

RECENT INSIGHTS IN THE BAYESIAN AND DETERMINISTIC CRB FOR BLIND SIMO CHANNEL ESTIMATION

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ABSTRACT

The performance of channel estimation is often assessed by deriving the proper Cramér-Rao Bound (CRB). Depending on how to treat the symbols and the channel, we have previously derived different versions of CRB. Specifically, we have dealt with the cases where the symbols and/or the channel are assumed to be either deterministic unknowns or random. Moreover, the symbols have been considered to be either jointly estimated with the channel or marginalized. All in all, we have derived six different versions of Bayesian and deterministic CRBs. However, we have shown that many of these CRBs are too optimistic in the sense that they are not strict enough to be attained by any deterministic or Bayesian estimator. In this paper we propose modified versions of those loose CRBs in the context of SIMO FIR system that are valid at least in the moderate and high SNR regimes. The analytical formulas for the lower bounds introduced are validated by some Monte-Carlo simulations.

1. INTRODUCTION

Traditionally, the transmitter sends some known information to the receiver to aid the latter in estimating the channel. However, in wireless communication the channel varies rapidly with time and as a consequence more training sequence/pilots are required. This process wastes a lot of bandwidth as a result of augmenting the transmission rate to maintain the throughput. In the last two decades a new branch of channel estimation has emerged focusing on accomplishing this task blindly i.e. without the need for a training sequence. Nevertheless, most wireless standards that have evolved during this period are still relying on the training sequence/pilots to estimate the channel. This is due probably to the unsatisfactory results of the blind channel estimation algorithms. On the other hand, some powerful channel estimation algorithms that take advantage of both aforementioned techniques have been also developed during the same era. These are known as semi-blind where a superior performance is achieved although few training sequence/pilots are transmitted. As usual the performance of these algorithms are lower bounded by the most famous lower bound namely, CRB. In [1] we have derived the CRBs that correspond to the different algorithms elaborated in [2]. Unfortunately, many of these CRBs are shown to be loose since they don't take into consideration the coupling between the channel and the symbol estimates at the level of the Fisher

Information Matrix (FIM). In this paper we propose modified versions of these CRBs that are tighter than those derived in [1]. We will show analytically that these CRBs constitute valid lower bounds in the moderate and high SNR regimes. This paper is organized as follows: In section II we develop the SIMO FIR transmission system model, while in section III we show a general framework that permits the derivation of the different CRBs. In section IV we make use of the framework developed in section III to derive the different modified CRBs. In section V we conduct some Monte-Carlo simulations to pictorially compare different modified CRBs with their corresponding estimators as well as with their corresponding traditional CRBs.

2. SIMO FIR TX SYSTEM MODEL

Consider a linear digital modulation over a linear channel with additive noise so that the received signal $y(t)$ has the following form:

$$y(t) = \sum_k h(t - kT)a(k) + v(t). \quad (1)$$

In (1) $a(k)$ are the transmitted symbols, T is the symbol period and $h(t)$ is the channel impulse response. The channel is assumed to be FIR with length NT . If the received signal is oversampled at the rate $\frac{m}{T}$ (or if m different samples of the received signal are captured by m sensors every T seconds, or a combination of both), the discrete input-output relationship can be written as:

$$\mathbf{y}(k) = \sum_{i=0}^{N-1} \mathbf{h}(i)a(k-i) + \mathbf{v}(k) = \mathbf{H}A_N(k) + \mathbf{v}(k) \quad (2)$$

where $\mathbf{y}(k) = [y_1^H(k) \cdots y_m^H(k)]^H$, $\mathbf{h}(i) = [h_1^H(i) \cdots h_m^H(i)]^H$, $\mathbf{v}(k) = [v_1^H(k) \cdots v_m^H(k)]^H$, $\mathbf{H} = [\mathbf{h}(N-1) \cdots \mathbf{h}(0)]$, $A_N(k) = [a(k-N+1) \cdots a(k)]^H$ and superscript H denotes Hermitian transpose. Let $\mathbf{H}(z) = \sum_{i=0}^{N-1} \mathbf{h}(i)z^{-i} = [\mathbf{H}_1^H(z) \cdots \mathbf{H}_m^H(z)]^H$ be the SIMO channel transfer function, and $\mathbf{h} = [\mathbf{h}^H(N-1) \cdots \mathbf{h}^H(0)]^H$. Consider additive independent white Gaussian circular noise $\mathbf{v}(k)$ with $r_{\mathbf{v}\mathbf{v}}(k-i) = E \mathbf{v}(k)\mathbf{v}(i)^H = \sigma_v^2 I_m \delta_{ki}$. Assume we receive M samples:

$$\mathbf{Y}_M(k) = \mathcal{T}_M(\mathbf{h}) A_{M+N-1}(k) + \mathbf{V}_M(k) \quad (3)$$

where $\mathbf{Y}_M(k) = [\mathbf{y}^H(k-M+1) \cdots \mathbf{y}^H(k)]^H$ and similarly for $\mathbf{V}_M(k)$, and $\mathcal{T}_M(\mathbf{h})$ is a block Toeplitz matrix with M block rows

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and $[\mathbf{H} \ 0_{m \times (M-1)}]$ as first block row. We shall simplify the notation in (3) with $k = M-1$ to

$$\begin{aligned} \mathbf{Y} = \mathcal{T}(\mathbf{h})\mathbf{A} + \mathbf{V} &= \mathcal{T}_K(\mathbf{h})\mathbf{A}_K + \mathcal{T}_U(\mathbf{h})\mathbf{A}_U + \mathbf{V} \\ &= \mathcal{A}_K\mathbf{h} + \mathcal{A}_U\mathbf{h} + \mathbf{V}. \end{aligned} \quad (4)$$

Where $\mathcal{T}_K(\mathbf{h})$ and $\mathcal{T}_U(\mathbf{h})$ represent respectively the portions of $\mathcal{T}(\mathbf{h})$ that correspond to A_k (M_K known symbols) and A_U (M_U unknown symbols), see Figure 5 in [1]. On the other hand, \mathcal{A} is a block Toeplitz matrix filled with the elements of \mathbf{A} while \mathcal{A}_K and \mathcal{A}_U are block Toeplitz matrices filled with the elements of A_K and A_U respectively. Here we assume for simplicity that the known symbols are gathered at the beginning of the block.

3. A UNIFIED FRAMEWORK FOR DIFFERENT CRBS

We have presented in [1] a complete framework that permits the handling of the different cases of the channel and the symbols estimation. Here we shall present briefly the main results. In [1] the different estimation cases have been classified into two main categories. In the first category the channel and the unknown symbols are estimated jointly by making some assumptions on the channel and the unknown symbols. If we denote by θ the unknown parameters to be estimated then it is given by:

$$\theta = [A_U^H, \mathbf{h}^H]^H \quad (5)$$

Applying the log function to the joint probability density function (pdf), we get [1]:

$$\ln[f(\mathbf{Y}, A_U, \mathbf{h})] = \ln[f(\mathbf{Y}/A_U, \mathbf{h})] + \ln[f(A_U)] + \ln[f(\mathbf{h})] \quad (6)$$

where $f(\mathbf{Y}, A_U, \mathbf{h})$ and $f(\mathbf{Y}/A_U, \mathbf{h})$ denote respectively the joint and conditional pdf. Now, let J represents the Fisher Information matrix (FIM), it is given by [3]:

$$\begin{aligned} J_{\theta\theta} &= \mathbf{E} \left(\frac{\partial \ln[f(\mathbf{Y}, A_U, \mathbf{h})]}{\partial \theta^*} \right) \left(\frac{\partial \ln[f(\mathbf{Y}, A_U, \mathbf{h})]}{\partial \theta} \right)^H \\ &= -\mathbf{E} \frac{\partial}{\partial \theta^*} \left(\frac{\partial \ln[f(\mathbf{Y}, A_U, \mathbf{h})]}{\partial \theta} \right)^H \end{aligned} \quad (7)$$

As we shall observe later, since we are treating complex parameters we also need, besides $J_{\theta\theta}$, $J_{\theta\theta^*}$ which is defined by:

$$\begin{aligned} J_{\theta\theta^*} &= \mathbf{E} \left(\frac{\partial \ln[f(\mathbf{Y}, A_U, \mathbf{h})]}{\partial \theta^*} \right) \left(\frac{\partial \ln[f(\mathbf{Y}, A_U, \mathbf{h})]}{\partial \theta} \right)^H \\ &= -\mathbf{E} \frac{\partial}{\partial \theta} \left(\frac{\partial \ln[f(\mathbf{Y}, A_U, \mathbf{h})]}{\partial \theta^*} \right)^H \end{aligned} \quad (8)$$

When $J_{\theta\theta^*} \neq 0$ we shall resort to θ_R defined below:

$$\theta_R = \begin{bmatrix} \text{Re}(\theta) \\ \text{Im}(\theta) \end{bmatrix} = \mathcal{M} \begin{bmatrix} \theta \\ \theta^* \end{bmatrix}, \mathcal{M} = \frac{1}{2} \begin{bmatrix} I & I \\ -jI & jI \end{bmatrix} \quad (9)$$

Knowing that $J_{\theta\theta} = J_{\theta^*\theta^*}^*$ and $J_{\theta\theta^*} = J_{\theta^*\theta}^*$ then (9) yields:

$$J_{\theta_R\theta_R} = \mathcal{M} \begin{bmatrix} J_{\theta\theta} & J_{\theta\theta^*} \\ J_{\theta^*\theta^*}^* & J_{\theta^*\theta}^* \end{bmatrix} \mathcal{M}^H \quad (10)$$

On the other side, when $J_{\theta\theta^*} = 0$ then $J_{\theta_R\theta_R}$ is defined totally by $J_{\theta\theta}$. This holds true for all the cases where we jointly estimate the channel and the symbols as we shall notice later. Under some assumptions and regularity conditions [4], the error covariance matrix of an unbiased channel estimator $\hat{\mathbf{h}}(\mathbf{Y})$, which is defined as:

$$C(\hat{\mathbf{h}}) = \mathbf{E} \left\{ \left[\hat{\mathbf{h}}(\mathbf{Y}) - \mathbf{h} \right] \left[\hat{\mathbf{h}}(\mathbf{Y}) - \mathbf{h} \right]^H \right\} \quad (11)$$

satisfies the following inequality:

$$C(\hat{\mathbf{h}}) \geq \{J_{\theta_R\theta_R}\}^{-1} \triangleq CRB \quad (12)$$

We usually focus on comparing the Mean Square Error, $\text{MSE} = \text{tr} \{C(\hat{\mathbf{h}})\}$ to the minimum error variance which is defined by $\text{tr} \{CRB\}$ where tr stands for the trace of a matrix. However, in the second category the channel and the noise variance are the only parameters to be estimated while the symbols are supposed to be marginalized during the estimation process.

$$\theta = [\mathbf{h}^H, \sigma_v^2]^H \quad (13)$$

Again, when we apply the log function to the joint pdf, we get:

$$\ln[f(\mathbf{Y}, \mathbf{h}, \sigma_v^2)] = \ln[f(\mathbf{Y}/\mathbf{h}, \sigma_v^2)] + \ln[f(\mathbf{h})] + \ln[f(\sigma_v^2)] \quad (14)$$

As for FIM, both (7) and (8) are still applicable where only θ is redefined as in (13).

4. DERIVATIONS OF MODIFIED CRBS

We have derived in [1] six different Bayesian and deterministic CRBs. Four out of those six CRBs are shown to be loose. We shall develop in this section modified versions of those loose CRBs. This will be done by exploiting the framework introduced in the previous section. For explanation on the way by which we call the different CRBs, we refer the reader to [1]. As we have observed in (7) and (8), there is an expectation operator in the definition of the FIM. When both the channel and the unknown symbols are deterministic, this expectation operator can be written as $\mathbf{E}_{Y/A, \mathbf{h}}$. In this case the expectation means averaging over the noise which is the only random vector. However, when either the channel or the unknown symbols or even both of them are considered as random, the expectation operator means averaging over the different realizations of the noise, the channel and the unknown symbols. Therefore, the expectation operator becomes as follows (Baye's Theorem):

$$\begin{aligned} \mathbf{E}_{Y, \mathbf{h}/A_U} &= \mathbf{E}_{Y/A_U, \mathbf{h}} \mathbf{E}_{\mathbf{h}} \quad (\mathbf{h} \text{ is random}). \\ \mathbf{E}_{Y, A_U/\mathbf{h}} &= \mathbf{E}_{Y/A_U, \mathbf{h}} \mathbf{E}_{A_U} \quad (\mathbf{A}_U \text{ is random}). \\ \mathbf{E}_{Y, \mathbf{h}, A_U} &= \mathbf{E}_{Y/A_U, \mathbf{h}} \mathbf{E}_{\mathbf{h}} \mathbf{E}_{A_U} \quad (\mathbf{h} \text{ and } \mathbf{A}_U \text{ are random}). \end{aligned} \quad (15)$$

As we have noticed in [1], the deficiency of the traditional Bayesian and deterministic CRBs is a direct consequence of the implementation of $\mathbf{E}_{\mathbf{h}}$ and/or \mathbf{E}_{A_U} in the FIM formulas. Specifically, these expectations make the channel and the unknown symbols estimates decoupled although in reality they are coupled. Our main idea is to postpone the implementation of these expectation operators so that we compute the inverse of the FIM first then we apply them in the second step to get the modified CRB. Based on this introduction, the question now, does this modified CRB still constitute a lower bound for the channel estimate? Another question that poses itself, is there any proof that this modified CRB is tighter than the traditional one? To address the first question, we note that the main drawback of postponing the implementation of the expectation operators ($\mathbf{E}_{\mathbf{h}}$ and/or \mathbf{E}_{A_U}) is that our modified CRBs would correspond to estimators that should be unbiased for every channel and/or unknown symbols realizations.

When both the channel and the symbols are considered as deterministic unknowns (see SB-ML-ML in [2]), the channel estimate has been traditionally considered as unbiased. However, in [5] it has been shown that the unbiasedness doesn't hold at low SNR!

As a consequence, even the traditional CRB is no longer a valid lower bound in this SNR regime. Moreover, theoretically once the channel and/or the symbols are considered as random, the channel estimate should be biased even at high SNR due to the usage of the prior information of the channel and/or the unknown symbols. However, as our simulations have shown, the bias is negligible at moderate and high SNR. This encourages us to proceed in our idea since now J_{hh}^{-1} (where $E_{Y/A,h}$ is used instead of $E_{Y,A/h}$, $E_{Y,h/A}$ and $E_{Y,A,h}$) can be considered as a valid lower bound for every channel and/or unknown symbols realization. This is true at least in the moderate and high SNR regimes. Hence, we have $C(\hat{\mathbf{h}}) \geq J_{hh}^{-1}$. Now if apply the expectation operators on both sides we get $E_{\mathbf{h}} E_A J_{hh}^{-1} \geq E_{\mathbf{h}} E_A C(\hat{\mathbf{h}})$. In other words, the modified CRB is a valid lower bound for the mean of the channel estimation error. The mean here is computed by averaging over the different channel and symbols realizations. As for answering the second question raised above, we need to prove that $E J_{hh}^{-1} \geq \{E J_{hh}\}^{-1}$. In [6] it has been shown after a tedious derivation that for any positive definite matrix \mathbf{B} we have:

$$\text{tr} (E \{[\mathbf{B}]^{-1}\}) \geq \text{tr} ([E \{\mathbf{B}\}]^{-1}) \quad (16)$$

However, we will present here a much simpler proof that this inequality holds also without the trace operator. Suppose we have a matrix which is formed as $\mathbf{G} = [\mathbf{B}^{\frac{H}{2}} \quad \mathbf{B}^{-\frac{1}{2}}]^H$, the inner product of this matrix is given by $\langle \mathbf{G}, \mathbf{G} \rangle = E \mathbf{G} \mathbf{G}^H \geq \mathbf{0}$ where the inequality stems from the non degeneracy property of the inner product [7]. Developing the inner product yields:

$$E \begin{bmatrix} E \mathbf{B} & I \\ I & E \mathbf{B}^{-1} \end{bmatrix} \geq 0 \quad (17)$$

Now applying the Schur's complement we get: $E \mathbf{B} - (E \mathbf{B}^{-1})^{-1} \geq \mathbf{0} \Rightarrow E \mathbf{B} \geq (E \mathbf{B}^{-1})^{-1} \Rightarrow E \mathbf{B}^{-1} \geq (E \mathbf{B})^{-1}$. It is obvious now that (16) follows directly. Hence, we can allege that our modified Bayesian and deterministic CRBs that are based on inverting the matrix first then applying the expectation operators are tighter than the traditional ones. This fact is going to be elaborated in the sequel where the formulas of the modified Bayesian and deterministic CRBs are derived.

4.1. DCRB_{sto,joint} and MDCRB_{sto,joint}

This estimator also belongs to the first category, thus the joint pdf is given by (6). Moreover, $f(A_U) = \frac{1}{(\pi\sigma_a^2)^{M+N-1-M_K}} \exp[-\frac{A_U^H A_U}{\sigma_a^2}]$ and $f(\mathbf{h}) = \mathbf{h}^o \delta(\mathbf{h} - \mathbf{h}^o)$. It is obvious here that $\ln[f(\mathbf{h})]$ can be omitted without affecting the computation of FIM. Hence, (7) and (15) yield:

$$J_{\theta\theta} = E_{A_U} \frac{1}{\sigma_v^2} \begin{bmatrix} \mathcal{T}_U^H(\mathbf{h})\mathcal{T}_U(\mathbf{h}) + \frac{\sigma_v^2}{\sigma_a^2} I_{M_U} & \mathcal{T}_U^H(\mathbf{h})\mathcal{A} \\ \mathcal{A}^H \mathcal{T}_U(\mathbf{h}) & \mathcal{A}^H \mathcal{A} \end{bmatrix} \quad (18)$$

Furthermore, we can write $J_{\theta\theta} = E_{A_U} \mathbf{B}$ where \mathbf{B} denotes the matrix in the square brackets in (18) multiplied by the inverse of the variance of the noise. Denoting $E_{A_U} \{\mathcal{A}\} = \mathcal{A}'_K$ and $E_{A_U} \{\mathcal{A}^H \mathcal{A}\} = C_k$ where $C_k = \mathcal{A}'_K{}^H \mathcal{A}'_K + M_U \sigma_a^2 I_{m_N}$ and noting that $J_{\theta\theta^*} = 0$, then by inverting $J_{\theta\theta}$ in (18) which is composed of four blocks, we get again a matrix composed of four blocks. $DCRB_{sto,joint}$ is given by the block in the lower right corner of

that matrix:

$$DCRB_{sto,joint} = \sigma_v^2 \left(C_K - \mathcal{A}'_K{}^H \mathcal{T}_U(\mathbf{h}) [\mathcal{T}_U^H(\mathbf{h})\mathcal{T}_U(\mathbf{h}) + \frac{\sigma_v^2}{\sigma_a^2} I]^{-1} \mathcal{T}_U^H(\mathbf{h})\mathcal{A}'_K \right)^{-1} \quad (19)$$

Now, if we follow the discussion presented in section IV, the modified version of this CRB ($MDCRB_{sto,joint}$) is given by the block in the lower right corner of $E_{A_U} \mathbf{B}^{-1}$.

$$MDCRB_{sto,joint} = E_{A_U} \sigma_v^2 \left(\mathcal{A}^H \left(I - \mathcal{T}_U(\mathbf{h}) [\mathcal{T}_U^H(\mathbf{h})\mathcal{T}_U(\mathbf{h}) + \frac{\sigma_v^2}{\sigma_a^2} I]^{-1} \mathcal{T}_U^H(\mathbf{h}) \right) \mathcal{A} \right)^{-1} \quad (20)$$

4.2. BCRB_{sto,joint} and MBCRB_{sto,joint}

In this lower bound both the channels and the unknown symbols are assumed random with Gaussian distribution and are supposed to be estimated jointly. Hence, this lower bound in its turn belongs to the first category and its joint pdf is given by (6). By substituting the terms in (6) by their corresponding functions and make use of (15), we deduce the corresponding FIM as follows:

$$J_{\theta\theta} = E_{\mathbf{h}} E_{A_U} \frac{1}{\sigma_v^2} \begin{bmatrix} \mathcal{T}_U^H(\mathbf{h})\mathcal{T}_U(\mathbf{h}) + \frac{\sigma_v^2}{\sigma_a^2} I_{M_U} & \mathcal{T}_U^H(\mathbf{h})\mathcal{A} \\ \mathcal{A}^H \mathcal{T}_U(\mathbf{h}) & \mathcal{A}^H \mathcal{A} + \sigma_v^2 C_h^{-1} \end{bmatrix} \quad (21)$$

However, we can write $J_{\theta\theta} = E_{A_U} E_{\mathbf{h}} \mathbf{B}$ where \mathbf{B} denotes the matrix in the square brackets in (21) multiplied by the inverse of the variance of the noise. Assuming that both the channel and the unknown symbol distributions have a zero mean as stated above, we get [1]:

$$BCRB_{sto,joint} = \sigma_v^2 \left(C_K + \sigma_v^2 C_h^{-1} \right)^{-1} \quad (22)$$

For the reasons discussed in [1], this CRB is considered to be too optimistic. Once again here following the discussion presented in section IV, the modified version of this CRB ($MBCRB_{sto,joint}$) is given by the block in the lower right corner of $E_{A_U} E_{\mathbf{h}} \mathbf{B}^{-1}$.

$$MBCRB_{sto,joint} = E_{A_U} E_{\mathbf{h}} \sigma_v^2 \left(\mathcal{A}^H \mathcal{A} + \sigma_v^2 C_h^{-1} - \mathcal{A}^H \mathcal{T}_U(\mathbf{h}) \left[\mathcal{T}_U^H(\mathbf{h})\mathcal{T}_U(\mathbf{h}) + \frac{\sigma_v^2}{\sigma_a^2} I \right]^{-1} \mathcal{T}_U^H(\mathbf{h})\mathcal{A} \right)^{-1} \quad (23)$$

4.3. BCRB_{det,joint} and MBCRB_{det,joint}

In this lower bound we consider the unknown symbols to be deterministic unknowns while the channel is considered to be random with Gaussian distribution, $f(\mathbf{h}) = \frac{1}{(\pi)^{m_N} |C_h|} \exp[-\mathbf{h}^H C_h^{-1} \mathbf{h}]$. However, the unknown symbols are considered as deterministic to be jointly estimated with the channel hence, this estimator belongs to the first category where the joint pdf is given by (6). Moreover,

here again $\ln[f(A_U)]$ has no effect on computing FIM so it can be omitted. Therefore, (7) and (15) yield:

$$J_{\theta\theta} = E_{A_U} \frac{1}{\sigma_v^2} \begin{bmatrix} \mathcal{T}_U^H(\mathbf{h})\mathcal{T}_U(\mathbf{h}) & \mathcal{T}_U^H(\mathbf{h})\mathcal{A} \\ \mathcal{A}^H\mathcal{T}_U(\mathbf{h}) & \mathcal{A}^H\mathcal{A} + \sigma_v^2 C_h^{o-1} \end{bmatrix} \quad (24)$$

Again, we can write $J_{\theta\theta} = E_h \mathbf{B}$ where \mathbf{B} denotes the matrix in the square brackets in (24) multiplied by the inverse of the variance of the noise. Assuming that the channel distribution has a zero mean as stated above we get:

$$BCRB_{det,joint} = \sigma_v^2 \left(\mathcal{A}^H \mathcal{A} + \sigma_v^2 C_h^{o-1} \right)^{-1} \quad (25)$$

This CRB is also too optimistic [1]. Once again here following the discussion presented in section IV, the modified version of this CRB ($MBCRB_{det,joint}$) is given by the block in the lower right corner of $E_h \mathbf{B}^{-1}$.

$$MBCRB_{det,joint} = E_h \sigma_v^2 \left(\mathcal{A}^H \mathcal{A} + \sigma_v^2 C_h^{o-1} - \mathcal{A}^H \mathcal{T}_U(\mathbf{h}) \left[\mathcal{T}_U^H(\mathbf{h})\mathcal{T}_U(\mathbf{h}) \right]^{-1} \mathcal{T}_U^H(\mathbf{h})\mathcal{A} \right)^{-1} \quad (26)$$

4.4. $BCRB_{sto,marg}$ and $MBCRB_{sto,marg}$

This lower bound belongs to the second category since the symbols are supposed to be eliminated. The joint pdf is given by (14) but this time $\ln[f(\mathbf{h})]$ can't be omitted. Substituting the terms in (14) by their corresponding functions and following the same steps mentioned in $DCRB_{sto,marg}$ in [1] section IV, we get:

$$\begin{cases} J_{\theta\theta} &= E_h \{ J_{\theta\theta}^{sto} \} + \begin{bmatrix} C_h^{o-1} & 0 \\ 0 & 0 \end{bmatrix} \\ J_{\theta\theta^*} &= E_h \{ J_{\theta\theta^*}^{sto} \} \end{cases} \quad (27)$$

where $J_{\theta\theta}^{sto}$ denotes the FIM that correspond to $DCRB_{sto,marg}$ in [1]. Now we can resort to (10) to compute $J_{\theta_R \theta_R}$. Consequently, we can extract easily $BCRB_{sto,marg}$ from $J_{\theta_R \theta_R}^{-1}$. As for $MBCRB_{sto,marg}$, we make use of the modified versions of $J_{\theta\theta}$ and $J_{\theta\theta^*}$ in (27). This modification stems from postponing the implementation of E_h . After that we proceed in the same way as in the case of $BCRB_{sto,marg}$ and compute the modified $J_{\theta_R \theta_R}^{-1}$. Now, $MBCRB_{sto,marg}$ follows directly from $E_h J_{\theta_R \theta_R}^{-1}$.

5. SIMULATIONS

The goal of this section is to validate numerically the ideas presented throughout this paper. In each Monte-Carlo simulation we generate different realizations of the noise and when required for the channel and the symbols too. As for the channel, we generate a Rayleigh fading channel with a constant power delay profile (PDP). Specifically, C_h^o is an identity matrix. As for the symbols, we generate random 8PSK symbols to reflect the real world case. The performance of the different CRBs is evaluated by means of the Normalized MSE (NMSE) vs. SNR. The SNR is defined as: $SNR = \frac{\|\mathcal{T}(\mathbf{h})\mathcal{A}\|^2}{mM\sigma_v^2}$. The

NMSE is defined as $\frac{avg \text{tr}(CRB)}{avg \|\hat{\mathbf{h}}\|^2}$ where avg stands for average. In

figure 1, we plot the square norm of the bias ($\|\mathbf{h} - E_{Y/A_U, \mathbf{h}} \hat{\mathbf{h}}\|^2$) versus SNR for different semi-blind channel estimators presented in [2]. It is worthy noting that in this plot only one randomly chosen channel and unknown symbols realization has been generated. However, the Monte-Carlo simulations have been run over 100 noise realizations. The bias of SB-ML-ML which has been thought generally

as unbiased can be used as a reference. As we have indicated before, we remark that at moderate and high SNR the biases for all estimators are almost negligible whereas at low SNR they are prominent. In the subfigures of figure 2, we plot the traditional Bayesian and deterministic CRBs along with their modified versions elaborated in section IV. Furthermore, we plot also in the same figures the NMSE for their corresponding semi-blind channel estimators presented in [2]. It is obvious that our modified CRBs are tighter than their traditional versions. Moreover, our modified CRBs are attainable by their corresponding estimators at high SNR. Another remark can be drawn from these plots namely, our modified CRBs match with their traditional versions at very low SNR.

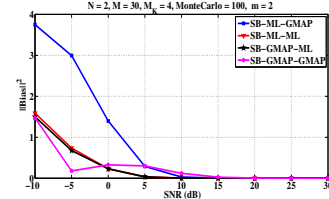


Fig. 1. $\|Bias\|^2$ vs. SNR for different estimators

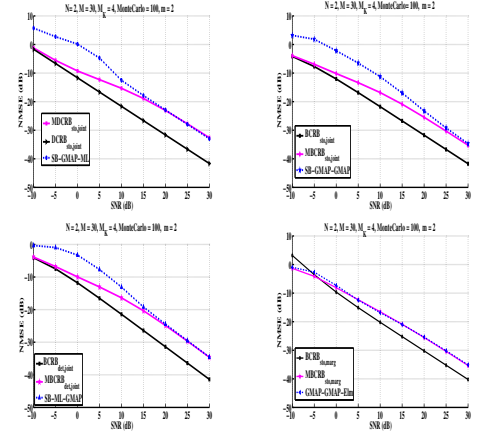


Fig. 2. NMSE vs SNR for different traditional and modified CRBs along with their corresponding estimators.

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