text{SINR} \leq \alpha\} = c \left(\frac{\alpha}{\rho}\right)^k = \left(\frac{\alpha}{g \rho}\right)^k$

$k =$ diversity order, $g = c^{-1/k} =$ coding gain (reduction in SNR required for identical outage probability)

- suboptimal Rx (LE, DFE): channel-equalizer cascade $= \text{AWGN}$ channel with Mutual Information $C = \log(1 + \text{SINR})$
  also true for MFB, by using non-causal UMMSE DFE
- MMSE ZF design: Gaussianity OK
- MMSE design: Gaussianity OK if input Gaussian, or ISI negligible effect at high SNR for QAM inputs
Outage-Rate Tradeoff (2)

- at high SNR $\rho$, $C = \log(1 + \text{SINR}) \approx \log \rho$

So consider rate $R = r \log \rho$ (nats),

$r \in [0, 1]$ is the normalized rate.

Then the outage probability at high SNR is

$$P_o = \text{Prob}\{C < R\} = \text{Prob}\{\log(1 + \text{SINR}) < \log(\rho^r)\}$$

$$= \text{Prob}\{\rho \gamma < \rho^r - 1\} = \text{Prob}\{\gamma < \frac{1}{\rho(1-r)} - \frac{1}{\rho}\}$$

$$= \text{Prob}\{\gamma < \frac{1}{\rho(1-r)}\}, \text{ for } r > 0$$

$$= c \frac{1}{\rho^{(1-r)k}} = \frac{1}{(g \rho)^{(1-r)k}}$$
Outage-Rate Tradeoff (3)

- Hence for $r \in (0, 1]$:

\[
d(r) = (1 - r)^k, \quad g(r) (\text{dB}) = -\frac{10}{(1 - r)^k} \log_{10} c
\]

$d(r) = \text{diversity(order)-rate tradeoff}$,
$g(r) = \text{tradeoff dependent coding gain}$.

As $c > 1$ usually, the coding gain is actually a coding loss that decreases with increasing diversity order $k$ and decreasing rate $r$.
The case $r = 0$ (fixed rate) requires separate investigation.
Optimal Outage-Rate Tradeoff

- MI with white Gaussian input

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \log(1 + \rho \| h(f) \|^2) df = \log(1 + \text{SINR}_{\text{DFE}}^{\text{MMSE}})
\]

[MedlesSlock:isit04]: \( \text{SINR}_{\text{DFE}}^{\text{MMSE}} \geq \beta_L \text{ MFB} \Rightarrow k = p(L+1) \)

and

\[
d^*(r) = (1 - r) p(L + 1), \quad r \in [0, 1]
\]

which is valid for Rayleigh \( h \) with non-singular \( R_{hh} \).

- This tradeoff can be achieved by transmitting i.i.d. QAM symbols from a constellation of size \( e^R = \rho^r \) and using a MMSE DFE Rx. [MedlesSlock:ITsubm04] \Rightarrow \text{at high SNR the probability of (symbol or frame) error is dominated by the outage probability.}
Outage Analysis of Suboptimal Rx SINR

- A perfect outage occurs when $\text{SINR} = 0$
- For the MFB this can only occur if $\mathbf{h} = 0$.
- For a suboptimal Rx however, the SINR can vanish for any $\mathbf{h}$ on the Outage Manifold $\mathcal{M} = \{\mathbf{h} : \text{SINR}(\mathbf{h}) = 0\}$.
- At fixed rate $R$, the diversity order = codimension of (the tangent subspace of) the outage manifold, assuming this codimension is constant almost everywhere and assuming a channel distribution with finite positive density everywhere (e.g. Gaussian with non-singular covariance matrix).
  For example, for the MFB the outage manifold is the origin, the codimension of which is the total size of $\mathbf{h}$. 
Outage Analysis of Suboptimal Rx SINR (2)

- The codimension is the (minimum) number of complex constraints imposed on the complex elements of $\mathbf{h}$ by putting $\text{SINR}(\mathbf{h}) = 0$.

- An actual outage occurs whenever $\mathbf{h}$ lies in the Outage Shell, a (thin) shell containing the outage manifold. The thickness of this shell shrinks as the rate increases.

- More precisely, the diversity order may depend on the channel distribution details within the outage shell.
LE in SC-CP systems

- after cyclic prefix (CP) insertion, block of N symbols: beq

\[ Y = HA + V \]

\[
H = \begin{bmatrix}
    h_0 & & & h_L \\
    \vdots & & \ddots & \vdots \\
    h_L & & & h_0 \\
\end{bmatrix}
\]

- applying DFT at Rx

\[
\begin{bmatrix}
    F_{N,p}Y \\
    U
\end{bmatrix} = \begin{bmatrix}
    F_{N,p}H \overset{F_N^{-1}}{\mathcal{H}} \overset{F_N A}{X} + F_{N,p}V \\
    W
\end{bmatrix}
\]

where \( F_{N,p} = F_N \otimes I_p \),

\( \mathcal{H} = \text{blockdiag}\{h_0, \ldots, h_{N-1}\} \) with \( h_n = h(f_n), f_n = \frac{n}{N} \).

At tone \( n \):

\[ u_n = h_n x_n + w_n. \]
**LE in SC-CP systems (2)**

- ZF ($\delta = 0$) or MMSE ($\delta = 1$) LE produces per tone
  $\hat{x} = (h^H h + \frac{\delta}{\rho})^{-1} h^H u$ from which $\hat{a}$ is obtained after IDFT with

  \[
  \text{SINR}_{\text{CP-LE}}^{\delta} = \rho \left( \frac{1}{N} \sum_{n=0}^{N-1} (\|h_n\|^2 + \frac{\delta}{\rho})^{-1} \right)^{-1} - \delta
  \]

- For $N \geq L+1$, the (ZF) outage manifold is the collection of
  manifolds for which $h_n = 0$ for some $n$ (SINR = 0). As a result
  the codimension is $p$. Hence

  \[
  d_{\text{CP-LE}}^{\text{ZF}}(r) = (1 - r)p , \quad r \in [0, 1]
  \]

  any frequency diversity is lost! Also for MMSE, except

  $d_{\text{CP-LE}}^{\text{MMSE}}(0) = p (L + 1)$: the MMSE has full diversity at constant
  rate $R$ (at finite SNR, some $r > 0$ also).
LE in SC-CP systems (3)

\[ d(r) \]

\[ p(L + 1) \]

OPTIMAL

\[ p \]

MMSE LE

MMSE ZF LE

0

0

1

r
Non-Causal Infinite Length Linear Equalizer

- \( \text{SINR}_{LE}^{ZF} = \frac{\rho}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\|h(f)\|^2} df} \)

- The outage manifold is clearly \( \{ h : h(f) = 0 \text{ for any } f \} \). For any given \( f \) the codimension is again \( p \).

- In spite of the ambiguity on \( f \), the diversity order is \( p \).

Consider e.g. the case \( L = 1 \): \( \{ h : h_0 = h_1 e^{-j(2\pi f+\pi)} \} \) which for the SISO case becomes \( |h_0| = |h_1| \), for which it is easy to verify that the diversity order is 1.
FIR Linear Equalization

- FIR LE of length $N$.
- SIMO channels: $\exists$ ZF FIR equalizers of length $N$ for FIR channels (Bezout identity) if $N \geq \frac{L}{p-1}$. LE design is based on a banded block Toeplitz input-output matrix

$$
\mathbf{H} = \begin{bmatrix}
\mathbf{h}_L & \cdots & \mathbf{h}_0 \\
\mathbf{h}_L & \cdots & \cdots \\
& \cdots & \cdots & \cdots \\
\mathbf{h}_L & \cdots & \mathbf{h}_0
\end{bmatrix}
$$

- for a certain equalizer delay $d$

$$
\text{SINR}_{FIR-LE}^\delta + \delta = \frac{\rho}{e_d^H \left( \mathbf{H}^H \mathbf{H} + \frac{\delta}{\rho} \right)^{-1} e_d} = \frac{\rho}{\sum_i \frac{1}{\lambda_i + \frac{\delta}{\rho}} |V_{i,d}|^2}
$$

where $e_d = [\underbrace{0 \cdots 0}_{d} 1 0 \cdots 0]^T$,

SVD $\mathbf{H}^H \mathbf{H} = \mathbf{V} \Lambda \mathbf{V}^H = \sum_i \lambda_i \mathbf{V}_i \mathbf{V}_i^H$. 

FIR Linear Equalization (2)

• The (ZF) outage manifold is determined (again) by $\lambda_{min} = 0$. For $N \geq \frac{L}{p-1}$, $\mathbf{H}^H \mathbf{H}$ singular $\iff \mathbf{H}$ loses full column rank $\iff \mathbf{h}[z_o] = 0$ for some $z_o$: the subchannel transfer functions have a zero in common. This imposes on the $p-1$ other subchannels to have a zero equal to a zero of the first subchannel. Hence the codimension of the outage manifold is $p-1$. So

$$d_{FIR-LE}(r) = (1 - r)(p - 1), \quad r \in (0, 1), \quad d \in [0, L+N]$$

$$d_{MMSE}(0) = p \min\{N, L+1\} \text{ for appropriate } d$$

• $V_{min} = \sqrt{\frac{1-|z_o|^2}{1-|z_o|^2(N+L)}} \begin{bmatrix} 1 & z_o & z_o^2 & \cdots & z_o^{N+L-1} \end{bmatrix}^T$ associated to $\lambda_{min} = 0$. For $r = 0$, ZF, diversity jumps from $p-1$ to $p$ as $N \to \infty$ since weighting $|V_{min,d}|^2 \to 0$ unless $|z_o| = 1$. 
DFE with Ideal FFF and Reduced FBF

- DFE reaches full diversity as long as the feedback order \( M = L \).
- Consider the MSE in a DFE design, after optimization of the unconstrained feedforward filter, with the feedback filter \( b(f) \) still to be designed

\[
\text{MSE} = \sigma_v^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|b(f)|^2}{\| h(f) \|^2 + \frac{\delta}{\rho}} \, df
\]

With \( M = 0 \) (\( b(f) = 1 \)), the MSE explodes whenever \( h(f) = 0 \) (outage manifold LE). When \( M = L \), as \( h(f) = 0 \) can be zero only in at most \( L \) frequencies, the optimized feedback filter \( b(f) \) will put zeros in those frequencies and hence can always prevent the MSE from exploding. When \( M < L \), we have an outage whenever \( h(f) \) has \( M+1 \) zeros.

- So the diversity for any \( M \) is \( d_{DFE}^{ZF}(r) = p(M+1)(1-r) \).
DFE in Single Carrier Cyclic Prefix Systems

- The problem of infinite length non-causal feedforward filters (FFFs) in the DFE can be overcome by introducing a CP and performing the FFF’ing in the frequency domain (SC-CP) and the feedback filtering in the time domain, see [Falconer:WhitePaper02] (where oversampling leads to increased DFT size which is not necessary).

- The same expressions as for the infinite length FFF case are obtained by replacing integration in the frequency domain by averaging over tones. The same diversity results hold.
FIR Decision-Feedback Equalization

- Consider now a FFF of length $N$, feedback filter order $M = L$ and equalization delay equal to $N - 1$. 

FIR FFF DFE design is based on a banded block Toeplitz matrix

$$
\overline{H} = \begin{bmatrix}
    h_L & \cdots & h_0 \\
    h_L & \cdots & h_0 \\
    \vdots & \ddots & \vdots \\
    h_L & \cdots & h_0 \\
\end{bmatrix}, \quad N > L+1
$$

outage: last column of $\overline{H}$ is $\perp$ w.r.t. other columns fades

- For $N = 1$, $d^{ZF}(0) = p$ whereas $d^{ZF}(0) = p(L+1)$ for $N \to \infty$. For intermediate $N$, intermediate diversity orders are obtained.

- For instance with $L = 1$, for $N = 2$, $d^{ZF}(0) = 2p-1$ and fractional diversities $\in (2p-1, 2p)$ are obtained for $N > 2$. 
MIMO Channel Model

\[ H(q) = \sum_{l=0}^{L-1} H_l q^{-l}, \quad q^{-1} x_k = x_{k-1}. \]

\[ H_l: N_r \times N_t. \]

\( L \): channel delay spread. SNR \( \rho = \frac{P}{N_t \sigma_v^2} \).

\[ y_k = H(q) a_k + v_k = \sum_{l=0}^{L-1} H_l a_{k-l} + v_k, \]

entries of \( H_l, l = 0, \ldots, L - 1 \): i.i.d. Gaussian \( H_l^{rt} \sim \mathcal{CN}(0, 1) \).

Notation: SIMO: \( L+1 \) \quad \rightarrow \quad MIMO: \( L \)
Diversity vs Multiplexing Background

[Zheng&Tse’03]

- A coding scheme $\mathcal{C}(\rho)$ is a family of codes of block length $T$, that supports a bit rate $R(\rho)$.

- **Spatial multiplexing** $r$: $\lim_{\rho \to \infty} \frac{R(\rho)}{\ln(\rho)} = r$.

- **Diversity gain** $d$: $\lim_{\rho \to \infty} \frac{\ln P_e(\rho)}{\ln(\rho)} = -d$.

$d^*(r)$ denotes the supremum of the diversity advantage achieved over all possible schemes.

In practice: $P_e(\rho) = P_{out}(\rho)$, $d^*(r) = d_{out}^*(r)$.

Proof of $d_{out}(r)$ corrected here w.r.t. [Medles&Slock’ISIT05].
Diversity vs Multiplexing Background (2)

- **Flat MIMO channel** ($L = 1$) For $T \geq N_t$, the optimal trade-off curve $d^*(r)$ is given by the piecewise-linear function connecting the points $(k, d^*(k))$, $k = 0, 1, \ldots, p$, where

\[
d^*(k) = (p - k)(q - k),
\]

\[
p = \min\{N_r, N_t\},
\]

\[
q = \max\{N_r, N_t\}.
\]

Achieved by the family of codes with non-vanishing determinant [Elia, Pawar et al’ ALLERTON04].

- **SIMO/MISO frequency selective channel** The optimal trade-off curve is given by the linear function $d^*(r) = Lq(1 - r)$ [Grokop & Tse’ISIT04]. For SIMO achieved by using QAM at Tx and MMSE DFE at Rx [Medles & Slock’ISIT04].
Diversity vs Multiplexing Background (3)

Diversity Gain: $d^*(r)$

Spatial Multiplexing Gain: $r$

(0, $N_r, N_t$)

(1, ($N_r - 1$)($N_t - 1$))

(2, ($N_r - 2$)($N_t - 2$))

($r$, ($N_r - r$)($N_t - r$))

(min($N_r, N_t$), 0)

Diversity vs. Multiplexing optimal tradeoff for flat MIMO channel
Div. vs Mul. for MIMO FS Channel

- Assume $T >> L$, mutual information for white input

$$I_T(H) \approx I(H) = \frac{1}{2\pi j} \int \frac{dz}{z} \ln \det(I + \rho H(z)H^\dagger(z)).$$

- The behavior of $I(H)$ is characterized by

$$I(H) \doteq \ln \det(I + \rho \bar{H}\bar{H}^H), \text{ where } \bar{H} = \begin{bmatrix} H_0 \\ \vdots \\ H_{L-1} \end{bmatrix} \text{ for } N_t \leq N_r,$$

$$\bar{H} = [H_0, H_1, \ldots, H_{L-1}] \text{ for } N_t \geq N_r.$$
The optimal trade-off curve $d^*(r)$ is given by the piecewise-linear function connecting the points $(k, d^*(k)), k = 0, 1, \ldots, p$, where

$$d^*(k) = (Lq - k)(p - k),$$

$$p = \min\{N_r, N_t\},$$

$$q = \max\{N_r, N_t\}.$$

For $N_t \leq N_r$, diversity is the same as for a flat MIMO channel with $N'_t = N_t$ and $N'_r = LN_r$.

Coding over $L$ independent OFDM subcarriers (spacing of $\frac{1}{L}$):

$$d(r) = L(q - r)(p - r) \leq d^*(r) \rightarrow \text{suboptimal}.$$

Difference $d^*(r) - d(r) = (L - 1) r (p - r)$, peaks at $r = \frac{p}{2}$.

For large $L$ and $N_r = N_t$, $d^*(p/2) \approx 2 d(p/2)$. 
Div. vs Mul. for MIMO FS Channel (3)

Diversity Gain: \( d^* (r) \)

Spatial Multiplexing Gain: \( r \)

(0, \( L.N_r.N_t \))

(1, \((L.N_r - 1)(N_t - 1)\))

(2, \((L.N_r - 2)(N_t - 2)\))

(\( r, (L.N_r - r)(N_t - r) \))

(\( N_t, 0 \))

Diversity vs. Multiplexing optimal tradeoff for MIMO FS channel with \( N_t \leq N_r \)
Outage Manifolds Analysis

- parameterization FIR channel of rank $k \leq p = N_t \leq N_r = q$

$$\begin{align*}
\text{H}(z) &= \text{H}(z) \begin{bmatrix} I_k & \text{H} \end{bmatrix} \mathcal{P} \\
&= \text{FIR}_L \times p \quad \text{FIR}_L \times k \quad k \times (p-k) \quad \text{permutation} \\
&= \text{constant} \\
\end{align*}$$

degrees of freedom in $q \times p$ rank-$k$ FIR-$L$ manifold:

$$\begin{align*}
qkL + (p-k)k &= qL - (qL - k)(p - k) \\
\end{align*}$$

- To send at rate $k$, need to be guaranteed rank $k$. The diversity degree is the remaining number of degrees of freedom in $\text{H}(z)$:

$$d^*(k) = qL - (qL - (qL - k)(p - k)) = (qL - k)(p - k)$$
Concluding Remarks

- Existing diversity-rate tradeoff
  - at high SNR
  - focuses only on diversity order and not on coding gain/SNR offset

To observe the MIMO FS tradeoff, need to go to very high SNR (e.g. 50dB).
Also: MMSE LE.

- Work at finite SNR required.