Pairwise Error Probability Analysis for Power Delay Profile Fingerprinting based Localization

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***Abstract*— Although most of the conventional localization algorithms rely on LOS conditions, it is possible to do positioning with Power Delay Profile-Fingerprinting (PDP-F) in multipath and even in NLOS environments. Many algorithms for position fingerprinting have been developed, but analytical investigation in this area is still not matured yet. In this paper we aim to find the pairwise error probability (PEP) for PDP-F based localization systems. The objective is to see the performance of PDP-F algorithms under different cost functions and also under different path amplitude assumptions. By PEP, what is meant is the same as in the PEP analysis in digital communication channels. Hence the approach is similar for PDP-F. However its analysis is not as straightforward as it is for the digital communication channel case. We investigate and show the results for least squares (LS) based algorithm under deterministic path amplitude modeling and Gaussian Maximum Likelihood (GML) based algorithm for the Rayleigh fading modeling of the path amplitudes.**

***Index Terms*— fingerprinting, localization, pairwise error probability, least squares, Gaussian Maximum Likelihood**

I. Introduction

Location fingerprinting (LF) (introduced by U.S. Wireless Corp. of San Ramon, Calif.) relies on signal structure characteristics [1], [2]. It exploits the multipath nature of the channel hence the NLOS conditions. By using multipath propagation pattern, the LF creates a signature unique to a given location. The position of the mobile is determined by matching measured signal characteristics from the BS-MT link to an entry of the database. The location corresponding to the highest match of the database entry is considered as the location of the mobile. For LF, it is enough to have only one BS-MT link (multiple BSs are not required) to determine the location of the mobile. Also LF is classified among Direct Location Estimation (DLE) techniques. Ahonen and Eskelinen suggest using the measured Power Delay Profiles (PDPs) in the database [3] for fingerprints. In [4], authors provide deterministic and Bayesian methods for PDP-F based localization. The Gaussian Maximum Likelihood (GML) based PDP-F revealed in that article is one of the techniques that we analyze in this paper.

What is meant by PEP is the same as in the PEP analysis for digital communication channels. In that case the aim is to find the probability of error when a vector of symbols $\mathbf{s_i}$ is transmitted but another vector $\mathbf{s_j}$ is detected at the receiver. We will pursue a similar approach for PDP-F PEP analysis. However its analysis is not as straightforward as for the digital communication channel case. The difficulty arises from the structure of the problem as will be clear soon. The objective is to determine the probability of error (the probability that wrong entry in the database is selected instead of the true position) when the channel estimates from the MT-BS link is matched with a wrong entry of the database. Hence position estimation error occurs as a result. We will investigate two different algorithms under different path amplitude modeling.

Notations: upper-case and lower-case boldface letters denote matrices and vectors, respectively. $(.)^T$ and $(.)^H$ represent the transpose and the transpose-conjugate operators. $E\{.\}$ is the statistical expectation, $\Re\{.\}$ is the real part and $tr\{.\}$ is the trace operator defined for square matrices.

II. Channel Model and Analytical Expressions of PEP for the LS Technique under Deterministic Path Amplitude Modeling

We start with the channel model because PDP is just the magnitude squared version of the channel impulse response (CIR). But before using the measured PDPs, it is classically averaged over some time duration. However, if the mobile moves rapidly and/or some paths are not resolvable (due to the limited bandwidth of the pulse-shape $p(t)$, path contributions can overlap), the averaging gives a poor PDP estimation.

$$h(t,\tau) = \sum_{i=1}^{N_p} A_i(t)\ p(\tau - \tau_i(t)) \tag{1}$$

where N_p denotes the number of paths (rays), $p(t)$ is the convolution of the transmit and receive filters (pulse shape), $\tau_i(t)$, $A_i(t)$ denote delay and complex attenuation coefficient (amplitude and phase of the ray) of the i^{th} path respectively. We can write the complex path amplitude of path i in polar form as $A_i(t) = a_i(t)e^{j\phi_i(t)}$. It is reasonable to assume that path delays and amplitudes vary slowly with the position. Let us now consider sampling the CIR with a sampling period

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of τ_s leading to N_τ samples and stacking them in a vector as follows:

$$\mathbf{h}(t) = \begin{bmatrix} h(\tau_s, t) \\ h(2\tau_s, t) \\ \vdots \\ h(N_\tau \tau_s, t) \end{bmatrix} = \sum_{i=1}^{N_p} A_i(t)\ \mathbf{p}_{\tau_i}, \quad (2)$$

where $\mathbf{p}_\tau$ is defined as: $\mathbf{p}_\tau = \begin{bmatrix} p(\tau_s - \tau) \\ p(2\tau_s - \tau) \\ \vdots \\ p(N_\tau \tau_s - \tau) \end{bmatrix}$ which is the sampled complex pulse shape vector having a delay equal to the delay of the path in samples and has N nonzero samples. If we write Equation (2) in matrix notation and include the channel estimation noise, we obtain the estimated CIR vector as:

$$\hat{\mathbf{h}}(t) = \underbrace{\left[\mathbf{p}_{\tau_1} \cdots \mathbf{p}_{\tau_{N_p}}\right]}_{\mathbf{P}_\tau} \underbrace{\begin{bmatrix} A_1(t) \\ \vdots \\ A_{N_p}(t) \end{bmatrix}}_{\mathbf{a}(t)} + \mathbf{v}(t). \quad (3)$$

where $\mathbf{v}(t)$ is the complex additive white Gaussian noise vector with covariance matrix $\sigma_v^2 \mathbf{I}$. We estimate the PDP as:

$$\widehat{\mathbf{PDP}} = \frac{1}{T} \sum_{t=1}^{T} \left| \hat{\mathbf{h}}(t) \right|^2 \quad (4)$$

where T is the number of channel observations. There is one thing that needs to be clarified that the absolute squaring operation is element-wise. Hence the resulting PDP estimate is another vector having the same length as the channel estimates. For the path amplitudes, there can be two possibilities:

- deterministic model: $A_i(t)$ deterministic unknowns
- Gaussian model: $A_i(t)$ Gaussian with zero mean, characterized by a power (variance) i.e. $var(A_i) = \sigma_i^2$, which corresponds to Rayleigh fading for the magnitudes.

We will consider the first case now where the path amplitudes are considered as deterministic unknowns. The PEP can be defined as follows when the LS criteria is the cost function:

$$PEP = Pr\left(||\widehat{\mathbf{PDP}} - \mathbf{PDP}_F|| < ||\widehat{\mathbf{PDP}} - \mathbf{PDP}_T||\right), \quad (5)$$

where $\widehat{\mathbf{PDP}}$ is the measured PDP vector defined in Equation (4), $\mathbf{PDP}_T$ is the true PDP vector which is computed off-line from the stored database and $\mathbf{PDP}_F$ is the PDP vector to be erroneously detected. Every position in the database (Ray Tracing database or any other pre-computed database) is distinguishable from each other, e.g. they have either different number of paths, or path delays or amplitudes (variances) are different. Hence there are unique entries in the database so that fingerprinting can work correctly. $\mathbf{PDP}_T$ and $\mathbf{PDP}_F$ are given as:

$$\mathbf{PDP}_T = \mathbf{PDP}_{true} + \sigma_v^2 \mathbf{1}, \quad \mathbf{PDP}_F = \mathbf{PDP}_{false} + \sigma_v^2 \mathbf{1}, \quad (6)$$

where $\mathbf{1}$ is a vector of all 1's which is added to include the effects of the noise. $\mathbf{PDP}_{true}$ and $\mathbf{PDP}_{false}$ are computed with delays and amplitudes of paths. For example in case of non-overlapping pulses (pulses from different paths not overlapping with each other), they would be given as:

$$\mathbf{PDP}_{true} = \sum_{i=1}^{N_p} a_i^2 \left|\mathbf{p}_{\tau_i}\right|^2, \quad \mathbf{PDP}_{false} = \sum_{i=1}^{L} b_i^2 \left|\mathbf{p}_{\zeta_i}\right|^2, \quad (7)$$

Based on Equation (4), $\widehat{\mathbf{PDP}}$ can be calculated as:

$$\begin{aligned} \widehat{\mathbf{PDP}} &= \frac{1}{T} \sum_{t=1}^{T} \left(|\mathbf{v}(t)|^2 + 2(\Re\mathbf{h}(t) \odot \Re\mathbf{v}(t) + \Im\mathbf{h}(t) \odot \Im\mathbf{v}(t)) \right) \\ &+ \mathbf{PDP}_{true}, \end{aligned} \quad (8)$$

where $\odot$ stands for the element-wise Hadamard multiplication. In this case, it is used to multiply the corresponding elements of the real and imaginary parts of the noise and channel vectors. In fact in the equation above, we made a little approximation coming from the last term. We implicitly assumed that averaging gives a good PDP estimate ($\mathbf{PDP}_{true}$ plus the noise terms). For example in case of a non-overlapping pulse assumption, it would not be an approximation. We will make an important simplification in Equation (8) and assume that the terms in the innermost parentheses tend to go to $\mathbf{0}$. In other words they will be replaced by their expected values as the noise is a zero mean process. In fact it is a reasonable assumption when the number of observations T is high. Hence $\widehat{\mathbf{PDP}}$ is now approximated as:

$$\widehat{\mathbf{PDP}} \approx \mathbf{PDP}_{true} + \frac{1}{T} \sum_{t=1}^{T} |\mathbf{v}(t)|^2. \quad (9)$$

After these calculations and definitions we can turn back to the PEP formulation. In fact PEP can also be stated equivalently as:

$$PEP = Pr\left(||\widehat{\mathbf{PDP}} - \mathbf{PDP}_F||^2 < ||\widehat{\mathbf{PDP}} - \mathbf{PDP}_T||^2\right). \quad (10)$$

This equivalent formulation is easier to deal with. For simplicity of notation, let us call $\widehat{\mathbf{PDP}}$ as $\mathbf{x}$, $\mathbf{PDP}_{false}$ as $\mathbf{y}$ and $\mathbf{PDP}_{true}$ as $\mathbf{z}$. Then PEP becomes:

$$Pr\left(\mathbf{x}^T(\mathbf{z} - \mathbf{y}) < \frac{(\mathbf{z} + \sigma_\mathbf{v}^2\mathbf{1})^T(\mathbf{z} + \sigma_\mathbf{v}^2\mathbf{1}) - (\mathbf{y} + \sigma_\mathbf{v}^2\mathbf{1})^T(\mathbf{y} + \sigma_\mathbf{v}^2\mathbf{1})}{2}\right). \quad (11)$$

If we check Equation (9), we immediately recognize that first term of the equation is deterministic while the second term is the random part. If we do the algebra, we can reorganize Equation (11) as:

$$PEP = Pr\left((\mathbf{z} - \mathbf{y})^T \sum_{t=1}^{T} |\mathbf{v}(t)|^2 < \frac{T}{2}(2k_2 - k_1 - k_3 + 2M)\right) \quad (12)$$

where $k_1 = \mathbf{z}^T\mathbf{z}$, $k_2 = \mathbf{y}^T\mathbf{z}$, $k_3 = \mathbf{y}^T\mathbf{y}$ and $M = \sigma_v^2 \mathbf{1}^T(\mathbf{z} - \mathbf{y})$. Here k_2 is an important parameter which gives information about the overlapping between the vectors. As it is clear, it is always non-negative. It can be 0 if and only if the vectors do not overlap with each other at all. Mathematical formulation of PEP is almost complete. When we explore Equation (12), it is a summation of random variables on the left hand side. We can divide the analysis for each turn of T. Let us call the random variable as W_i for the i^{th} loop. So the left hand side as a result becomes a random variable W which is $W = \sum_{i=1}^{T} W_i$. However finding the distribution of W is not easy as we will see later. Therefore we will just compute the distribution of a W_i (W_1 without loss of generality). And then we will call the central limit theorem (CLT) for W as all the W_i's are identically distributed. Remember that $\mathbf{v}(t)$ is a complex white Gaussian noise vector. Hence each element of the $|\mathbf{v}(t)|^2$ vector is composed of sums of squares of two Gaussian random variables. It is well known that this leads to the exponential distribution with mean $1/\lambda = \sigma_v^2$, i.e. $f_M(m) = \lambda e^{-m\lambda}$, $m \geq 0$ [5]. Therefore W_1 will be a summation of exponential random variables. However they all have different parameters (different λ's) which makes the calculation of the overall distribution more difficult. In other words it would be a summation of independent but not identically distributed exponential random variables. If all had the same parameters, we know that this leads to the Erlang distribution [6]. The distribution of W_1 which is a summation of K exponential random variables with means $1/\lambda_i$'s is derived as (proof is omitted due to lack of space):

$$f_{W_1}(u) = \left(\prod_{i=1}^{K} \lambda_i\right) \left(\sum_{j=1}^{K} \frac{e^{-\lambda_j u}}{\prod\limits_{\substack{l=1\\ l\neq j}}^{K} (\lambda_l - \lambda_j)}\right). \tag{13}$$

Deriving the distribution of W which is a summation of T of these random variables (W_i's) will be more challenging. We can also compute the probability for $T = 1$ with the obtained derivation. However in that case the assumption that we have done in Equation (9) will be disturbed. Due to these reasons, we will call the CLT for these T (T being large) independent and identically distributed random variables as we mentioned before. Before applying the CLT, we have to know the mean and variance of W_i's. By using Equation (12), we determine the mean and variance of W_i's as follows:

$$\begin{aligned} \mu_{W_i} &= M, & (14)\\ \sigma_{W_i}^2 &= \sigma_v^4 \left(k_1 + k_3 - 2k_2\right). & (15) \end{aligned}$$

By CLT, $\dfrac{W - T\mu_{W_i}}{\sigma_{W_i}\sqrt{T}}$ will tend to have a standard normal distribution ($\mathcal{N}(0,1)$) when T is large. Hence PEP can be reformulated as:

$$\begin{aligned} PEP &= Pr\left(\frac{W - T\mu_{W_i}}{\sigma_{W_i}\sqrt{T}} < \frac{\sqrt{T}}{2\sigma_{W_i}}\left(2k_2 - k_1 - k_3\right)\right) \\ &= Q\left(\frac{\sqrt{T}}{2\sigma_v^2}\sqrt{k_1 + k_3 - 2k_2}\right) & (16)\\ &= Q\left(\frac{\sqrt{T}}{2\sigma_v^2}||\mathbf{z} - \mathbf{y}||\right). & (17) \end{aligned}$$

And by using the Chernoff bound for the Q function, we can bound the PEP as:

$$PEP \leq \frac{1}{2} e^{-\frac{T}{8\sigma_v^4}||\mathbf{z}-\mathbf{y}||^2}. \tag{18}$$

We see that PEP decreases when the norm of the difference between the true and false PDPs increase. In fact it is a reasonable result. When they become more and more apart from each other, one can expect that it will be less likely to confuse the true PDP with the false one. The interesting thing is that we reached this result after the approximation given by Equation (9) and by the use of the CLT.

III. Analytical Expressions of PEP for the GML Technique for Rayleigh Fading Modeling of the Path Amplitudes

In this part, we investigate the PEP analysis for the GML based PDP-F technique. We also have a different assumption for the complex path amplitudes $A_i(t)$. Instead of modeling them as deterministic unknowns, we now model them as complex Gaussian random variables (Rayleigh distribution for the magnitudes). For a complete description of this PDP-F method, readers can refer to [4]. The channel model that we have proposed in the previous section is still valid and given by Equation (3). We now assume that pulses from different paths are non-overlapping ($\mathbf{P}_\tau$ is an orthogonal matrix) to simplify the analysis which is a reasonable assumption in high bandwidth systems. The matching criteria is based on Gaussian log-likelihood. Hence formulation of the PEP is such that the probability that the log-likelihood performed in the true position is lower than the log-likelihood in the false position which results in the false position to be selected. If we have multiple channel estimates, the log-likelihood can be expressed as:

$$\mathcal{LL} \propto -\ln\left(\det\left(\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}\right)\right) - tr\left(\hat{\mathbf{C}}\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}^{-1}\right) \tag{19}$$

where $\hat{\mathbf{C}} = \frac{1}{T}\sum_{i=1}^{T}\left(\hat{\mathbf{h}}_{\mathbf{i}} - \mu\right)\left(\hat{\mathbf{h}}_{\mathbf{i}} - \mu\right)^H$ is the sample covariance matrix obtained from channel estimates. Since the complex path amplitudes $A_i(t)$ and the noise have both zero mean, channel estimates have also zero mean, i.e., $\mu = \mathbf{0}$. For simplicity of notation, let us call $\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}$ as $\mathbf{C_T}$ which denotes the covariance matrix calculated with the true positions' entries. By using Equation (3), we have $\mathbf{C_T} = \mathbf{P}_\tau\mathbf{C_a}\mathbf{P}_\tau^{\mathbf{H}} + \sigma_{\mathbf{v}}^{\mathbf{2}}\mathbf{I}$ where $\mathbf{C}_a$ is a diagonal matrix having $\left[\sigma_{a_1}^2, \sigma_{a_2}^2, \cdots, \sigma_{a_{N_p}}^2\right]$ on its diagonal ($var(A_i) = \sigma_{a_i}^2$). We also introduce $\mathbf{C_F}$ for the covariance matrix computed with the false positions' entries as $\mathbf{C_F} = \mathbf{P}_\zeta\mathbf{C_b}\mathbf{P}_\zeta^{\mathbf{H}} + \sigma_{\mathbf{v}}^{\mathbf{2}}\mathbf{I}$ where $\mathbf{C}_b$ is a diagonal matrix having $\left[\sigma_{b_1}^2, \sigma_{b_2}^2, \cdots, \sigma_{b_L}^2\right]$ on its diagonal and $\mathbf{P}_\zeta$ is defined similarly as $\mathbf{P}_\tau$. After giving the necessary information, we can state PEP as:

$$PEP = Pr\left(\mathcal{LL}_T < \mathcal{LL}_F\right). \quad (20)$$

We know that there are many scenarios to investigate. However we will try to explore the scenario where the error probability is more likely to occur. And also the scenario proposed will also simplify the analysis.

Scenario: The scenario can be summarized as follows:

1) Number of paths are equal, i.e., $L = N_p$.
2) Path delays are equal, i.e., $\tau_i = \zeta_i \ \forall i$.
3) There is no delay synchronization error.

Under these assumptions $\mathbf{P}_\zeta = \mathbf{P}_\tau$. We see that the only differences between the true and the false positions' parameters are the path amplitude variances. By using Equation (19) we can restate PEP as:

$$PEP = Pr\left(\ln\left(\det \mathbf{C_T}/\mathbf{det}\mathbf{C_F}\right) > tr\left(\hat{\mathbf{C}}\mathbf{C_F^{-1}}\right) - tr\left(\hat{\mathbf{C}}\mathbf{C_T^{-1}}\right)\right). \quad (21)$$

Under the assumption that $\mathbf{P}_\tau$ being an orthogonal matrix, the determinants can be easily calculated.

$$\begin{aligned}\det\left(\mathbf{C}_T\right) &= \sigma_v^{2N_\tau} \det\left(\mathbf{I} + \frac{1}{\sigma_v^2}\mathbf{P}_\tau \mathbf{C}_a \mathbf{P}_\tau^H\right) \\ &= \sigma_v^{2N_\tau} \prod_{i=1}^{N_p}\left(1 + \frac{e_p \sigma_{a_i}^2}{\sigma_v^2}\right), \end{aligned} \quad (22)$$

where we have used the Sylvester's determinant theorem, $\det(\mathbf{I} + \mathbf{AB}) = \det(\mathbf{I} + \mathbf{BA})$ and $e_p = S$ is the pulse energy. One thing to note is that the determinant does not depend on the path delays when the pulses are non-overlapping. Similarly we obtain $\det\left(\mathbf{C}_F\right)$. Hence left hand side of Equation (21) is:

$$\ln\left(\det \mathbf{C_T}/\mathbf{det}\mathbf{C_F}\right) = \sum_{i=1}^{N_p} \ln\left(\frac{\sigma_v^2 + e_p \sigma_{a_i}^2}{\sigma_v^2 + e_p \sigma_{b_i}^2}\right) = f_1. \quad (23)$$

For the inversion of $\mathbf{C}_T$ and $\mathbf{C}_F$ we will use the Woodbury's matrix inversion lemma. We get:

$$\mathbf{C}_T^{-1} = \sigma_v^{-2}(\mathbf{I} - \mathbf{P}_\tau \mathbf{D}_a \mathbf{P}_\tau^H) \quad (24)$$
$$\mathbf{C}_F^{-1} = \sigma_v^{-2}(\mathbf{I} - \mathbf{P}_\tau \mathbf{D}_b \mathbf{P}_\tau^H) \quad (25)$$

where $\mathbf{D}_a$ and $\mathbf{D}_b$ are diagonal matrices having $[\mathbf{D}_a]_{ii} = \dfrac{\sigma_{a_i}^2}{\sigma_v^2 + e_p \sigma_{a_i}^2}$ and $[\mathbf{D}_b]_{ii} = \dfrac{\sigma_{b_i}^2}{\sigma_v^2 + e_p \sigma_{b_i}^2}$ on their diagonals respectively. By using Equation (3) we can write $\hat{\mathbf{C}}$ as:

$$\begin{aligned}\hat{\mathbf{C}} &= \frac{1}{T}\sum_{i=1}^{T} \hat{\mathbf{h}}_\mathbf{i} \hat{\mathbf{h}}_\mathbf{i}^H = \frac{1}{T}\sum_{i=1}^{T} (\mathbf{P}_\tau \mathbf{a}_i + \mathbf{v}_i)(\mathbf{P}_\tau \mathbf{a}_i + \mathbf{v}_i)^H \\ &= \frac{1}{T}\sum_{i=1}^{T}\left(\mathbf{P}_\tau \mathbf{a}_i \mathbf{a}_i^H \mathbf{P}_\tau^H + \mathbf{P}_\tau \mathbf{a}_i \mathbf{v}_i^H + \mathbf{v}_i \mathbf{a}_i^H \mathbf{P}_\tau^H + \mathbf{v}_i \mathbf{v}_i^H\right) \\ &\approx \frac{1}{T}\sum_{i=1}^{T}\left(\mathbf{P}_\tau \mathbf{a}_i \mathbf{a}_i^H \mathbf{P}_\tau^H + \mathbf{v}_i \mathbf{v}_i^H\right) \end{aligned} \quad (26)$$

where in the last equation we have made an approximation based on the fact that noise samples and channel coefficients are uncorrelated zero mean Gaussian random variables. Hence for large T we replaced them with their expectations resulting in $\mathbf{0}$. With these at hand, the trace functions can be evaluated by using Equation (24), (25) and (26). By exploiting the properties of the trace function and also the orthogonality of $\mathbf{P}_\tau$ we obtain:

$$\begin{aligned} tr\left(\hat{\mathbf{C}}\mathbf{C_T^{-1}}\right) = \frac{\sigma_v^{-2}}{T}\Bigg[& e_p\, tr\left(\sum_{i=1}^{T} \mathbf{a}_i \mathbf{a}_i^H\right) + tr\left(\sum_{i=1}^{T} \mathbf{v}_i \mathbf{v}_i^H\right) \\ & - e_p^2\, tr\left(\sum_{i=1}^{T} \mathbf{a}_i \mathbf{a}_i^H \mathbf{D}_a\right) - tr\left(\sum_{i=1}^{T} \mathbf{v}_i \mathbf{v}_i^H \mathbf{P}_\tau \mathbf{D}_a \mathbf{P}_\tau^H\right)\Bigg]. \end{aligned}$$

We assume that random variables are uncorrelated in time. It is well known that distribution remains the same under orthonormal transformations. Therefore we realize that $\mathbf{v}_i$ and $\mathbf{w}_i = \frac{1}{\sqrt{e_p}}\mathbf{P}_\tau^H \mathbf{v}_i$ have the same distribution ($\mathcal{N}(0, \sigma_v^2 \mathbf{I})$). However size of the vector changes (size of $\mathbf{I}$ also changes). By this transformation we rewrite the above equation:

$$\begin{aligned} tr\left(\hat{\mathbf{C}}\mathbf{C_T^{-1}}\right) = \frac{\sigma_v^{-2}}{T}\Bigg[& e_p\, tr\left(\sum_{i=1}^{T} \mathbf{a}_i \mathbf{a}_i^H\right) + tr\left(\sum_{i=1}^{T} \mathbf{v}_i \mathbf{v}_i^H\right) \\ & - e_p^2\, tr\left(\sum_{i=1}^{T} \mathbf{a}_i \mathbf{a}_i^H \mathbf{D}_a\right) - e_p\, tr\left(\sum_{i=1}^{T} \mathbf{w}_i \mathbf{w}_i^H \mathbf{D}_a\right)\Bigg]. \end{aligned}$$

Similarly we can derive $tr\left(\hat{\mathbf{C}}\mathbf{C_F^{-1}}\right)$. The term we need in Equation (21) is:

$$tr\left(\hat{\mathbf{C}}\mathbf{C_F^{-1}} - \hat{\mathbf{C}}\mathbf{C_T^{-1}}\right) = \frac{e_p\, \sigma_v^{-2}}{T}\left(e_p \sum_{i=1}^{T} \mathbf{a}_i^H \mathbf{D} \mathbf{a}_i + \sum_{i=1}^{T} \mathbf{w}_i^H \mathbf{D} \mathbf{w}_i\right). \quad (27)$$

where $\mathbf{D} = \mathbf{D}_a - \mathbf{D}_b$ being another diagonal matrix. Each element of $\mathbf{w}_k$ and $\mathbf{a}_k$ are complex Gaussian random variables with mean 0. For $\mathbf{w}_k$, every entry has the variance σ_v^2 while i^{th} entry of $\mathbf{a}_k$ has a variance of $\sigma_{a_i}^2$. The matrix $\mathbf{D}$ being diagonal simplifies the analysis substantially. It prevents the coupling of the cross elements of the vectors. Therefore Equation (27) represents a summation of squares of Gaussian random variables weighted by $\mathbf{D}$. Since $\mathbf{a}_k^H \mathbf{D} \mathbf{a}_k = \sum_{j=1}^{N_p} [\mathbf{D}]_{jj} |a_{kj}|^2$ and $\mathbf{w}_k^H \mathbf{D} \mathbf{w}_k = \sum_{j=1}^{N_p} [\mathbf{D}]_{jj} |w_{kj}|^2$, each loop of (by loop any of the T iterations is meant) Equation (27) is composed of summation of non-identically distributed exponential random variables. One important thing to mention is that in order to consider it as a summation we implicitly assume that $\mathbf{D}$ has all positive elements on its diagonal meaning that $\sigma_{a_i}^2 > \sigma_{b_i}^2 \forall i$. In the previous section that distribution was calculated and given by Equation (13). As we have done in the previous section, let us call this distribution as W_i, and let $W = \sum_{i=1}^{T} W_i$ (all W_i's identically distributed). Since the derivation of the distribution of summation of T of them (W) will be difficult, we will call the CLT again for T being large. Before that we need the mean and variance of W_i which is calculated as:

$$\mu_{W_i} = \frac{e_p}{T}\sum_{i=1}^{N_p}[\mathbf{D}]_{ii}\left(e_p\,\sigma_v^{-2}\sigma_{a_i}^2+1\right) = \frac{e_p}{T}\,f_2, \quad (28)$$

$$\sigma_{W_i}^2 = \frac{e_p^2}{T^2}\sum_{i=1}^{N_p}[\mathbf{D}]_{ii}^2\left(e_p^2\,\sigma_v^{-4}\sigma_{a_i}^4+1\right) = \frac{e_p^2}{T^2}\,f_3. \quad (29)$$

We know that $\dfrac{W - T\mu_{W_i}}{\sigma_{W_i}\sqrt{T}}$ will tend to have a standard normal distribution ($\mathcal{N}(0,1)$) when T is large. Hence PEP can be reformulated as:

$$\begin{aligned} PEP &= Pr\left(\frac{W - T\mu_{W_i}}{\sigma_{W_i}\sqrt{T}} < \frac{\sqrt{T}}{e_p\sqrt{f_3}}\,(f_1 - e_p\,f_2)\right) && (30)\\ &= Q\left(\frac{\sqrt{T}}{e_p\sqrt{f_3}}\,(e_p\,f_2 - f_1)\right). && (31) \end{aligned}$$

And we can use the Chernoff bound for the Q function bounding the PEP as:

$$PEP \leq \frac{1}{2}e^{-\frac{T}{2\,e_p^2\,f_3}(e_p\,f_2 - f_1)^2}. \quad (32)$$

In the general ergodic case, using the CLT, we get for the PEP:

$$PEP = Q\left(\frac{tr\{\mathbf{C_T}\mathbf{C_F^{-1}} - \mathbf{I}\} - \ln\det(\mathbf{C_T}\mathbf{C_F^{-1}})}{\sqrt{\frac{1}{T}tr\{(\mathbf{C_T}\mathbf{C_F^{-1}} - \mathbf{I})^{\mathbf{2}}\}}}\right) \quad (33)$$

from which we see that a mismatch in every path contributes separately to decreasing the PEP when the path delays are well separated (the numerator of the argument of the Q function is a form of the Itakura-Saito distance between covariance matrices).

For the non-ergodic case in which the channel $\mathbf{h}$ remains constant in the T estimates $\hat{\mathbf{h}}_{\mathbf{i}}$, the PEP using the CLT becomes:

$$PEP = E_{\mathbf{h}}\,Q\left(\frac{\mathbf{h}^H\mathbf{A}\mathbf{h} + \sigma_v^2 tr(\mathbf{A}) + \ln(\det\mathbf{C_F}) - \ln(\det\mathbf{C_T})}{\frac{\sigma_v^2}{\sqrt{T}}||\mathbf{A}||_F}\right) \quad (34)$$

where $\mathbf{A} = \mathbf{C_F^{-1}} - \mathbf{C_T^{-1}}$.

IV. Conclusion

In this contribution we derived approximate analytic results for the PEP for PDP-F. To the best of our knowledge, there has not been any work for the computation of PEP for fingerprinting applications so far. Hence the effect of the pulse shape and other parameters on PEP are explicitly shown. As expected we have shown that the PEP decreases with the increasing T. In the asymptotic case, PEP goes to 0 for both of the algorithms investigated. Possible extensions of this work might be to investigate specific cases, e.g., all path amplitudes being equal, and etc.

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