# Normalized Nash Equilibrium for Power Allocation in Femto Base Stations in Heterogeneous Network

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***Abstract*—We consider heterogeneous networks with multiple femtocells and macrocells. Femto-base stations (femto-BS) are constrained to allocate transmitting powers such that the total interference at each macro-user terminal (macro-UT) is below a given threshold. We formulate a power allocation problem as a concave game with femto-BSs as players and multiple macro-UTs enforcing coupled constraints. Equilibrium selection is based on the concept of normalized Nash equilibrium (NNE) . When the interference at a femto-user terminal (femto-UT) from adjacent femto-BSs is negligible, for any strictly concave nondecreasing utility the NNE is unique and the NNE is the solution of a concave potential game. We also propose a distributed algorithm which converges to the unique NNE. When the interference is not negligible, an NNE may not be unique and the computation of NNE has exponential complexity . We introduce the concept of *weakly normalized Nash equilibrium* (WNNE) which keeps the most of NNEs' interesting properties but, in contrast to the latter, the WNNE can be determined with low complexity. We show the usefulness of the WNNE concept for the relevant case of Shannon capacity as femto-BS's utility.**

## I. Introduction

Intercell interference coordination (ICIC) in heterogeneous networks was already recognized as a crucial issue in 4G wireless network standardization (LTE-Advanced) and significant standardization efforts were devoted towards devising protocols to support ICIC schemes. Nowadays, network heterogeneity is recognized as a driving feature in the conception of 5G wireless networks and it will be an intrinsic characteristic of future generation networks. Therefore, efficient, scalable, and low complexity interference coordination techniques are needed to improve the existing LTE-Advanced systems and to inspire protocol definition for 5G networks. In [1], a study by Zhao et al. identified the interference caused by femtocell communications to downlink communications in macrocells as the most detrimental kind of interference in standard heterogeneous networks, i.e. networks consisting of coexisting femto-cells and macro-cells. For an overview of interference management techniques for femto-cells the interested reader can refer to [2]. Power allocation is an effective ICIC approach that allows universal frequency reuse with the well-known benefits in terms of system spectral efficiency. It can be performed in a centralized or distributed way. In this paper, we focused on distributed techniques based on non-cooperative games. Various kinds of non-cooperative games have been proposed to address the power allocation problem in this heterogeneous network setting. In [4], [13], [14], the power allocation problem is formulated as a Stackelberg game with macro-UTs as leaders and femto-BSs as followers. The Stalckelberg game framework combined with stochastic learning methods is adopted in [5]. Due to space limitation, an exhaustive overview of the game theoretical approaches to power allocation in heterogeneous networks exceeds the scope of this work and the interested reader can refer to [4].

In this paper, we formulate the power allocation problem among femto-BSs as a coupled constrained concave game [7] with femto-BSs as players. Thus, in contrast to Stackelberg games considered in above literature, macro-UTs do not need to know the utility functions of femto-BSs. As a result, our model is readily scalable and implementable. In order to design distributed algorithms converging to a well defined and unique NE, as in [3], we resort to the concept of NNE introduced by Rosen in [7]. In contrast to [3], we relax the restrictive assumptions of a single macro-UT and interference-free femto cells and we address the general problem with $M$ macro-UTs and presence of co-channel interference among femto-cells. We contribute in this space. The presence of multiple macro-cells in the system raises a problem of computational complexity in determining the unique NNE: the NNE computation does not boil down to an ordinary water-filling problem as in [3], by applying standard techniques it turns out that the problem has exponential complexity in the number of macro-UTs and femto-BSs and we are not aware of the techniques that solve it with low complexity. Along with increasing complexity, co-channel interference among femto-cells has the effect of destroying the property of uniqueness of NNE always satisfied in the setting considered in [3]. To address the latter issue, we identify a class of utility functions which admits a unique NNE also in presence of co-channel interference among femto-cells. Then, we introduce the concept of WNNE in Section V which enables us to extend the scope of NNE for a specific utility set to any other set of strictly increasing utility functions.

Low complexity methods to determine an NNE or WNNE in presence of multiple constraints and co-channel interference are presented in Section VI by leveraging on the properties of constrained concave potential games [6]. We show that, when the NNE is unique, solving a potential game is equivalent to determining the NNE. Additionally, we introduce a distributed algorithm converging to the NNE when the game is strictly concave potential. In the distributed algorithm macro-UTs only need to track the total interference from all femto-BSs and femto-BSs only need to know their own channel parameters.

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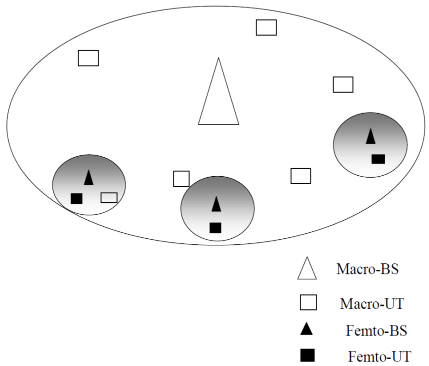


Fig. 1. Macro-BS, macro-UTs, femto-BSs and femto-UTs in a region. Circles represent the range of base stations. Macro-BS has higher coverage compared to femto-BS. Femto-UTs are placed close to the femto-UT. Femto-BSs can cause interference at a macro-UT if it is in the range of femto-BS.

We show that the NNE can be implemented in a distributed manner for a large class of utility functions even in presence of co-channel interference among femto cells.

In Section VI-D the significance of WNNE is illustrated by the analysis of a specific game with the maximum achievable rate as utility function for each femto-UTs in presence of co-channel interference among femto-cells. In general, this game does not admit a unique NNE or a potential function. Standard techniques to compute a NNE have exponential complexity. We leverage on the WNNE, which can be obtained using the strictly concave constrained potential game, in order to implement an NE in a distributed fashion which retains desirable properties of NNE. Finally we numerically evaluate various properties of NNE solution for some well known utility functions in Section VII.

*Notation.* Vectors and matrices are denoted by bold lower case and bold capital letters, respectively; $\cdot^T$ denotes the transpose operator; the notation $\mathbf{x} \succeq \mathbf{0}$ stays for componentwise inequality; $\mathbf{1}_M$ is the $M$-dimensional column vectors of ones. The vector $\mathbf{v}_{-i}$ is obtained from vector $\mathbf{v}$ by suppressing the $i$th component. The matrix operator $\odot$ denotes the Hadamard or component wise product; $\mathbf{D} = \mathrm{diag}(\mathbf{v})$ maps vector $\mathbf{v}$ onto a diagonal matrix with diagonal component $d_{\ell,\ell} = v_\ell$. The set of nonnegative real numbers in denoted by $\mathbb{R}_+$.

## II. System Model

We consider a heterogeneous network consisting of $F$ femto-BSs equipped with single antenna and serving a single femto-UT per channel use and $M$ macro-UTs (Fig. 1). The macro-UTs can be served by a single macro-BS or different macro-BSs. We do not make any assumptions regarding the distribution of femto-BSs, femto-UTs and macro-UTs except the fact that femto-UTs are located close to the femto-BSs. Let $h_f$ be the channel gain between femto-BS $f$ and femto-UT $f$ and $\widehat{h}_m^f$ the channel gain between femto-BS $f$ and macro-UT $m$. Finally, $\widetilde{h}_k^f$ is the channel gain between femto-BS $k$ and femto-UT $f$. The femto-BS $f$ transmits with power $p_f \geq 0$. For future use, it is convenient to define the following vectors $\mathbf{p} = (p_1, p_2, \ldots, p_F)^T$, $\mathbf{h} = (h_1, h_2, \ldots, h_F)^T$, $\widehat{\mathbf{h}}_m = (\widehat{h}_m^1, \widehat{h}_m^2, \ldots, \widehat{h}_m^F)^T$, with $m = 1, \ldots, M$, $\widehat{\mathbf{h}}^f = (\widehat{h}_1^f, \widehat{h}_2^f, \ldots, \widehat{h}_M^f)^T$, and $\widetilde{\mathbf{h}}^f = (\widetilde{h}_1^f, \widetilde{h}_2^f, \ldots, \widetilde{h}_{f-1}^f, \widetilde{h}_{f+1}^f, \ldots \widetilde{h}_F^f)^T$, with $f = 1, \ldots, F$.

The interference from femto-BSs at macro-UT $m$ is given by

$$I_m = \mathbf{p}^T \widehat{\mathbf{h}}_m \qquad m = 1, \ldots, M. \tag{1}$$

Note that in this setting we are interested only in keeping the interference caused by femto-base stations to macro-UT below a certain acceptable limit. In fact, in practice, the interference from macro-BSs to macro-UTs is efficiently controlled by proper beamforming design and coordinated beamforming.

The signal to interference and noise ratio (SINR) at femto-UT $f$ is given by

$$\gamma_f = \frac{p_f h_f}{\sigma^2 + \mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f} \tag{2}$$

where $\sigma^2$ is the variance of the additive white Gaussian noise that accounts also for interference from macro-BSs. In general $\gamma_f$ is a function of the power vector $\mathbf{p}$. When it is convenient, we explicitly point out this dependence by writing $\gamma_f(\mathbf{p})$, otherwise we omit it and use the short notation $\gamma_f$. When the interference from adjacent femto-BSs is negligible, i.e. $\widetilde{\mathbf{h}}^f \cong \mathbf{0}$, the SINR reduces to the signal to noise ratio (SNR) $\gamma_f' = \dfrac{p_f h_f}{\sigma^2}$.

In the following, we assume that each femto-BS has complete knowledge of the channel, i.e. it knows $\mathbf{h}$, $\widehat{\mathbf{h}}_m$, for $m = 1, \ldots, M$ and $\widetilde{\mathbf{h}}^f$, for $f = 1, \ldots, F$. It is worth noticing that the acquisition of these pieces of information is costly in terms of both bandwidth and signal processing efforts. Then, it is convenient to reduce the necessary information, for example by approximating $\mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f$ with a constant value and modeling it as white noise. In this case, the SINR reduces to the expression of the SNR with an increased variance for the Gaussian noise. Alternatively, we introduce distributed algorithms based on local information in Section VI.

In order to keep the quality of the downlink communications in each macro-UT acceptable, total interference from all the femto-BSs must be below the following acceptable limit

$$\mathbf{p}^T \widehat{\mathbf{h}}_m \leq I_T \qquad m = 1, \ldots, M. \tag{3}$$

We consider identical threshold at different macro-UTs in order to keep notations simple. The extension to the general case with different thresholds is straightforward.

Additionally, the transmit powers are constrained to a maximum value $P_{\mathrm{MAX}}$ such that

$$\mathbf{p} \preceq P_{\mathrm{MAX}} \mathbf{1}_F. \tag{4}$$

## III. Problem Formulation and Solution concept

### A. Problem Formulation

Each femto-BS performs a power control with the objective of maximizing the quality of its communication in downlink. Its communication quality is characterized by a function $U_f(\gamma_f)$, where $U_f(\cdot)$ is a concave nondecreasing function.

We formulate the power allocation at femto-BSs as a noncooperative game where each femto-BS aims to maximize its utility $U_f(\gamma_f)$ under constraints[1](3).

[1]Throughout the rest of this paper, to keep notation and equations compact, we do not consider constraints (4). They are orthogonal and can be immediately embedded in the proposed game theoretical framework. The extension of all the results presented here to the case including these additional constraints is straightforward.

More specifically, we define this non-cooperative game in a strategic form as

$$\mathcal{G} = \{\mathcal{F}, \mathcal{P}, \{u_f(\mathbf{p})\}_{f\in\mathcal{F}}\} \tag{5}$$

where the elements of the game are

- Player set: Set of the femto-BSs $\mathcal{F} = \{1, ..., F\}$;
- Strategy set: $\mathcal{P} = \{\mathbf{p}|\mathbf{p} \in \mathbb{R}_+^F$ and $\mathbf{p}^T\widehat{\mathbf{h}}_m \leq I_T,\ m = 1, \ldots M\}$, where $\mathbb{R}_+^F$ is the product space of $F$ nonnegative real spaces $\mathbb{R}_+$.
- Utility set: the functions $u_f(\mathbf{p})$ are defined as $u_f(\mathbf{p}) \equiv U_f(\gamma_f(\mathbf{p})) = U_f\left(\frac{p_f h_f}{\sigma^2+\mathbf{p}_{-f}^T\widetilde{\mathbf{h}}^f}\right)$, being $U_f(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ a concave nondecreasing function in $\mathbb{R}_+$.

We adopt a NE of the non-cooperative game $\mathcal{G}$ as a power allocation policy for the femto-BSs of the heterogeneous network. More specifically, the power allocation vector $\mathbf{p}^*$ is a Nash Equilibrium (NE) if for every $f \in \mathcal{F}$

$$U_f\left(\frac{p_f^* h_f}{\sigma^2 + \mathbf{p}_{-f}^{*T}\widetilde{\mathbf{h}}^f}\right) \geq U_f\left(\frac{p_f h_f}{\sigma^2 + \mathbf{p}_{-f}^{*T}\widetilde{\mathbf{h}}^f}\right) \tag{6}$$

for all $p_f$ such that $(p_1^*, \ldots p_{f-1}^*, p_f, p_{f+1}^*, p_F^*) \in \mathcal{P}$.

By observing that each $U_f\left(\frac{p_f h_f}{\sigma^2+\mathbf{p}_{-f}^T\widetilde{\mathbf{h}}^f}\right)$ is continuous in $\mathbb{R}_+^F$ and concave for $p_f \in \mathbb{R}_+$ and the set $\mathcal{P}$ is convex and closed, we conclude that $\mathcal{G}$ falls in the class of concave games with coupled constraints studied in [7] and a NE exists [7].

### *B. Normalized Nash Equilibrium*

The strategy set $\mathcal{P}$ is convex and bounded Boundedness is immediately verified when we assume that each femto-BS's transmitted signal impinges on at least one of the receiving macro-BSs, i.e. there exists no femto-BS $f$ such that $\widehat{h}_m^f = 0$ for all $m = 1, \ldots, M$.Thus, sufficient conditions for strong duality, the so-called constraint qualifications, are satisfied (see e.g. [8], [9]).

Under the further assumption that the functions $U_f$, for all $f = \mathcal{F}$, possess continuous first derivatives, we can use the necessary and sufficient KKT conditions for constrained maxima (see e.g. [8]) to obtain conditions satisfied by a NE $\mathbf{p}^*$. If $\mathbf{p}^*$ is a NE, then, there exist $F$ vectors $\boldsymbol{\lambda}^f = (\lambda_1^f, \lambda_2^f \ldots, \lambda_M^f)$ with $\boldsymbol{\lambda}^f \geq \mathbf{0}$ such that $\mathbf{p}^* \in \mathcal{P}$ satisfies the following system of equations

$$\lambda_m^f(\mathbf{p}^T\widehat{\mathbf{h}}_m - I_T) = 0 \qquad m = 1, \ldots M \text{ and } f = 1, \ldots F \tag{7}$$

$$\frac{\partial U_f(\gamma_f)}{\partial p_f} - \sum_{m=1}^{M} \lambda_m^f \frac{\partial}{\partial p_f}(\mathbf{p}^T\widehat{\mathbf{h}}_m - I_T) = 0 \quad f = 1, \ldots F \tag{8}$$

We can write (8) as

$$U_f^{'}(\gamma_f)\frac{\partial \gamma_f}{\partial p_f} - \boldsymbol{\lambda}^{f\,T}\widehat{\mathbf{h}}^f = 0, \qquad f = 1, \ldots F \tag{9}$$

$\mathbf{p}^*$ is a normalized Nash equilibrium (NNE) if KKT conditions in (7)-(9) are satisfied with $\boldsymbol{\lambda}^f = \boldsymbol{\lambda}$ for all $f \in \mathcal{F}$ i.e. the Lagrangians are identical for each player. It has several advantages: First, the Lagrangian multiplier $\lambda_m^f$ can be viewed as the price per unit of interference caused by player $f$ at macro-UT $m$. In the case of NNE, all the femto-BSs pay the same price for the interference caused to a certain macro-UT. Thus, a macro-UT does not have to select different prices for different players. Additionally, it will be clear from the decentralized implementation proposed in Section VI, the above property considerably reduces the cost and the complexity of the signaling among macro-UTs and femto-BSs. Second, In the distributed algorithm thanks to the properties of the Lagrangian multipliers for a NNE, each Macro-UT only needs to track the sum of the interference in order to calculate the price and doe not need to track interference from each player which is not feasible in practice.

Since KKT conditions are necessary, thus, once the $\boldsymbol{\lambda}$ is specified each player's decision is also specified by the KKT conditions. Thus, if the optimal $\boldsymbol{\lambda}$ is found, each player can individually obtain its own NNE strategy using that $\boldsymbol{\lambda}$.

Since NNE has favorable properties to be implemented in a decentralized fashion, we henceforth examine the NNE.

## IV. On the Uniqueness of a NNE

The uniqueness of a NNE for concave games with coupled constrained has been studied in [7]. We shortly summarize the results obtained in [7] in the following

**Proposition 1.** *[7] Let*

$$\mathbf{G}(\mathbf{p}) = \begin{pmatrix} \dfrac{\partial^2 u_1}{\partial p_1^2} & \dfrac{\partial^2 u_1}{\partial p_2 \partial p_1} & \cdots & \dfrac{\partial^2 u_1}{\partial p_F \partial p_1} \\ \dfrac{\partial^2 u_2}{\partial p_1 \partial p_2} & \dfrac{\partial^2 u_2}{\partial p_2^2} & \cdots & \dfrac{\partial^2 u_2}{\partial p_F \partial p_2} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial^2 u_F}{\partial p_1 \partial p_F} & \dfrac{\partial^2 u_F}{\partial p_2 \partial p_F} & \cdots & \dfrac{\partial^2 u_F}{\partial p_F^2} \end{pmatrix}. \tag{10}$$

*If the symmetric matrix $\mathbf{G}(\mathbf{p}) + \mathbf{G}^T(\mathbf{p})$ is negative definite for all $\mathbf{p} \in \mathcal{P}$, then there exists a unique vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$ such that a unique NE $\mathbf{p}^*$ satisfies all the KKT conditions in (7)-(9) with $\lambda_m^f = \lambda_m$ for all $f \in \mathcal{F}$ and $m = 1, \ldots, M$, i.e. NNE is unique* [2].

In order to study the uniqueness of an NNE, throughout this section, we assume that the utility functions $U_f(\cdot)$ are twice differentiable and strictly concave and we analyze under which conditions the game $\mathcal{G}$ defined for the heterogeneous network with femto-BSs as players admits a unique NNE.

First, we focus on the case when the interference from adjacent femto-BSs is negligible at all the femto-UT, i.e. $u_f(\mathbf{p}) = U_f(\gamma_f') = U_f\left(\frac{p_f h_f}{\sigma^2}\right)$. Then,

$$\mathbf{G}(\mathbf{p}) = \operatorname{diag}\left(\frac{h_1^2}{\sigma^4}U_1''\left(\frac{p_1 h_1}{\sigma^2}\right), \frac{h_2^2}{\sigma^4}U_2''\left(\frac{p_2 h_2}{\sigma^2}\right), \ldots \frac{h_F^2}{\sigma^4}U_F''\left(\frac{p_F h_F}{\sigma^2}\right)\right). \tag{11}$$

[2]The condition is sufficient, but not necessary. In [7] a weaker condition is provided for the uniqueness of NNE. We do not consider that property since it is very difficult to verify that condition in practice.

Thanks to the assumption of strict concavity of $U_f(\cdot)$, for $f \in \mathcal{F}$, for every $\mathbf{p} \in \mathcal{P}$ , all the diagonal elements of the matrix $\mathbf{G}(\mathbf{p})$ are strictly negative and according to Proposition 1, NNE is unique.

Now, consider the case where the interference from adjacent cells is not negligible. In this case,

$$\frac{\partial^2 U_f}{\partial p_f^2} = \kappa_f U_f''(\gamma_f) h_f \tag{12}$$

$$\frac{\partial^2 U_f}{\partial p_\ell \partial p_f} = -\kappa_f \widetilde{h}_\ell^f \left( \gamma_f U_f''(\gamma_f) + U_f'(\gamma_f) \right) \tag{13}$$

with $\kappa_f = \dfrac{h_f}{(\sigma^2 + \mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f)^2}$. Then, the general expression of matrix $\mathbf{G}(\mathbf{p})$ is presented in (14) at the bottom of next page and the properties of the matrix $\mathbf{G}(\mathbf{p})$ strictly depend on the selected functions $U_f(\cdot)$, $f \in \mathcal{F}$, on the realizations of the channels $\mathbf{h}$ and $\widetilde{\mathbf{h}}^f$, $f \in \mathcal{F}$. In general, it is not clear that the matrix $\mathbf{G}(\mathbf{p}) + \mathbf{G}^T(\mathbf{p})$ is negative definite for every $\mathbf{p} \in \mathcal{P}$ and thus, the uniqueness of the normalized Nash equilibrium is not guaranteed. Even for given functions $U_f(\cdot)$, $f \in \mathcal{F}$, and channel $\mathbf{h}$ and $\widetilde{\mathbf{h}}^f$, it might be impossible to show that the condition of Proposition 1 is satisfied.

## V. Weakly normalized NE

In order to have an equilibrium selection criterion yielding a unique NE, whose uniqueness can be simply verified, we introduce the concept of weakly normalized Nash equilibrium with respect to a set of functions $\{V_f(\cdot) | \, V_f : \mathbb{R}_+ \to \mathbb{R}_+\}$ by leveraging on the following proposition.

**Proposition 2.** *Let $\mathcal{G} \equiv \{\mathcal{F}, \mathcal{P}, \{U_f(\gamma_f)\}_{f\in\mathcal{F}}\}$ and $\mathcal{G}' \equiv \{\mathcal{F}, \mathcal{P}, \{V_f(\gamma_f)\}_{f\in\mathcal{F}}\}$ be two games of the kind defined in (5) with identical player and strategy sets and different utility sets. Let the functions $V_f$ and $U_f$, $f \in \mathcal{F}$, be strictly increasing functions. Then, if $\mathbf{p}^*$ is a NE of game $\mathcal{G}'$, then it is also a NE for game $\mathcal{G}$.*

*Proof:* Assume that $\mathbf{p}^*$ is a NE for $\mathcal{G}'$ but not for $\mathcal{G}$. Then, there exists a $p_f$ such that $(p_f, \mathbf{p}^*_{-f}) \in \mathcal{P}$ and

$$U_f(\gamma_f(p_f, \mathbf{p}^*_{-f})) > U_f(\gamma_f(\mathbf{p}^*)).$$

Since $U_f$ is increasing then $\gamma_f(p_f, \mathbf{p}^*_{-f}) > \gamma_f(\mathbf{p}^*)$. But also $V_f$ is an increasing function in $\gamma_f$ and

$$V_f(\gamma_f(p_f, \mathbf{p}^*_{-f})) > V_f(\gamma_f(\mathbf{p}^*)).$$

This contradicts the assumption that $\mathbf{p}^*$ is a NE for $\mathcal{G}'$. Thus, $\mathbf{p}^*$ is a NE for both $\mathcal{G}$ and $\mathcal{G}'$. ■

Now, we formally define the weakly normalized NE (WNNE) using the above proposition:

**Definition 1.** *Let game $\mathcal{G}'$ with utility set $\mathcal{V} \equiv \{V_f(\gamma_f) | f \in \mathcal{F}\}$ and strictly increasing $V_f(\cdot)$ have a NNE $\overline{\mathbf{p}}$. Then $\overline{\mathbf{p}}$ is also a NE of the game $\mathcal{G}$ with utility set $\mathcal{U} \equiv \{U_f(\gamma_f) | f \in \mathcal{F}\}$ and strictly increasing $U_f(\cdot)$. This NE $\overline{\mathbf{p}}$ will be denoted as the* Weakly Normalized Nash equilibrium *(WNNE) of $\mathcal{G}$ with respect to the utility set $\mathcal{V}$.*

Note that WNNE depends on the set of utility functions $\mathcal{V}$. If the game $\mathcal{G}'$ with the specified set of utility function $\mathcal{V}$ admits a unique NNE, then there is a unique WNNE of the game $\mathcal{G}$ with respect to the utility set $\mathcal{V}$.

For some game $\mathcal{G}$ the NNE may not be unique and the computation of an NNE can be highly costly in terms of the computational complexity, e.g. exponential complexity in the number of femto-BSs and macro-UTs, while it can be possible to identify a WNNE with respect to a different set of utility functions with lower complexity, e.g. polynomial complexity in the number of femto-BSs and macro-UTs. Also WNNE retains some favorable properties of NNE as it caters to distributed setting discussed in Section III-B. Section VI-D will enlighten the benefit of the WNNE concept related to the computational complexity in determining an NNE.

In our setting, the concept of WNNE can be illustrated by selecting the functions $V_f(x) = \log(x)$ and defining the utility set

$$\mathcal{V} = \left\{ v_f(\mathbf{p}) = \log(\gamma) = \log\left( \frac{p_f h_f}{\sigma^2 + \mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f} \right) \right\}. \tag{15}$$

Note that it is worthwhile to consider such a utility function thanks to the following features:

- $\sum_{f\in\mathcal{F}} \log(\gamma_f)$ is the utility function underlying a proportionally fair SINR allocation.
- When $\mathrm{SINR} \gg 1$, then the maximum achievable rate of each femto-UT, shortly referred as Shannon capacity, $\log(1 + \mathrm{SINR})$ can be approximated by $\log(\mathrm{SINR})$, i.e. $\log(1 + \mathrm{SINR}) \approx \log(\mathrm{SINR})$.
- For certain applications (e.g. voice transmission) the utility functions of interest increase with SINR in a logarithmic way.

By observing that

$$x \frac{d^2 \log(x)}{dx^2} + \frac{d \log(x)}{dx} = 0$$

the matrix $\mathbf{G}(\mathbf{p})$ in (14) reduces to a diagonal matrix with strictly negative diagonal elements for every $\mathbf{p} \in \mathcal{P}$ with utility functions $\mathcal{V}$. Thus, by Proposition 1 the NNE $\overline{\mathbf{p}}$ is unique for the above game. We can adopt $\overline{\mathbf{p}}$ as a unique WNNE induced by the set $\mathcal{V}$ to any game $\mathcal{G}$ of the kind defined in (5) with strictly increasing utility functions $U_f(\cdot)$, $f \in \mathcal{F}$.

## VI. Computing a Normalized Nash Equilibrium

irst, we discuss coupled constrained potential games. Subsequently, we show that finding NNE is equivalent to solving the concave potential game. We propose a distributed algorithm for strictly concave potential game. Subsequently, we identify a class of utility functions which admit strictly concave potential functions. Finally, we discuss the significance of obtaining an NNE in such a game as it induces a WNNE in a broad class of games, as shown in the previous section.

### A. Constrained Concave Potential Games

We first point out the novel contributions of our work on constrained concave potential games compared to the works [12] and [6] on the topic of constrained potential games: To the best of our knowledge our work is the first to provide a relationship between potential games and with NNE in a heterogeneous network with multiple femto-BSs, femto-UTs and macro-UTs. In fact [6] and [12] considered a multiple access channel (MAC) channel. In contrast, the channel model we consider here is an interference channel which presents much higher complexity than a MAC. For example, utility functions like the Shannon capacity that in a MAC setting admits potential functions, but does not admit a potential function in presence of inter-femto cells interference for an interference channel. Additionally, we provide a broad class of utility functions which admit concave potential games. Finally, we propose a distributed algorithm by leveraging on the concave potential game. This distributed algorithm enables us to implement NNE or WNNE in a distributed fashion in interference channel which was not considered in [6] and [12]. Let us first introduce the following definitions used throughout.

**Definition 2.** *[12] A non-cooperative game $\mathcal{G}$ with utility set $\{u_f(\mathbf{p})|f \in \mathcal{F}\}$ is a potential game if there exists a function $\Phi(\mathbf{p})$ such that for all $f \in \mathcal{F}$ and $(p_f, \mathbf{p}_{-f})$, $(p'_f, \mathbf{p}_{-f}) \in \mathcal{P}$:*

$$\Phi(p_f, \mathbf{p}_{-f}) - \Phi(p'_f, \mathbf{p}_{-f}) = u_f(p_f, \mathbf{p}_{-f}) - u_f(p'_f, \mathbf{p}_{-f}).$$

**Definition 3.** *[12] A potential game is called a concave potential game if the potential function $\Phi(\mathbf{p})$ is concave in $\mathbf{p} \in \mathcal{P}$. If $\Phi(\mathbf{p})$ is strictly concave, it is called as strictly concave game.*

**Remark 1.** *For a differentiable utility function $u_f(\cdot)$, $\Phi(\cdot)$ is a potential function of the game if and only if (iff) [10]*

$$\frac{\partial u_f}{\partial p_f} = \frac{\partial \Phi}{\partial p_f} \qquad \forall f \in \mathcal{F}. \tag{16}$$

The utility of introducing the concave potential game is shown in the following proposition.

**Proposition 3.** *Suppose there exists a potential function $\Phi(\mathbf{p})$ of the game $\mathcal{G}$ defined in (5) as $\mathcal{G} = \{\mathcal{F}, \mathcal{P}, \{u_f(\mathbf{p})|f \in \mathcal{F}\}\}$. The solution of the following optimization problem referred to as CCPG is an NNE.*

$$\begin{array}{lll} \textit{CCPG} & \textit{maximize}_{\mathbf{p}} & \Phi(\mathbf{p}) \\ & \textit{subject to} & \mathbf{p} \in \mathcal{P} \end{array}$$

*Proof.* Let $\mathbf{p}^*$ be an optimal solution to CCPG. First, note that $\mathbf{p}^*$ is an NE. If it was not, then there would exist a $p'_f$ such that $(p'_f, \mathbf{p}_{-f}) \in \mathcal{P}, f \in \mathcal{F}$ such that

$$u_f(p^*_f, \mathbf{p}^*_{-f}) < u_f(p'_f, \mathbf{p}^*_{-f}). \tag{17}$$

Since $\Phi(\cdot)$ is a potential function, (17) implies that $\Phi(p^*_f, \mathbf{p}^*_{-f}) < \Phi(p'_f, \mathbf{p}^*_{-f})$. This contradicts the fact that $\mathbf{p}^*$ is an optimal solution.

Since $\mathbf{p}^*$ is an optimal solution, thus, according to the KKT conditions there exists a $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_M)^T \succeq \mathbf{0}_M$ such that

$$\frac{\partial \Phi}{\partial p_f} - \boldsymbol{\nu}\widehat{\mathbf{h}}^f = 0 \qquad \forall f \in \mathcal{F} \tag{18}$$

at $\mathbf{p} = \mathbf{p}^*$, with

$$\nu_m(\mathbf{p}^T\widehat{\mathbf{h}}_m - I_T) = 0 \quad \forall m. \tag{19}$$

Identifying $\boldsymbol{\lambda}$ in (7)–(9) with $\boldsymbol{\nu}$ we can easily discern that $\mathbf{p}^*$ is indeed an NNE. Hence, the result follows. □

**Remark 2.** *Note that if an NNE is unique then the solution of CCPG is the unique NNE. If a coupled constraint concave game $\mathcal{G}$ admits a potential function, in general, not every NNE can be expressed as a solution of CCPG. However, if the potential function is concave, then,each NNE is a solution of CCPG since KKT conditions are also sufficient and necessary for optimality in the concave potential games.*

### *B. A Distributed Algorithm to Determine an NNE*

In Section VI-A we showed that when a coupled constrained concave game $\mathcal{G}$ has a unique NNE and admits a potential function, we can solve the potential game in order to achieve the unique NNE. When the potential function $\Phi(\cdot)$ is strictly concave and $\dfrac{\partial \Phi}{\partial p_f}$ only depends on $p_f, \forall f \in \mathcal{F}$, then there exists a distributed algorithm which converges to the unique optimal solution $\mathbf{p}^*$ and the dual optimal solution $\boldsymbol{\lambda}^*$. The distributed algorithm is the following:

**Algorithm DIST**

Initially macro-UT $m$ selects $\boldsymbol{\lambda}^0 \in \mathbb{R}^M_+ \setminus \{\mathbf{0}\}$ randomly[3]

At iteration $k+1 = 1, 2, \ldots,$ the following tasks are performed:

1) Each femto-BS $f$ sets

$$p_f^{k+1} = \operatorname*{argmax}_{p_f \geq 0} \Phi(p_f) - p_f \boldsymbol{\lambda}^{k,} \hat{\mathbf{h}}^f \tag{20}$$

Then, all the femto-BSs transmit with updated power level $\mathbf{p}^{k+1} = (p_1^{k+1}, p_2^{k+1}, \ldots p_F^{k+1})$.

2) Macro-UT $m$ sets

$$\lambda_m^{k+1} = (\lambda_m^k + \delta(\widehat{\mathbf{h}}_m^T \mathbf{p}_f^{k+1} - I_T))^+ \tag{21}$$

where $\delta > 0$ is a sufficiently small constant[4]. Macro-UT $m$ reports the updated cost $\lambda_m^{k+1}$ to all the femto-BSs.

Since solution of CCPG is a NNE and $\Phi(\cdot)$ is strictly concave, thus, the convergence of Algorithm DIST follows immediately from known results in [11] and it is stated in the following proposition.

**Proposition 4.** *Algorithm DIST converges to the unique optimal primal solution $\mathbf{p}^*$ and dual solution $\boldsymbol{\lambda}^*$ when $\Phi(\cdot)$ is strictly concave in $\mathbf{p}$.*

[3] $\boldsymbol{\lambda}^0$ is initialized arbitrarily.

[4] If there is no communication is possible between macro-UT and femto-BS, then instead of macro-UT, macro-BS can compute the price for its macro-UT and sends it to femto-BS.

$$\mathbf{G}(\mathbf{p}) = \begin{pmatrix} \kappa_1 h_1 & -\kappa_1 \widetilde{h}_2^1 & \ldots & -\kappa_1 \widetilde{h}_F^1 \\ -\kappa_2 \widetilde{h}_1^2 & \kappa_2 h_2 & \ldots & -\kappa_2 \widetilde{h}_F^2 \\ \vdots & \vdots & \ddots & \vdots \\ -\kappa_F \widetilde{h}_1^F & -\kappa_F \widetilde{h}_2^F & \ldots & -\kappa_F h_F \end{pmatrix} \bigodot \begin{pmatrix} U_1''(\gamma_1) & \gamma_1 U_1''(\gamma_1) + U_1'(\gamma_1) & \ldots & \gamma_1 U_1''(\gamma_1) + U_1'(\gamma_1) \\ \gamma_2 U_2''(\gamma_2) + U_2'(\gamma_2) & U_2''(\gamma_2) & \ldots & \gamma_2 U_2''(\gamma_2) + U_2'(\gamma_2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_F U_F''(\gamma_F) + U_F'(\gamma_F) & \gamma_F U_F''(\gamma_F) + U_F'(\gamma_F) & \ldots & U_F''(\gamma_F) \end{pmatrix}. \tag{14}$$

In Algorithm DIST, each femto-BS needs to solve an optimization problem. Since $\frac{\partial \Phi}{\partial p_f} = \frac{\partial U_f}{\partial p_f}$ does not depend on $\mathbf{p}_{-f}$ $\forall f \in \mathcal{F}$, then a femto-BS can solve the optimization problem using local parameters measurable at the femto-BS without need for costly feedback exchanges. Thus, even though the solution of potential game CCPG requires to know other femto-BSs information, NNE can be implemented in a distributed manner where each femto-BS only needs to know its own transmission parameter. Also note that each macro-UT only needs to know the total interference it is experiencing which is readily available[5]. It is practically infeasible to track interference from each femto-BS to a macro-UT. Thus, the distributed algorithm that we have provided is readily scalable and implementable without the need of a central controller. In the following Section VI-C and VI-D we will introduce large classes of potential games where $\frac{\partial \Phi}{\partial p_f}$ does not depend on $\mathbf{p}_{-f}, \forall f \in \mathcal{F}$. Thus, the import of the results is that we can implement a distributed algorithm which converges to a unique NNE for a large class of utility functions.

### C. Negligible Inter-Femtocell Interference

In this section, we show that when the inter femto-cell interference is negligible, then the game is a strictly concave potential game. We have already shown in Section IV that the NNE is unique in this setting. Thus, from Proposition 3, we can obtain the unique NNE by solving the potential game CCPG.

In this setting, SINR reduces to the SNR $\gamma'_f(\mathbf{p}) = \frac{p_f h_f}{\sigma^2}, \forall f \in \mathcal{F}$. Thus, femto-BS $f$'s utility $U_f(\gamma_f) = U_f(\gamma'_f)$ is only a function of $p_f$ in this scenario. Next lemma shows that in *this scenario* every game $\mathcal{G}$ defined in (5) is a potential game and if $U_f(\cdot)$ is strictly concave, then the potential game is also strictly concave.

**Lemma 1.** *Let $\Phi(\mathbf{p}) = \sum_f U_f(\gamma'_f(\mathbf{p}))$. Then, $\Phi(\mathbf{p})$ is a potential function for game $\mathcal{G}$ defined in (5) with utility set $\{U_f(\gamma'_f) | f \in \mathcal{F}\}$. Moreover, if $U_f(\cdot)$ is strictly concave, then so $\Phi(\cdot)$ is.*

*Proof.* Since $U_f(\gamma'_f)$ does not depend on $\mathbf{p}_{-f}$, for any $f \in \mathcal{F}$

$$\Phi(p_f, \mathbf{p}_{-f}) - \Phi(p'_f, \mathbf{p}_{-f}) = U_f(\gamma'_f(p_f, \mathbf{p}_{-f})) - U_f(\gamma'_f(p'_f, \mathbf{p}_{-f}))$$

for any $(p_f, \mathbf{p}_{-f}), (p'_f, \mathbf{p}_{-f}) \in \mathcal{P}$. This proves that $\Phi(\cdot)$ is a potential function.

By the definition of $\Phi(\cdot)$ and $\gamma'_f$, it is clear that if $U_f(\cdot)$ is strictly concave $\forall f \in \mathcal{F}$, then so $\Phi(\cdot)$ is. $\square$

Since $\frac{\partial \Phi}{\partial p_f}$ does not depend on $\mathbf{p}_{-f}$ in this setting, thus we can apply Algorithm DIST, which will converge to optimal $\mathbf{p}^*$ and dual variable $\boldsymbol{\lambda}^*$.

### D. Presence of Inter-Femtocell Interference

When the interference from adjacent femto-BSs is not negligible at all the femto-UTs, then we have already seen that an NNE may be not unique. However, in Section IV we have identified a utility set $\mathcal{V}$ defining a game $\mathcal{G}$, whose NNE is unique and can be used to define unique WNNE for games with different utility sets. In this section, we show that game with utility set $\mathcal{V}$ admits a potential function.

Let us consider again the utility set $\mathcal{V}$ defined in (15). The following lemma shows that the game $\mathcal{G}$ with the utility set $\mathcal{V}$ defined in (15) is a potential game.

**Lemma 2.** *Let $\Phi(\mathbf{p}) = \sum_f \log(p_f)$. $\Phi(\mathbf{p})$ is a potential function for the game $\mathcal{G}$ defined in (5) and the utility set (15). Moreover, the potential game is strictly concave.*

*Proof.* Note that

$$\Phi(p_f, \mathbf{p}_{-f}) - \Phi(p'_f, p_{-f}) = \log(p_f) - \log(p'_f). \quad (22)$$

But,

$$\log(\gamma_f(p_f, \mathbf{p}_{-f})) - \log(\gamma_f(p'_f, p_{-f})) = \log\left(\frac{p_f h_f}{\mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f + \sigma^2}\right) - \log\left(\frac{p'_f h_f}{\mathbf{p}_{-f}^T \widetilde{\mathbf{h}}^f + \sigma^2}\right) = \log(p_f) - \log(p'_f). \quad (23)$$

Thus, comparing (22) and (23) we conclude that $\Phi(\mathbf{p})$ is a potential function. It is easy to verify that $\Phi(\mathbf{p})$ is a strictly concave function in $\mathbf{p}$. $\square$

Hence, the solution of CCPG will provide the unique NNE to game $\mathcal{G}$ defined in (5) with the utility set (15). We can use Algorithm DIST described in Section VI-B to obtain the unique NNE for a game $\mathcal{G}$ with utility set (15) since $\frac{\partial \Phi}{\partial p_f}$ does not depend on $\mathbf{p}_{-f}$ for all $f$. Since $\Phi(\mathbf{p})$ is different from $\sum_{f=1}^{F} V_f(\mathbf{p})$, thus, NNE may not be an optimal solution of the optimization problem where the objective is to maximize the sum of the players' utilities.

Now, we show that, in general, a potential game does not exist in this setting by using the following result

**Proposition 5.** *[10] For a twice continuously differentiable utility functions $u_f, u_\ell, f, \ell \in \mathcal{F}$, there exists a potential function iff $\frac{\partial^2 u_f}{\partial p_\ell \partial p_f}(\mathbf{p}) = \frac{\partial^2 u_\ell}{\partial p_f \partial p_\ell}(\mathbf{p})$, $\forall \mathbf{p} \in \mathcal{P}$.*

It is easy to verify from (13) that in general the utilities $U_f(\gamma_f)$ do not satisfy the conditions stated in Proposition 5. There exist large classes of function $U_f(\gamma_f(\mathbf{p}))$ which are strictly concave functions of $p_f$ but still they do not admit a potential function. One such examples is $U_f(\gamma_f) = \log(1 + \gamma_f)$.

For all these cases when the functions $U_f(\cdot)$ are strictly increasing, it is convenient to appeal to Proposition 2 and resort to the concept of WNNE with respect to the utility set $\mathcal{V}$ in (15). We have already shown in Lemma 2 that the game $\mathcal{G}$ with utility set $\mathcal{V}$ in (15) is a strictly concave potential game and thus, we can implement the distributed algorithm proposed in Section VI-B to compute the unique NNE with low complexity. Hence, we can easily obtain the WNNE for

[5]Note that the femto-BSs which are situated at a far-off location from a macro-UT, we may assume that the interference from that femto-BS is negligible in order to avoid communication overhead

the game $\mathcal{G}$ with $U_f(\gamma_f) = \log(1+\gamma_f)$ even though this latter game is not a potential game.

## VII. Numerical Results

We numerically evaluate the characteristics of an NNE strategy profile for several utilities. We consider two scenarios: i) Interference at each femto-UT from adjacent femto-BSs is negligible ($\gamma_i \approx \mathrm{SNR}_i$), ii) Interference is not negligible ($\gamma_i = \mathrm{SINR}_i$). To generate $\hat{\mathbf{h}}_m^i, \tilde{\mathbf{h}}_f^i$, we first randomly place femto-BSs and macro-UTs in a disc of radius $r_1$. Then, we randomly place a femto-UT in a disc of radius $r_2$ around each femto-BS (Fig. 1). We take $r_1 > r_2$ because in practice, a femto-UT is in a close vicinity of its femto-BS compared to the size of a macro-cell. We compute the channel gain between two nodes according to the formula: $d^{-\beta}$ where $d$ is the distance between two entities and $\beta$ is a positive constant with $\beta = 2$. For all simulations we take $r_1 = 20, r_2 = 2$, and $\sigma^2 = 1$. Throughout this section, we use the optimal value of the sum of the utilities as the reference metric:

$$U_{\mathrm{OPT}} = \max_{\mathbf{p}\in\mathcal{P}} \sum_{f\in\mathcal{F}} U_f(\gamma_f) \forall \mathbf{p} \in \mathcal{P} \tag{24}$$

### A. Negligible Inter femto-cells Interference

In this setting $\gamma_i, i \in \mathcal{F}$ is approximated as $\mathrm{SNR}_i$, i.e. $\gamma_i = \gamma_i' = \frac{p_i h_i}{\sigma^2}$.

We consider the following utility functions

1) *Shannon Capacity*: Here $U_f(\gamma_f') = B\log(1+\gamma_f')$.
2) *Bit Error Rate*: From [3] we can approximate bit error rate (BER) for $K$-QAM modulation at femto-BS $f$ as follows:

$$\mathrm{BER} = 0.2\exp\left(-\frac{3\gamma_f'}{2(K-1)}\right)$$

Since each femto-BS wants to minimize the BER, we define the utility function as

$$U_f(\gamma_f') = -0.2\exp\left(-\frac{3\gamma_f'}{2(K-1)}\right)$$

3) $\alpha-$*proportional fair utility in SNR*: Here

$$U_f(\gamma_f') = \frac{(\gamma_f')^{1-\alpha}}{1-\alpha}, \quad \alpha \neq 1 \tag{25}$$

**Remark 3.** *It is easy to verify that the above functions are strictly concave in $\gamma_f'$. Thus, from Section VI we know that an NNE is the optimal solution of CCPG with potential function $\sum_i U_i(\cdot)$. Thus, $U_{OPT}$ is attained at the NNE strategy profile.*

We set $I_T = 5dB$ and $B = 1MHz$ and assume a 4-QAM modulation for all the simulations in this subsection. First, we study the variation of $U_{OPT}$ with the number of macro-UTs for the Shannon capacity. Fig. 2 shows that, as the number of macro-UTs increases, $U_{OPT}$ and the individual utilities decrease. Intuitively, this happens because when the number of macro-UTs increases the strategy set $\mathcal{P}$ can only reduce. The decrement of $U_{OPT}$ becomes small as the number of macro-UTs increases, thus an increase in the number of macro-UTs does not affect the individual utility significantly above a certain number of macro-UTs.

Fig. 3 reveals that the mean BER and each femto-BS's BER increase as the number of macro-UTs increases since each femto-BS transmits lower power as the number of macro-UTs increases. But the rate of increment slows down as the number of macro-UTs increases. Intuitively, as the number of macro-UTs increases the strategy set $\mathcal{P}$ remains almost identical. Thus, the optimal power remains almost the same even when the number of macro-UTs increases.

Fig. 4 reveals that as the number of macro-UTs increases $U_{OPT}$ decreases when $U_f(\cdot)$ is according to equation (25). This is because the strategy space $\mathcal{P}$ decreases as the number of macro-UTs increases. Fig. 4 also reveals that only one femto-BS's payoff is significantly positive for a given number of macro-UT, other players' utilities are very low. Intuitively, when $\alpha$ is very small, the utility $U_f$ behaves like linear function of $\gamma_f'$. Thus, only the player which has high channel gain and does not interfere significantly with a macro-UT gets high utility, the other players' select very low power, thus their utilities are close to 0.

Fig.5 shows the convergence of Algorithm DIST for systems with different number of femto-BSs, $F = 3, 5$, and 8. Numerical computation reveals that the convergence rate is higher for smaller number of femto-BSs.

### B. Inter-Femtocell Interference Not Negligible

Here, $\gamma_i$ is equal to $\mathrm{SINR}_i$. When we adopt the following utility $U_f(\gamma_f) = \log(\gamma_f), f \in \mathcal{F}$, the game is a strictly concave potential game, as shown in Section VI-D, with potential function $\sum_i \log(p_i)$. It is easy to verify that the maximization of $\sum_i U_i(\gamma_i)$ for $\mathbf{p} \in \mathcal{P}$ is a geometric programming [15]. Hence, we can employ standard tools to compute $U_{\mathrm{OPT}}$. We use the following notation: $U_{\mathrm{NNE}}$ be the sum of utilities at the NNE strategy profile $\mathbf{p}^{\mathrm{NNE}}$.

Fig.6 reveals that as the number of femto-BSs increases the difference between $U_{\mathrm{OPT}}$ and $U_{\mathrm{NNE}}$ increases. Intuitively, when the number of femto-BSs is small, then the interference at a femto-BS is not significant, thus $U_{\mathrm{NNE}}$ closely matches $U_{\mathrm{OPT}}$. But as the number of femto-BSs increases, $U_{\mathrm{NNE}}$ decreases at a faster rate and the difference between $U_{\mathrm{NNE}}$ and $U_{\mathrm{OPT}}$ increases. Note that $U_{\mathrm{OPT}}$ also decreases with the number of femto-BSs. Intuitively, as $U_f(\gamma_f) = \log(\gamma_f)$, thus, $p_f > 0$ for any $f \in \mathcal{F}$ [6]. Thus as the number of femto-BSs increase the co-tier interference becomes significant as each additional femto-BS will transmit a significant amount of positive amount of power. Thus, $U_{OPT}$ decreases with the the number of femto-BSs as the interference from other femto-BSs become significant.

*Shannon capacity*: We consider utility functions, $U_f(\gamma_f) = \log(1+\gamma_f)$.

We numerically study the characteristics of the WNNE solutions. Let $U_{\mathrm{WNNE}}$ denote the total utility at a WNNE $p_{WNNE}$ i.e. $\mathbf{p}_{WNNE}$ is the solution of CCPG with $\Phi(\mathbf{p}) = \sum_f \log(\gamma_f)$ and $U_{\mathrm{WNNE}} = \sum_{f=1}^{F} \log(1+\gamma_f(\mathbf{p}_{WNNE}))$. Fig. 7 reveals that initially $U_{\mathrm{WNNE}}$ increases as the number of femto-BSs increases. Intuitively, when the number of femto-BSs is small, the co-tier interference is small and thus, total utility $U_{\mathrm{WNNE}}$ is high. But when the number of femto-BSs becomes large the co-tier interference becomes significant,

[6] Otherwise, $U_{OPT}$ would be negative infinity.

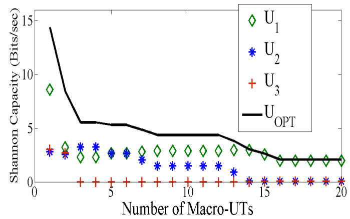


Fig. 2. $U_{OPT}$ and $U_f, f = 1, 2, 3$ versus number of macro-UTs for Shannon capacity and $F = 3$.

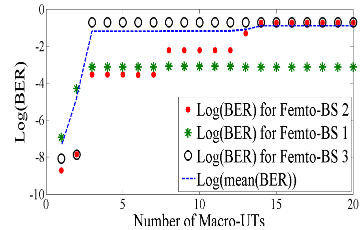


Fig. 3. $\log(\text{BER})$ versus number of macro-UTs for $F = 3$.

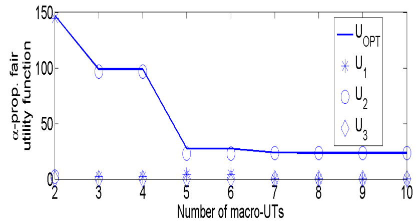


Fig. 4. $U_{OPT}$ and $U_f, f = 1, \ldots, 3$ versus the number of macro-UTs for functions in (25) ($\alpha = 0.1$).

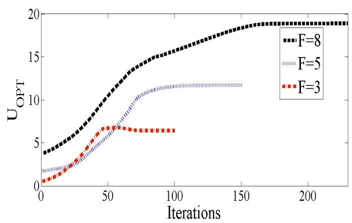


Fig. 5. Convergence analysis of Algorithm DIST for $F = 3, 5$ and 8 femto-BSs.

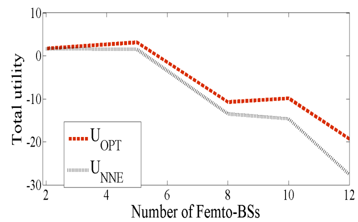


Fig. 6. $U_{OPT}$ and $U_{NNE}$ versus number of femto-BSs with $M = 5$.

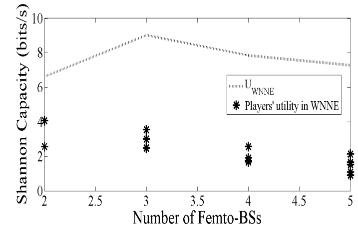


Fig. 7. $U_{WNNE}$ and individual utilities versus the number of femto-BSs for $I_T = 10$.

thus, the individual utilities decrease significantly and thus, the $U_{\text{WNNE}}$ also decreases. Fig.7 also shows that the fluctuation of utilities across players are very limited for power allocations based on the WNNE. Intuitively, $\sum_i \log(\gamma_i)$ is a utility function underlying a proportional fair SINR allocation, thus the players' utilities vary on a relatively smaller range for a power allocation based on an WNNE.

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