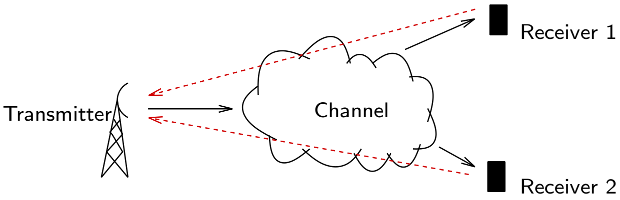
Memoryless Broadcast Channels with Feedback



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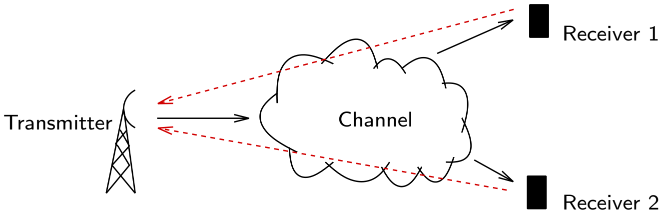
joint work with Michael Gastpar, Amos Lapidoth, Ofer Shayevitz

Eurecom, 1 July 2010, Nice, France

Broadcast Channels (BCs) with Feedback



- Two models:
  - Fading BC with memory and "state-feedback"
  - Memoryless BC with output feedback

In this Talk: Memoryless BCs with Output Feedback

Discrete Memoryless BCs

- Propose new coding scheme
- For some channels our scheme
  - Is optimal (in terms of rate)
  - Improves on optimal no-feedback scheme, even with noisy feedback

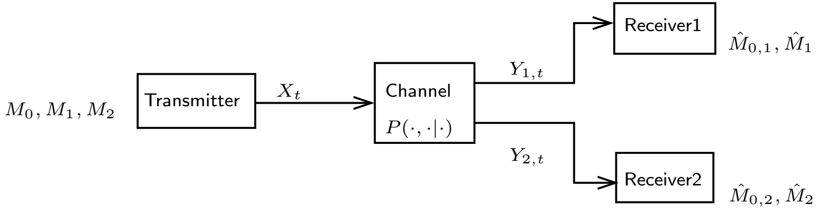
Scalar Gaussian BCs

- Feedback can double achievable throughput (sum-capacity)
- Feedback-gain susceptible to noise on feedback links

Part I:

Discrete Memoryless BCs

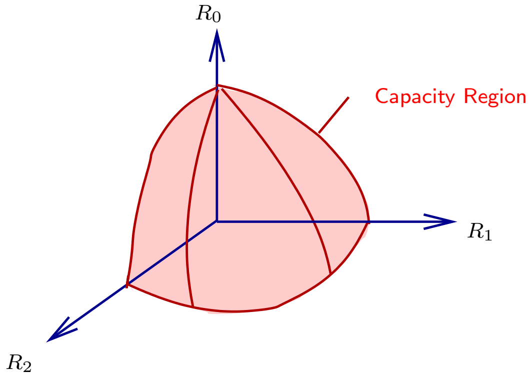
Discrete Memoryless BC without Feedback



- Rx 1 wants to learn Messages M_0, M_1; and Rx 2 Messages M_0, M_2
- Inputs $X_t = f_t(M_0, M_1, M_2)$
- Finite input and output alphabets $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$
- Channel memoryless $(Y_{1,t}, Y_{2,t}) - \circ - X_t - \circ - (X^{t-1}, Y_1^{t-1}, Y_2^{t-2})$
- Channel law: $P(y_1, y_2|x)$ of observing y_1 and y_2 given input x

Capacity Region/Sum-Capacity

- Rates of communication $R_0, R_1, R_2 \geq 0$
- Capacity region C: Set of (R_0, R_1, R_2) s.t. p(error) arbitrarily small
- Sum-capacity C_{Sum}: maximum throughput $(R_0 + R_1 + R_2)$ s.t. p(error) arbitrarily small



Capacity of discrete memoryless BC without Feedback

- Capacity region in general unknown
- Outer bound on capacity: augmented cutset bound
- Achievable region [Marton'79]:

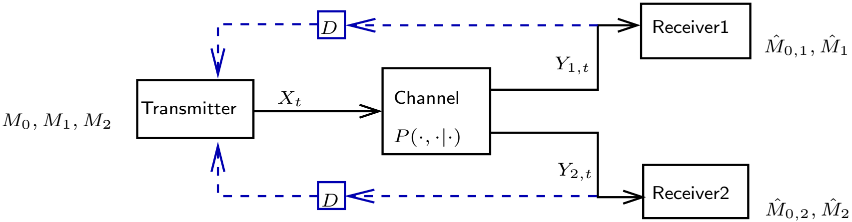
(R_0, R_1, R_2) achievable if:

$$\begin{aligned} R_0 &< \min_i I(U_0; Y_i) \\ R_0 + R_1 &< I(U_0, U_1; Y_1) \\ R_0 + R_2 &< I(U_0, U_2; Y_2) \\ R_0 + R_1 + R_2 &< I(U_1; Y_1 | U_0) + I(U_2; Y_2 | U_0) \\ &\quad + \min_i I(U_0; Y_i) - I(U_1; U_2 | U_0) \end{aligned}$$

for some (U_0, U_1, U_2) forming Markov chain $(U_0, U_1, U_2) - X - (Y_1, Y_2)$

- For some channels: Marton's region and augmented cutset bound tight

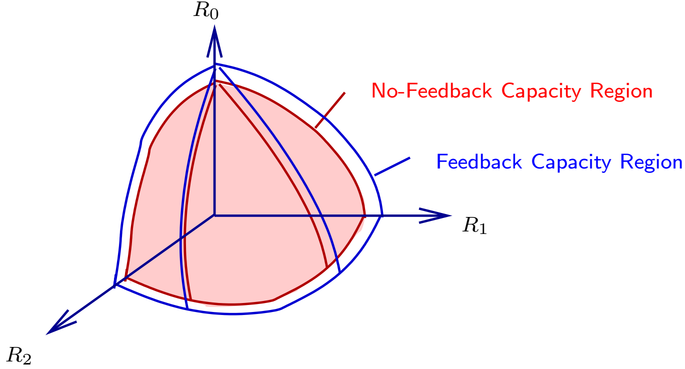
Noise-Free Feedback



- Transmitter causally observes channel outputs
- Can compute inputs as:

$$X_t = f_t(M_0, M_1, M_2, Y_1^{t-1}, Y_2^{t-1})$$

Does Noise-Free Feedback Increase Capacity?



$$C_{\mathsf{BC,NoFB}} = C_{\mathsf{BC,FB}} \qquad \text{or} \qquad C_{\mathsf{BC,NoFB}} \subsetneq C_{\mathsf{BC,FB}}?$$

- Recall: For memoryless point-to-point channels feedback does NOT increase capacity

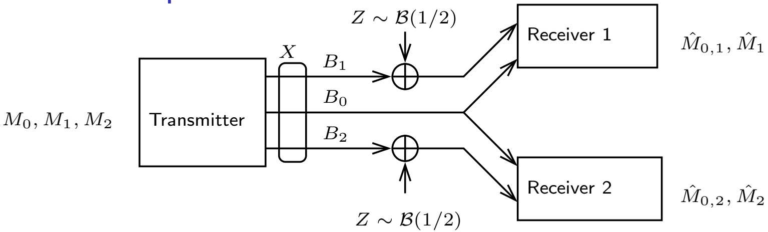
Previous Results on Capacity of DMBCs with Feedback

- El Gamal'78: No feedback-gain for physically degraded BCs
- Dueck'80, Kramer'00: Feedback-gain for specific discrete memoryless BCs
- Kramer'00: Multi-letter achievable region for general channels
- Capacity region not known in general

In this talk

Single-letter achievable region for general channels

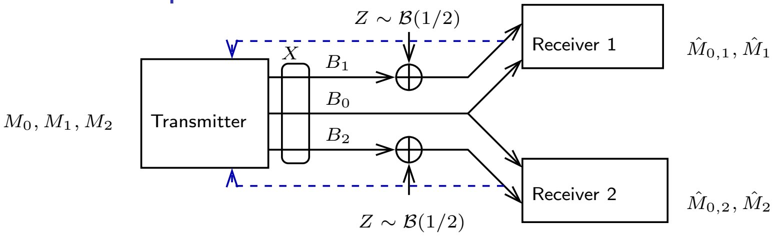
Dueck's Example



Without feedback:

- Top and bottom links useless
- No-feedback capacity: $0 \le R_0 + R_1 + R_2 \le 1$

Dueck's Example



Noise-free feedback:

- Transmitter learns noise and sends $B_{0,t} = Z_{t-1}$
- Feedback capacity: $0 \leq R_0 + R_1, R_0 + R_2 \leq 1$

Intuition why feedback helps

- "Actions" of channels $X \to Y_1$ and $X \to Y_2$ correlated
- Can send information useful to both receivers

Our Coding Scheme

- Block-Markov strategy
- Update info. about previous channel "actions" learned via feedback
- Fresh data/update info. sent with Marton's no-fb scheme
- Backward decoding:
  1. Block-b outputs improved with block-$(b+1)$ update info.
  2. Marton-decoding based on improved outputs

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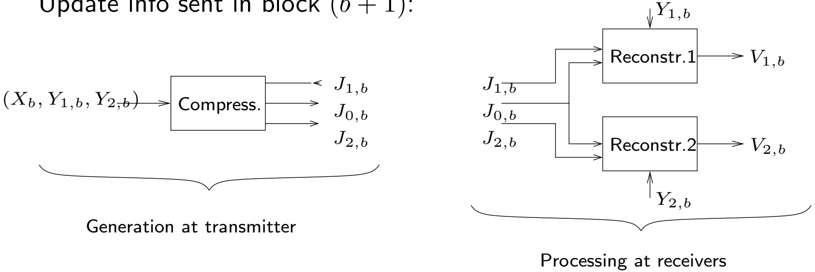
Update info.: Lossy GW-Compression of Channel Actions

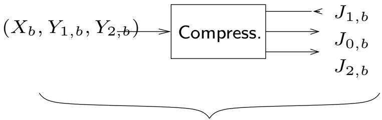
Update info sent in block $(b+1)$:



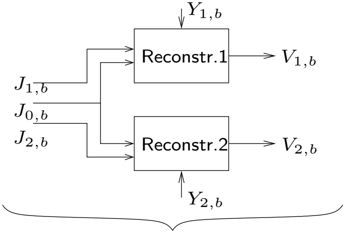
- $V_{1,b}, V_{2,b}$: lossy reconstructions of channel "actions"
- Indices $(J_{0,b}, J_{1,b}, J_{2,b})$ describe lossy compression of $(X_b, Y_{1,b}, Y_{2,b})$
- Goal: $(V_{i,b}, X_b, Y_{1,b}.Y_{2,b})$ jointly typical $\sim P_{V_i,X,Y_1,Y_2}$
- Improved block-b outputs at Rx i: $(V_{i,b}, Y_{i,b})$

Update info.: Lossy GW-Compression of Channel Actions

Update info sent in block $(b+1)$:



Generation at transmitter



Processing at receivers

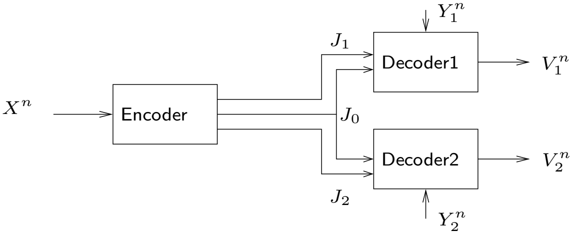
- $V_{1,b}, V_{2,b}$: lossy reconstructions of channel "actions"
- Indices $(J_{0,b}, J_{1,b}, J_{2,b})$ describe lossy compression of $(X_b, Y_{1,b}, Y_{2,b})$
- Goal: $(V_{i,b}, X_b, Y_{1,b}.Y_{2,b})$ jointly typical $\sim P_{V_i,X,Y_1,Y_2}$
- Improved block-b outputs at Rx i: $(V_{i,b}, Y_{i,b})$

Due to side-info, even independent $V_{1,b}$ and $V_{2,b}$ have interesting J_0

Lossy Gray-Wyner Source Coding with Side-Info.

Given: $P_{XY_1Y_2}P_{V_1V_2|X}$; $(X^n, Y_1^n, Y_2^n) \sim$ IID $P_{XY_1Y_2}$

Goal: V_i^n jointly typical with X^n according to P_{X,V_i}



A triplet (R_0, R_1, R_2) is achievable, if

$$R_0 > \max_i I(X; V_0|Y_i),$$
$$R_1 > I(X; V_1|V_0, Y_1)$$
$$R_2 > I(X; V_2|V_0, Y_2)$$

for some V_0 s.t. $(V_0, V_1, V_2) \multimap X \multimap (Y_1, Y_2)$ forms a Markov chain.

Our Achievable Region for the DMBC with Feedback

Theorem

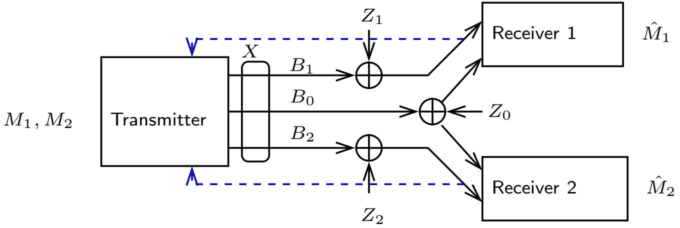
Nonnegative triplet (R_0, R_1, R_2) *achievable if,*

$$
\begin{aligned}
R_0 &\le \min_i I(U_0; Y_i, V_i) - \max_i I(V_0; X, Y_1, Y_2 | Y_i) \\
R_0 + R_1 &\le I(U_0, U_1; Y_1, V_1) - I(X, Y_2; V_1 | V_0, Y_1) - \max_i I(V_0; X, Y_1, Y_2 | Y_i) \\
R_0 + R_2 &\le I(U_0, U_2; Y_2, V_2) - I(X, Y_1; V_2 | V_0, Y_2) - \max_i I(V_0; X, Y_1, Y_2 | Y_i) \\
R_0 + R_1 + R_2 &\le I(U_1; Y_1, V_1 | U_0) + I(U_2; Y_2, V_2 | U_0) + \min_i I(U_0; Y_i, V_i) \\
&\quad - I(U_1; U_2 | U_0) - I(X, Y_1, Y_2; V_1 | V_0, Y_1) - I(X, Y_1, Y_2; V_2 | V_0, Y_2) \\
&\quad - \max_i I(V_0; X, Y_1, Y_2 | Y_i)
\end{aligned}
$$

for some $(U_0, U_1, U_2, V_0, V_1, V_2)$ *such that*

$$
\begin{aligned}
(U_0, U_1, U_2) &\multimap X \multimap (Y_1, Y_2) \\
(V_0, V_1, V_2) &\multimap (X, Y_1, Y_2) \multimap (U_0, U_1, U_2)
\end{aligned}
$$

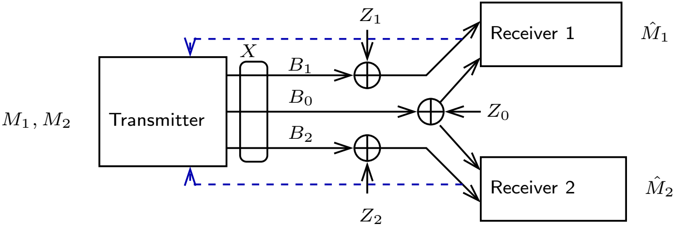
Capacity of Generalized Dueck-Example with Noise-Free Fb



- $B_0, B_1, B_2, Z_0, Z_1, Z_2$ binary
- Assumption: $H(Z_0, Z_1) \leq 1$ and $H(Z_0, Z_2) \leq 1$
- No feedback capacity: all pairs (R_1, R_2) s.t.

$$\begin{aligned} R_1 &\leq 2 - H(Z_0, Z_1) \\ R_2 &\leq 2 - H(Z_0, Z_2) \\ R_1 + R_2 &\leq 3 - H(Z_0, Z_1, Z_2) - I(Z_1; Z_2 | Z_0). \end{aligned}$$

Capacity of Generalized Dueck-Example with Noise-Free Fb

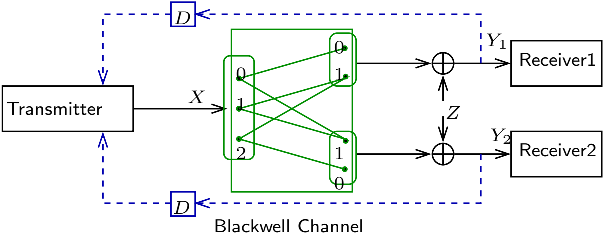


- $B_0, B_1, B_2, Z_0, Z_1, Z_2$ binary
- Assumption: $H(Z_0, Z_1) \leq 1$ and $H(Z_0, Z_2) \leq 1$
- Noise-free feedback capacity: all pairs (R_1, R_2) s.t.

$$R_1 \leq 2 - H(Z_0, Z_1)$$
$$R_2 \leq 2 - H(Z_0, Z_2)$$
$$R_1 + R_2 \leq 3 - H(Z_0, Z_1, Z_2)$$

$\rightarrow$ feedback helps iff $Z_1 -\!\!\circ\!\!- Z_0 -\!\!\circ\!\!- Z_2$ does NOT hold!

Another Example: Noisy Blackwell Channel



- Noise $Z \sim \mathcal{B}(p)$
- Both channel outputs corrupted by same noise (for simplicity)

Our Achievable Region for Noisy Blackwell Channel

(R_0, R_1, R_2) achievable if for some $\alpha, \beta \in [0,1]$:

$$R_0 \leq h_b\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{2}(h_b(\alpha) + h_b(\beta)) - \lambda(p,\alpha,\beta)$$

$$R_0 + R_1 \leq h_b\left(\frac{\alpha+\beta}{2}\right) - \lambda(p,\alpha,\beta)$$

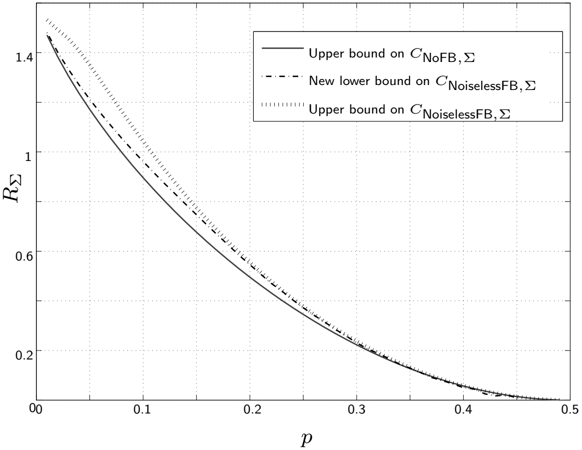
$$R_0 + R_2 \leq h_b\left(\frac{\alpha+\beta}{2}\right) - \lambda(p,\alpha,\beta)$$

$$R_0 + R_1 + R_2 \leq h_b\left(\frac{\alpha+\beta}{2}\right) + \frac{1-\beta}{2} h_b\left(\frac{\alpha}{1-\beta}\right) + \frac{1-\alpha}{2} h_b\left(\frac{\beta}{1-\alpha}\right) - \lambda(p,\alpha,\beta)$$

where

$$\lambda(p,q,\alpha,\beta) \triangleq h_b(p) + h_b\left(\frac{\alpha+\beta}{2}\right) - h_b\left(\left(\frac{\alpha+\beta}{2}\right) \star p\right)$$

Sum-Capacity of Noisy Blackwell Chan. with Noise-free Fb



Usefulness of Feedback

- For most values p noise-free feedback beneficial

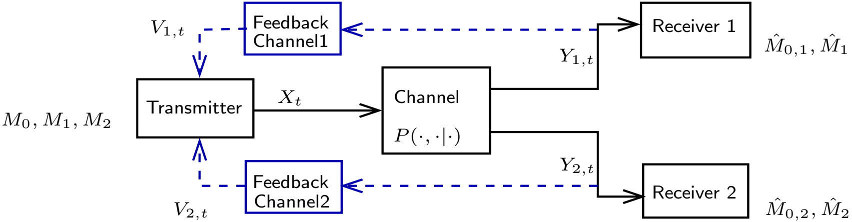
Improved Coding Scheme

- *Channels of interest* in Marton's scheme: $(U_0, U_1) \to Y_1$ and $(U_0, U_2) \to Y_2$

- Update info: lossy compression of these channel "actions", i.e., of $(U_0, U_1, U_2, Y_1, Y_2)$

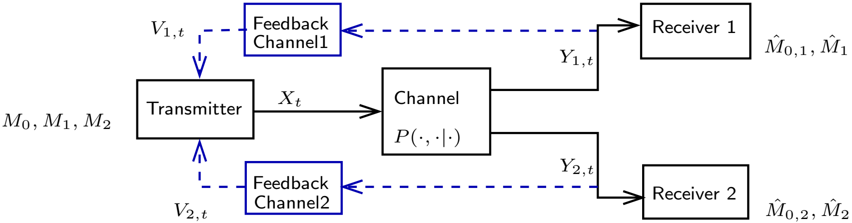
- At least as good as before. Better?

Noisy Feedback



- Noisy Feedback $(V_{1,t}, V_{2,t}) \multimap (Y_{1,t}, Y_{2,t}) \multimap (X^t, Y_1^{t-1}, Y_2^{t-1})$
- E.g., $V_{1,t} = Y_{1,t} + W_{1,t}$ and $V_{2,t} = Y_{2,t} + W_{2,t}$
- Inputs: $X_t = f_t(M_0, M_1, M_2, V_1{}^{t-1}, V_2^{t-1})$

Noisy Feedback



Our scheme still works:

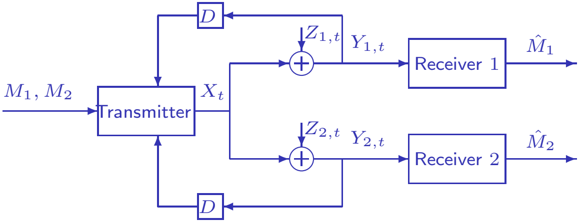
Exchange (Y_1, Y_2) by (V_1, V_2) when compressing channel-"actions"

- Noisy Blackwell channel: even with noisy feedback our scheme improves on no-feedback

Part II:

Scalar Gaussian Memoryless BC

Scalar Gaussian Broadcast Channel



- $X_t, Y_{1,t}, Y_{2,t} \in \mathbb{R}$
- $Y_{\nu,t} = X_t + Z_{\nu,t}$, $\qquad \begin{pmatrix} Z_{1,t} \\ Z_{2,t} \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} N_1 & \rho_z\sqrt{N_1 N_2} \\ \rho_z\sqrt{N_1 N_2} & N_2 \end{pmatrix}\right)$
- Noise correlation caused by interference
- Only private messages M_1 and M_2

No Feedback

Capacity Region when $N_1 \leq N_2$

All pairs (R_1, R_2) such that for some $\alpha \in [0, 1]$:

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{\alpha P}{N_1}\right)$$
$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{(1-\alpha)P}{\alpha P + N_2}\right).$$

- Achieved by superposition coding — a special case of Marton's scheme
- Capacity region independent of noise-correlation ρ_z!

Noise-Free Feedback

- Outer bound [Ozarow/Leung'84]
- Achievable regions [Ozarow/Leung'84] and [Bhaskaran'06]
- Capacity region not known, seems difficult

Other Questions:

- Can feedback *significantly* increase capacity?
- Feedback-gains in the high-SNR regime?

High-SNR Asymptotics: Degrees of Freedom

- Degrees of freedom:

$$\eta \triangleq \overline{\lim_{P\to\infty}} \frac{C_{\mathsf{Sum}}(P, N_1, N_2)}{1/2\log(P)}$$

- Classical result: MIMO Gaussian channel with independent noises

$$\begin{aligned}\eta_{\mathsf{MIMO}} &= \operatorname{rank}(\text{channel matrix}) \\ &\leq \min\{\#\text{tx-antennas}, \#\text{rx-antennas}\}\end{aligned}$$

Holds also with feedback!

- Intuition: $\eta_{\mathsf{MIMO}} = \#$ interference-free Gaussian channels in system

Degrees of Freedom of BC without Feedback

- Degrees of freedom: $\eta_{\mathsf{BC}} \triangleq \lim_{P \to \infty} \frac{C_{\mathsf{BC,Sum}}(P,\sigma_1^2,\sigma_2^2,\rho_z)}{\frac{1}{2}\log(P)} = 1$

- Degrees of freedom independent of ρ_z

- Only 1 "interference-free" link $\rightarrow$ receivers share 1 Gaussian channel

Capacity Results for Memoryless Networks at High SNR

- Memoryless single-user channel: no capacity gain with feedback (Shannon)
- Gaussian single-user channel with memory: at most 1/2 bit (Cover/Pombra)
- Two-user Gaussian MAC with memory: at most 1/2 bit (Thomas)
- For regular/non-regular fading channels: no change in high-SNR first order asymptotics (Lapidoth/Moser)

$\rightarrow$ For these settings feedback insignificant at high SNR

2 Degrees of Freedom Possible with 1 Transmit-Antenna

Theorem

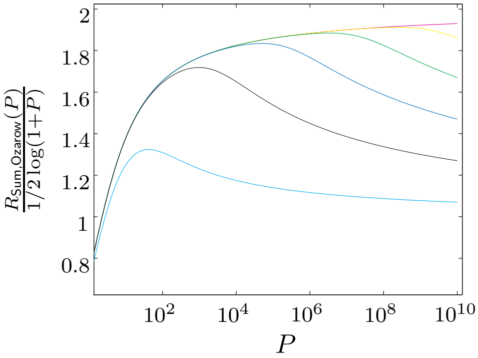
Degrees of freedom of scalar Gaussian BC with noise-free feedback

$$\eta_{\mathsf{BCFB}} = \begin{cases} 1, & |\rho_z| < 1, \\ 2, & \rho_z = -1. \end{cases}$$

- For $\rho_z = -1$:
  - Feedback doubles high-SNR capacity
  - At high SNR: as if transmitter had 2 separate links to receivers
  - η_{BCFB} can be larger than # tx-antennas

Correlation $\rho_z = -1$?

- Correlation $\rho_z = -1$ unlikely. Thermal noise at receivers!
- For finite powers: what if there is only small thermal noise?



Sum-rates for $\rho_z = -0.9, -0.999, -0.99999, -0.9999999, -0.999999999, -1.$

Feedback Gains for $\rho_z \neq -1$

- For $P \ggg 1$ and $\rho_z \approx 1 - \frac{1}{P^\alpha}$ sum-capacity multiplied by $(1+\alpha)$, $\alpha \in [0,1]$

Theorem

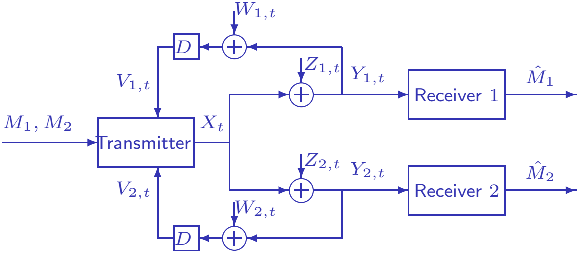
For $\rho_z(P)$ depending on power P:

$$\eta_{\mathsf{BCFB}} = \min\left\{2, 1 + \overline{\lim_{P\to\infty}} \frac{-\log(1+\rho_z(P))}{\log(P)}\right\}.$$

Thus, if $\rho_z(P) \sim (-1 + P^{-\alpha})$

$$\Rightarrow \eta_{\mathsf{BCFB}} = (1+\alpha), \quad \alpha \in [0,1].$$

Noisy Feedback



- $V_{\nu,t} = Y_{\nu,t} + W_{\nu,t}$, $\left\{ \begin{pmatrix} W_{1,t} \\ W_{2,t} \end{pmatrix} \right\}$ IID $\sim \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right)$
- $\{(W_{1,t}, W_{2,t})\}$ independent of $\{(Z_{1,t}, Z_{2,t})\}$
- Transmitter observes noisy feedback: $X_t = f_t(M_1, M_2, V_1^{t-1}, V_2^{t-1})$

Collapse in Degrees of Freedom with Noisy Feedback

Theorem

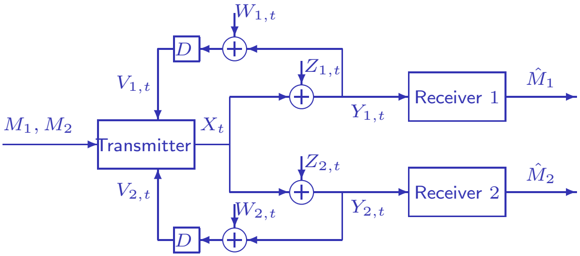
Degrees of freedom:

$$\eta_{\text{BC,NoisyFB}} = 1, \qquad \text{for all } \rho_z \in [-1, 1],$$

when σ_1^2, σ_2^2 fixed and positive.

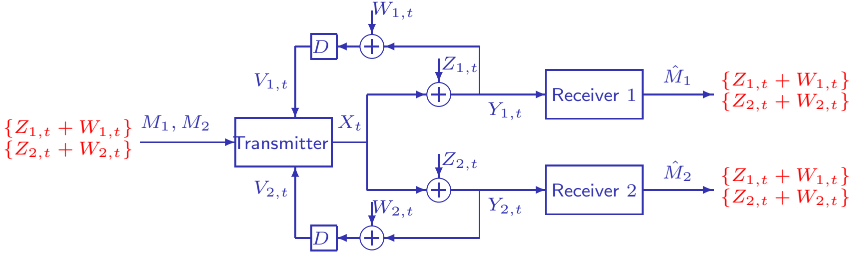
- Noise on feedback causes degrees of freedom to collapse!
- Promised gain very sensitive to feedback noise
- Engineering intuition "$\eta \leq \min\{\#\text{tx-antennas}, \text{ rx-antennas}\}$" OK!

Proof Idea why Degrees of Freedom Collapse



Proof idea: 3 transformations that don't decrease capacity

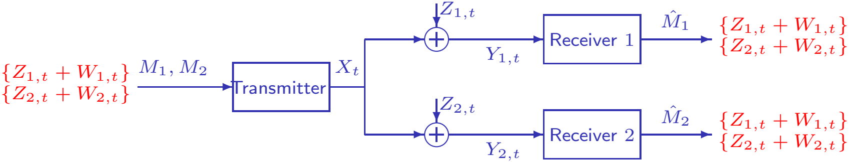
Proof Idea why Degrees of Freedom Collapse



Proof idea: 3 transformations that don't decrease capacity

1. Genie info $\{(Z_{1,t} + W_{1,t}, Z_{2,t} + W_{2,t})\}$ to tx and rxs $\rightarrow$ feedback useless

Proof Idea why Degrees of Freedom Collapse

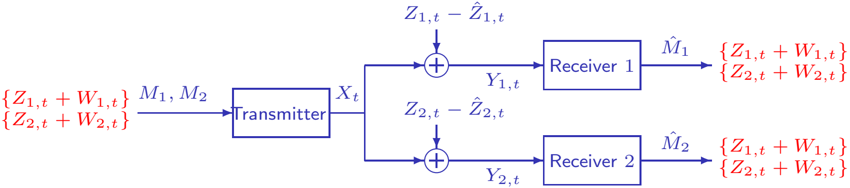


Proof idea: 3 transformations that don't decrease capacity

1. Genie info $\{(Z_{1,t} + W_{1,t}, Z_{2,t} + W_{2,t})\}$ to tx and rxs $\rightarrow$ feedback useless

2. Receiver i subtracts $\hat{Z}_{i,t} = \mathsf{E}[Z_{i,t} | Z_{1,t} + W_{1,t}, Z_{2,t} + W_{2,t}]$ from $Y_{i,t}$

Proof Idea why Degrees of Freedom Collapse



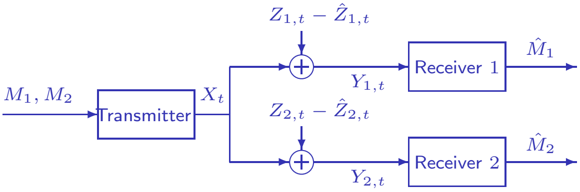
Proof idea: 3 transformations that don't decrease capacity

1. Genie info $\{(Z_{1,t} + W_{1,t}, Z_{2,t} + W_{2,t})\}$ to tx and rxs $\rightarrow$ feedback useless

2. Receiver i subtracts $\hat{Z}_{i,t} = \mathsf{E}[Z_{i,t} | Z_{1,t} + W_{1,t}, Z_{2,t} + W_{2,t}]$ from $Y_{i,t}$

3. Genie info $\{(Z_{1,t} + W_{1,t}, Z_{2,t} + W_{2,t})\}$ independ. of channel $\rightarrow$ useless

Proof Idea why Degrees of Freedom Collapse



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4. Resulting channel without fb/genie info: 1 degree of freedom

Summary

Discrete Memoryless BCs:

- New coding scheme for DMBC with feedback and new achievable region
- Simple example where our scheme yields noise-free feedback-capacity
- Noisy Blackwell channel: scheme improves on no-feedback capacity; even for noisy feedback

Scalar Gaussian BC:

- Noise-free feedback offers significant gains (doubles sum-capacity)
- Noise-free feedback: 2 degrees of freedom achievable with 1 tx-antenna ($\rho_z \approx -1$)
- Gains collapse when feedback noisy