*Future Network and MobileSummit 2010 Conference Proceedings*
*Paul Cunningham and Miriam Cunningham (Eds)*
*IIMC International Information Management Corporation, 2010*
*ISBN: 978-1-905824-16-8*

# Bayesian Equilibria in Slow Fading OFDM Systems with Partial Channel State Information

Sara Akbarzadeh, Laura Cottatellucci, Christian Bonnet
Eurecom, 2229 Route des Cretes, 06904 Sophia Antipolis Cedex, France
E-mail: sara.akbarzadeh@eurecom.fr

**Abstract:** We consider a slow frequency selective fading multiple access channel (MAC) where 2 independent transmitters are simultaneously communicating with a receiver using orthogonal frequency division multiplexing (OFDM) over $N$ subcarriers. Each transmitter has partial knowledge of the channel state. In such a context, the system is inherently impaired by a nonzero outage probability. We propose a low complexity distributed algorithm for joint rate and power allocation aiming at maximizing the individual throughput, defined as the successfully-received-information rate, under a power constraint. As well known, the problem at hand is non-convex with exponential complexity in the number of transmitters and subcarriers. Inspired by effective almost optimum recent results using the duality principle, we propose a low complexity distributed algorithm based on Bayesian games and duality. We show that the Bayesian game boils down to a two-level game, referred to as *per-subcarrier* game and *global* game. The per-subcarrier game reduces to the solution of linear system of equations while the global game boils down to the solution of several constrained submodular games. The provided algorithm determines all the possible Nash equilibria of the game, if they exist.

**Keywords:** Bayesian Games, OFDM, Nash equilibrium

## 1. Introduction

The main role played by OFDM in the 4th generation (4G) mobile networks fueled a very intense research on resource allocation algorithms for ODFM-based wireless networks. The information theoretical foundations of this problem are established in [1]. A large variety of studies deepened different aspects of this topic, just to mention some, joint rate and power allocation rather than suboptimal disjoint approaches, resource allocation for broadcast rather than multiple access channels, slow fading or fast fading channels, different modulation sets, etc. A complete overview exceeds the scope of this work and the interested reader could refer to [2, 3]. Approaches for joint power and rate allocation need to cope with the intrinsic high complexity of the problem which is exponential in both the number of subcarriers $N$ and the number of users. An effective answer to this issue has been proposed in [4] for OFDM system with a large number of subcarriers. In the asymptotic conditions when $N \rightarrow +\infty$ optimal resource allocation can be obtained with linear complexity in $N$ by making use of the properties of time-sharing functions in constrained optimization problems. The approach in [4] has been specialized to several different OFDM scenarios [5]. The next generation wireless network will be characterized by dynamic resource allocation from a common pool while maintaining decentralized control functions, high level of efficiency in the use of resources, and an acceptable signaling level. These requirements for future wireless networks shift the research interest onto distributed resource allocation algorithms (to decentralize the control functions) based on a limited amount of information (to limit the signalling). The Bayesian games provide a possible framework for a rigorous derivation of distributed resource allocation algorithms based on a partial knowledge of the channel state at the transmitters. Applications of the Bayesian game to wireless communications are limited to CDMA systems [6], two-user interference channels [7] and two-user multicarrier interference channels [8]. Bayesian games for OFDM multiple access channels are considered in [9] under the assumption of *fast* fading channels. In this article we consider the joint rate and power allocation in a two-user OFDM system with a large number of subcarriers and partial channel state information at the transmitters for *slow* frequency selective fading. Each transmitter has knowledge of its own link, which can be estimated locally, but no information about the other transmitter power attenuations. In these conditions, the transmitters are interested in maximizing the throughput, i.e. the rate of information successfully received, allowing for outage events. The total throughput of the system satisfies the time sharing conditions in [4] and the duality approach yields optimum resource allocation asymptotically as $N \to +\infty$. However, the complexity of an optimization algorithm is still significantly high. Then, we consider a Bayesian game based on suboptimal dual cost functions. The Bayesian game boils down into *per subcarrier* games and a *global* game. The first games determine Nash equilibria for power and rate allocation parametric in the Lagrangian coefficients of the dual utility functions. The following global game, based on the solution of a set of submodular games, provide the values of the Lagrangian coefficients at the Bayesian Nash equilibria. We propose an algorithm for the search of all the Bayesian Nash equilibria of the game. The performance of the joint power and rate allocation game is assessed and compared to the performance of the optimum power allocation and uniform power allocation for the two cases of complete and partial channel knowledge at the transmitters, respectively. Although for the whole dual game the existence of a Nash equilibrium is not proven, numerical simulations show that a Nash equilibrium exists for all the considered systems over a large range of power constraints.

Due to space constraints, proofs of lemmas, theorems and properties are omitted in this contribution.

## 2. System Model

We consider a frequency selective multiple access channel (MAC) with $K = 2$ independent transmitters and a receiver. Orthogonal frequency division multiplexing (OFDM) modulation over $N$ subcarriers is applied. In each subcarrier the channel is flat fading. The power attenuation of the channel between transmitter $k$ and the receiver over subcarrier $n$ is denoted by $g_k^n$. The channel attenuations take values in a discrete set $\Phi_k^n$ with a certain probability distribution $\gamma_k^n(g_k^n)$. We assume that the channel is block fading, i.e. the channel is constant during the transmission of a codeword and changes from a codeword to the following one. Furthermore, we assume that each transmitter has a perfect knowledge of the channel attenuations of its own link, i.e. transmitter $k$ knows exactly $g_k^n$, $n = 1, \dots N$, and has statistical knowledge of the channel attenuations on all the links, i.e. $\gamma_k^n(g_k^n)$, $k \in \{1, 2\}$ and $n = 1, \dots N$. Note that this is a realistic assumption for time division duplex (TDD) systems without feedback channels where the channels gains $g_k^n$ from transmitter $k$ to the destination can be estimated at the transmitter via the received signal from the destination assuming that the power attenuation in the two directions is identical (reciprocity principle). Let $\mathbb{R}^+$ be the set of nonnegative reals. We denote by $p_k^n \in \mathbb{R}^+$ the power transmitted by user $k$ on subcarrier $n$ and by $R_k^n \in \mathbb{R}^+$ the information rate over the same subcarrier. The signal at the receiver is impaired by additive Gaussian noise with variance $\sigma^2$ and the receiver adopts single user decoding on each subcarrier. When the realizations of the channel attenuation vector and the transmitted power vector on subcarrier $n$ are $\boldsymbol{g}^n = (g_1^n, g_2^n)$ and $\boldsymbol{p}^n = (p_1^n, p_2^n)$, respectively, the maximum achievable rate on subcarrier $n$ by user $k$ is[1]

$$r_k^n(\boldsymbol{p}^n, \boldsymbol{g}^n) = \log\left(1 + \frac{p_k^n g_k^n}{\sigma^2 + \sum_{j \neq k} p_j^n g_j^n}\right). \tag{1}$$

If transmitter $k$ transmits on subcarrier $n$ with a rate $R_k^n$ greater than $r_n^k(\boldsymbol{p}^n, \boldsymbol{g}^n)$ the transmitted information cannot be decoded reliably and an outage event happens. Because of the system assumptions, transmitter $k$ has only statistical knowledge of the interference term $\sum_{j \neq k} p_j^n g_j^n$ which can be arbitrarily large or bounded by a maximum value $I_{\text{MAX},k}$. For any finite rate $R_k^n > \log\left(1 + \frac{p_j^n g_j^n}{\sigma^2 + I_{\text{MAX},k}}\right)$ there is a nonzero outage probability

$$\Pr\left\{R_k^n > \log\left(1 + \frac{p_j^n g_j^n}{\sigma^2 + \sum_{j \neq k} p_j^n g_j^n}\right)\right\}. \tag{2}$$

If the transmitter can tolerate a nonzero information loss[2] and considers too restrictive the guaranteed transmission rate $\log\left(1 + \frac{p_j^n g_j^n}{\sigma^2 + I_{\text{MAX},k}}\right)$, it can transmits at a rate $R_k^n$ to attain a throughput

$$\rho_k^n = R_k^n \Pr\{R_k^n \leq r_k^n(\boldsymbol{g}^n, \boldsymbol{p}^n)\} \tag{3}$$

defined as the the average rate of information that can be successfully transmitted by transmitter $k$ over subchannel $n$.

In this context, we study joint power and rate allocation strategies for a transmitter $k$ under a power constraint for each transmitter[3] $k$

$$\sum_{n=1}^{N} \mathrm{E}_{g_k^n}\{p_k^n(g_k^n)\} \leq \overline{P}_k. \tag{4}$$

## 3. Optimum Joint Power and Rate Allocation

In the case of complete channel state information (CSI) at all the transmitters, it is well known (see e.g. [4]) that the optimum rate allocation is given by $R_k^n = r_k^n(\boldsymbol{g}^n, \boldsymbol{p}^n)$ and the joint source and rate allocation reduces to the power allocation for the following

[1]Throughout this work $\log(\cdot)$ is the natural logarithm and the rates are expressed in nat/sec.

[2]This depends typically on the services supported by the communication, for example voice services can tolerate a certain level of information loss.

[3]In the asymptotic case, $N \to \infty$, this is equivalent to $\sum_{n=1}^{N} p_k^n \leq \overline{P}_k$.

constrained optimization problem

$$\max_{\boldsymbol{p}} v(\boldsymbol{p}, \boldsymbol{g}) \tag{5}$$

$$\text{subject to} \sum_{n=1}^{N} p_k^n \leq \overline{P}_k \qquad k \in \{1, 2\} \tag{6}$$

where $\boldsymbol{p} = (\boldsymbol{p}^1, \dots \boldsymbol{p}^N)$, $\boldsymbol{g} = (\boldsymbol{g}^1, \dots \boldsymbol{g}^N)$ is given, and the objective function is defined as $v(\boldsymbol{p}, \boldsymbol{g}) = \sum_{k=1}^{2} \sum_{n=1}^{N} r_k^n(\boldsymbol{g}^n, \boldsymbol{p}^n)$. This problem is intrinsically non-convex and numerical optimization is difficult. As observed in [4], an exhaustive search would have a complexity exponential in the number of variables which is $2N$. In order to introduce a low complexity solution for this numerical problem we briefly recall the definition of time-sharing condition for an optimization problem of the form (5).

DEFINITION 1 *[4] Let $\mathbf{p}^*$ and $\mathbf{p}^\triangle$ be the optimal solutions of the optimization problem (5) with $\overline{\boldsymbol{P}} = (\overline{P}_1, \overline{P}_2)$ equal to $\overline{\boldsymbol{P}}^* = (\overline{P}_1^*, \overline{P}_2^*)$ and $\overline{\boldsymbol{P}}^\triangle = (\overline{P}_1^\triangle, \overline{P}_2^\triangle)$, respectively. An optimization problem of the form (5) satisfies the time sharing condition if for any $\overline{\boldsymbol{P}}^*$ and $\overline{\boldsymbol{P}}^\triangle$, and for any $0 \leq \nu \leq 1$, there always exists a feasible solution $\mathbf{p}^\diamond$ such that $\sum_n p_k^{n\diamond} \leq \nu \overline{\boldsymbol{P}}^* + (1-\nu)\overline{\boldsymbol{P}}^\triangle$, and $v(\boldsymbol{g}, \boldsymbol{p}^\diamond) \geq \nu v(\boldsymbol{g}, \boldsymbol{p}^*) + (1-\nu) v(\boldsymbol{g}, \boldsymbol{p}^\triangle)$.*

By observing that [4], (I) The dual problem (see e.g. [10] for a definition) of a primary problem of the form (5) has zero duality gap if the primary problem satisfies the time sharing conditions (see Theorem 1 in [4]); and (II) The problem (5) with $v(\boldsymbol{g}, \boldsymbol{p}) = \sum_k \sum_n r_k^n(\boldsymbol{g}^n, \boldsymbol{p}^n)$ satisfies the time sharing condition (see Theorem 2 in [4]) as $N \to \infty$, the optimization (5) reduces to the optimization over the dual problem as $N \to \infty$. The dual problem has linear complexity in the number of subcarriers. Note that the complexity is still exponential in the number of transmitters $K$.

In the case of partial channel knowledge at the transmitters the joint power and rate allocation is solution to the optimization problem

$$\max_{(\boldsymbol{p}, \boldsymbol{R})} u(\boldsymbol{p}, \boldsymbol{R}, \boldsymbol{g}) \tag{7}$$

$$\text{subject to} \sum_{n=1}^{N} p_k^n \leq \overline{P}_k \qquad k \in \{1, 2\} \tag{8}$$

where $\boldsymbol{R} = (\boldsymbol{R}^1, \dots, \boldsymbol{R}^N)$, $\boldsymbol{R}^n = (R_1^n, R_2^n)$, $\boldsymbol{g}_k = (g_k^1, \dots g_k^N)$ and

$$u(\boldsymbol{p}, \boldsymbol{R}, \boldsymbol{g}) = \sum_{k=1}^{2} \sum_{n=1}^{N} \mathrm{E}_{\boldsymbol{g}_k} \rho_k^n(g_k^n, p_k^n(\boldsymbol{g}_k), R_k^n(\boldsymbol{g}_k)). \tag{9}$$

Note that $\rho_k^n(g_k^n, p_k^n(\boldsymbol{g}_k), R_k^n(\boldsymbol{g}_k))$ coincides with the function defined in (3) but here we underline the dependence of the optimization variables $p_k^n$ and $R_k^n$ on $\boldsymbol{g}_k$, the partial knowledge of transmitter $k$ on the channel.

Similarly to the optimization (5), the optimization (7) is not convex and has exponential complexity in the variables $2N$. As in [4], a low complexity approach based on the dual problem can be proposed and justified by the following Theorem 1.

THEOREM 1 *The optimization problem (7) satisfies the time-sharing condition in the limit as $N \to \infty$.*

The proof of this theorem follows along the same lines as Theorem 1 in [4] and is omitted here.

Let us define the Lagrangian

$$L(\boldsymbol{p}, \boldsymbol{R}, \boldsymbol{\lambda}) = u(\boldsymbol{p}, \boldsymbol{R}, \boldsymbol{g}) + \sum_{k=1}^{K} \lambda_k (\overline{P}_k - \sum_{n=1}^{N} p_k^n), \tag{10}$$

with $\boldsymbol{\lambda} = (\lambda_1, \ldots \lambda_K)$, and the dual function

$$q(\boldsymbol{\lambda}) = \max_{(\boldsymbol{p},\boldsymbol{R})} L(\boldsymbol{p}, \boldsymbol{R}, \boldsymbol{\lambda}). \tag{11}$$

The dual optimization problem is defined as

$$\min_{\boldsymbol{\lambda}} q(\boldsymbol{\lambda}) \tag{12}$$

$$\text{subject to } \lambda_k > 0. \tag{13}$$

It is worth noticing that the optimization in (11) boils down to $N$ independent optimization problems

$$\max_{\boldsymbol{p}^n, \boldsymbol{R}^n} \sum_{k=1}^{K} \mathrm{E}_{\boldsymbol{g}^n} \left( \rho_k^n (g_k^n, p_k^n(\boldsymbol{g}_k), R_k^n(\boldsymbol{g}_k)) - \lambda_k p_k^n \right). \tag{14}$$

The optimization (14) focuses on power and rate allocation in a single subcarrier and $p_k^n$ and $R_k^n$ depend only on the knowledge of $g_k^n$. The optimization (14) is still complex. In order to further reduce the complexity of the problem we introduce a Bayesian game.

## 4. Equilibria for Joint Power and Rate Allocation

The previous resource allocation problem can be formulated as a 2-player Bayesian game $\mathcal{G} \equiv (\mathcal{S}, \mathcal{T}, \mathcal{D}, \mathcal{U}, \mathcal{P})$, where $\mathcal{S} \equiv \{1, 2\}$ is the set of players/transmitters, $\mathcal{T} \equiv \mathcal{T}_1 \times \mathcal{T}_2$ is the type set consisting of all possible realizations of the channel attenuation $\boldsymbol{g}$ with $\mathcal{T}_k = \{\boldsymbol{g}_k\}$ being the type set for transmitter $k$, $\mathcal{D}$ is the action set defined by

$$\mathcal{D} \equiv \bigcup_{k=1}^{K} \Big\{ \boldsymbol{d}_k | \boldsymbol{d}_k = (d_k^1, d_k^2, \ldots d_k^N), d_k^n = (R_k^n, p_k^n), R_k^n \in \mathbb{R}_+, p_k^n \in \mathbb{R}_+, \sum_{n=1}^{N} \mathrm{E}_{g_k^n} \{p_k^n\} \leq \overline{P}_k \Big\}. \tag{15}$$

Note that the set of strategies of each transmitter is orthogonal to the strategies of the others and consists of a vector of rate-power pairs, with the powers satisfying the average power constraint (4). In game $\mathcal{G}, \mathcal{U}$ is the set of payoff functions with the payoff for transmitter $k$ defined by

$$\rho_k(\boldsymbol{d}) = \mathrm{E}_{\boldsymbol{g}} \left( \sum_{n=1}^{N} R_k^n(\boldsymbol{g}_k) 1 \left\{ R_k^n(\boldsymbol{g}_k) \leq \log \left( 1 + \frac{p_k^n(\boldsymbol{g}_k) g_k^n}{\sigma^2 + \sum_{j \neq k} p_j^n(\boldsymbol{g}_j) g_j^n} \right) \right\} \right) \tag{16}$$

$$= \mathrm{E}_{\boldsymbol{g}_k} \left( \sum_{n=1}^{N} R_k^n(\boldsymbol{g}_k) \Pr \left\{ R_k^n(\boldsymbol{g}_k) \leq \log \left( 1 + \frac{p_k^n(\boldsymbol{g}_k) g_k^n}{\sigma^2 + \sum_{j \neq k} p_j^n(\boldsymbol{g}_j) g_j^n} \right) \middle| \boldsymbol{g}_k \right\} \right) \tag{17}$$

where $\boldsymbol{d} = (\boldsymbol{d}_1, \boldsymbol{d}_2)$, $1(\mathcal{E})$ is the indicator function equal to 1 if the event $\mathcal{E}$ is verified and equal to zero elsewhere. Finally, in $\mathcal{G}$, $\mathcal{P}$ is the probability set consisting of the probability functions of $g_k^n$.

Similarly to the optimization problem in Section 3., the game $\mathcal{G}$ is not convex and a numerical solution is too demanding. By following the same approach as in Section 3. we look at an approximation of the solutions of game $\mathcal{G}$ by considering the dual game $\mathcal{G}^D \equiv (\mathcal{S}, \mathcal{T}, \mathcal{D}^D, \mathcal{U}^D, \mathcal{P})$, where the set $\mathcal{U}^D$ consists of the cost functions

$$C_k^D(\boldsymbol{\lambda}) = \mathrm{E}_{\boldsymbol{g}_k} \max_{(\boldsymbol{R}_k(\boldsymbol{g}_k), \boldsymbol{p}_k(\boldsymbol{g}_k)) \in \mathcal{D}} L_k(\boldsymbol{p}, \boldsymbol{R}, \boldsymbol{\lambda}) \tag{18}$$

with

$$\begin{aligned} L_k(\boldsymbol{p}, \boldsymbol{R}, \boldsymbol{\lambda}) = & \sum_{n=1}^{N} R_k^n(\boldsymbol{g}_k) \Pr \left\{ R_k^n(\boldsymbol{g}_k) \leq \log \left( 1 + \frac{p_k^n(\boldsymbol{g}_k) g_k^n}{\sigma^2 + \sum_{j \neq k} p_j^n(\boldsymbol{g}_j) g_j^n} \right) \Bigg| \boldsymbol{g}_k \right\} \\ & + \lambda_k \left( \overline{P}_k - \sum_n \mathrm{E}_{g_k^n}\{p_k^n\} \right), \end{aligned} \tag{19}$$

and the action set $\mathcal{D}^D$ is based on the sets $\mathcal{D}_k^D = \{\lambda_k | \lambda_k > 0\}$. The dual game $\mathcal{G}^D$ is convex in $\boldsymbol{\lambda}$. The Nash equilibrium is the vector $\boldsymbol{\lambda}$ such that

$$C_k^D(\lambda_k^*, \boldsymbol{\lambda}_{-\boldsymbol{k}}{}^*) \leq C_k^D(\lambda_k, \boldsymbol{\lambda}_{-\boldsymbol{k}}{}^*), \qquad \forall \lambda_k > 0 \tag{20}$$

$$\boldsymbol{\lambda}_{-\boldsymbol{k}} \equiv (\lambda_1, ..., \lambda_{k-1}, \lambda_{k+1}, ..., \lambda_K). \tag{21}$$

Note that, for each strategy $\boldsymbol{\lambda}$, the solutions of the system given by

$$\max_{(\boldsymbol{R}_k(\boldsymbol{g}_k), \boldsymbol{p}_k(\boldsymbol{g}_k)) \in (D)} \mathrm{E}_{\boldsymbol{g}_k}\{L_k(\boldsymbol{p}, \boldsymbol{R}, \boldsymbol{\lambda})\}, \qquad \forall k \in \mathcal{S}. \tag{22}$$

are required. These solutions are the Nash equilibria of the game $\tilde{\mathcal{G}}_{\boldsymbol{\lambda}} \equiv (\mathcal{S}, \mathcal{T}, \mathcal{D}, \tilde{\mathcal{U}}, \mathcal{P})$, where the set of utility functions $\tilde{\mathcal{U}}$ consists of the functions $\mathrm{E}_{\boldsymbol{g}_k}\{L_k(\boldsymbol{p}, \boldsymbol{R}, \boldsymbol{\lambda})\}, k \in \mathcal{S}$.

By following the same lines as in the optimization problem, game $\tilde{\mathcal{G}}_{\boldsymbol{\lambda}}$ can be decomposed into $N$ games, one for each subcarrier. Therefore, the solution of the game $\mathcal{G}^D$ can be decomposed into the solutions of two level of games, a game for each subcarrier whose solutions are functions of the strategy $\boldsymbol{\lambda}$, and a global game based on the solutions of the games for the subcarriers. In the following, we analytically define these two level of games.

***Per Subcarrier Game*** – We define $N$ independent games, one for each subcarrier, in the parameter $\boldsymbol{\lambda}$, $\mathcal{G}_{\boldsymbol{\lambda}}^n \equiv (\mathcal{S}, \mathcal{T}^n, \mathcal{D}^n, \mathcal{U}_{\boldsymbol{\lambda}}^n, \mathcal{P}^n)$, where the type set of transmitter $k$ is the set of possible realizations of $g_k^n$ and $\mathcal{T}^n$ is the product of the type sets of all transmitters. The set of actions $\mathcal{D}^n$ is based on the feasible strategies of user $k$ on subcarrier $n$, $\mathcal{D}_k^n \equiv (d_k^n | d_k^n = (R_k^n, p_k^n), R_k^n, p_k^n \in \mathbb{R}_+)$. The set of payoffs $\mathcal{U}_{\boldsymbol{\lambda}}^n$ is given by

$$q_k^n(\boldsymbol{d}^n; \boldsymbol{\lambda}) = \mathrm{E}_{\boldsymbol{g}_k} \left\{ R_k^n(\boldsymbol{g}_k) \Pr \left\{ R_k^n(\boldsymbol{g}_k) \leq \log \left( 1 + \frac{p_k^n(\boldsymbol{g}_k) g_k^n}{\sigma^2 + \sum_{j \neq k} p_j^n(\boldsymbol{g}_j) g_j^n} \right) \Bigg| \boldsymbol{g}_k \right\} - \lambda_k p_k^n(\boldsymbol{g}_k) \right\} \tag{23}$$

Finally, the probability set $\mathcal{P}^n$ consists of the probability of channel attenuations $g_k^n$ for $k = 1, 2$.

***Global Game*** – It is the game defined by $\mathcal{G}_{\text{glob}} \equiv (\mathcal{S}, \mathcal{D}_{\text{glob}}, \mathcal{U}^{\mathcal{D}})$ where the cost function set is $\mathcal{U}^{\mathcal{D}}$. The action set is

$$\mathcal{D}_{\text{glob}} \equiv \left( \boldsymbol{\lambda} \middle| \lambda_k \in \mathbb{R}_+, k = 1, 2, \text{ and } \overline{P}_k - \sum_{n=1}^{N} \mathrm{E}_{g_k^n}\{p_k^n(g_k^n, \boldsymbol{\lambda})\} \geq 0 \right),$$

where $p_k^n(g_k^n, \boldsymbol{\lambda})$ are the solutions of the per subcarrier games parametric in $\boldsymbol{\lambda}$. Then, the definition of $\mathcal{D}_{\text{glob}}$ implies that only values of $\boldsymbol{\lambda}$ yielding solutions for $\mathcal{G}_{\boldsymbol{\lambda}}^n$ satisfying the constraint $\overline{P}_k - \sum_{n=1}^{N} \mathrm{E}_{g_k^n}\{p_k^n(g_k^n, \boldsymbol{\lambda})\} \geq 0$ are of interest for the game $\mathcal{G}_{\text{glob}}$. The cost functions $C_k^{\mathcal{D}}(\boldsymbol{\lambda})$ can be expressed as

$$C_k^{\mathcal{D}}(\boldsymbol{\lambda}) = \sum_{n=1}^{N} q_k^n(\overline{\boldsymbol{d}}^n(\boldsymbol{\lambda}); \boldsymbol{\lambda}) + \lambda_k \overline{P}_k \qquad k = 1, 2 \tag{24}$$

with $\overline{\boldsymbol{d}}^n(\boldsymbol{\lambda})$ being the solution of the per subcarrier game $\mathcal{G}_{\boldsymbol{\lambda}}^n$. In the following subsections 4.1 and 4.2 we analyze independently the games $\mathcal{G}_{\text{sub}}^n$ and the global game $\mathcal{G}_{\boldsymbol{\lambda}}$, respectively. In Section 5. we provide an algorithm to determine all the Bayesian-Nash equilibria.

### *4.1 Per Subcarrier Games* $\mathcal{G}_{\boldsymbol{\lambda}}^n$

For the sake of notation, we concatenate the power vectors of the two transmitters to form a 4-dimensional column vector, $\boldsymbol{p} = [p_{11}, p_{12}, p_{21}, p_{22}]^T$. The same notation is used later for channel gains and their probabilities, i.e. $\boldsymbol{g} = [g_{11}, g_{12}, g_{21}, g_{22}]^T$ and $\boldsymbol{\gamma} = [\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}]^T$. Moreover, we assume that the number of fading states per subcarrier per transmitter is equal to 2. Thus, $\mathcal{S} \equiv \{1, 2\}$, the type set of transmitter $k$ on subcarrier $n$ is $\mathcal{T}^n \equiv \{g_{k,1}^n, g_{k,2}^n\}$ and the corresponding probability set is $\mathcal{P}^n \equiv \{\gamma_{k1}^n, \gamma_{k2}^n\}$. We denote the user of interest by subscript $k \in \{1, 2\}$ and the interfering user by subscript $m \in \{1, 2\}, m \neq k$.

In this section, we focus on the resource allocation of any arbitrary subcarrier. The payoff function (23) specializes as follows

$$q_k^n(\boldsymbol{p}^n, \boldsymbol{R}^n; \boldsymbol{\lambda}) = \gamma_{k1}^n \mathcal{W}_{k1}^n(\boldsymbol{p}^n, \boldsymbol{R}^n, \boldsymbol{\lambda}) + \gamma_{k2}^n \mathcal{W}_{k2}^n(\boldsymbol{p}^n, \boldsymbol{R}^n, \boldsymbol{\lambda}) \tag{25}$$

with

$$\mathcal{W}_{kh}^n(\boldsymbol{R}^n, \boldsymbol{p}^n, \boldsymbol{\lambda}) = R_{kh}^n \left( \gamma_{m1}^n \mathbb{1}\left( R_{kh}^n \leq \log\left(1 + \frac{p_{kh}^n g_{kh}^n}{\sigma^2 + p_{m1}^n g_{m1}^n}\right)\right) + \gamma_{m2}^n \mathbb{1}\left( R_{kh}^n \leq \log\left(1 + \frac{p_{kh}^n g_{kh}^n}{\sigma^2 + p_{m2}^n g_{m2}^n}\right)\right)\right) - \lambda_k p_{kh}^n. \tag{26}$$

Here, $R_{kh}^n = R_k^n(g_{kh}^n, \boldsymbol{\lambda})$ and $p_{kh}^n = p_k^n(g_{kh}^n, \boldsymbol{\lambda})$ are the rate and power allocated by transmitter $k$ on subcarrier $n$ when the channel realization is $g_{kh}^n$, and $R^n$ and $p^n$ are the pairs $(R_{k1}^n, R_{k2}^n)$ and $(p_{k1}^n, p_{k2}^n)$. Throughout this section, we consider a single subcarrier and omit the index $n$.

By considering the possible values of the indicator functions, (26) boils down to the following piecewise function

$$\mathcal{W}_{kh}(\boldsymbol{p}, \boldsymbol{R}, \boldsymbol{\lambda}) = \begin{cases} R_{kh} - \lambda_k p_{kh} & \frac{g_{kh} p_{kh}}{e^{R_{kh}} - 1} - \sigma^2 \geq g_{mg} p_{mg} \\ \gamma_{ms} R_{kh} - \lambda_k p_{kh} & g_{m\ell} p_{m\ell} \leq \frac{g_{kh} p_{kh}}{e^{R_{kh}} - 1} - \sigma^2 \leq g_{mg} p_{mg} \\ -\lambda_k p_{kh} & \frac{g_{kh} p_{kh}}{e^{R_{kh}} - 1} - \sigma^2 \leq g_{m\ell} p_{m\ell} \end{cases} \tag{27}$$

where $g_{mg}p_{mg} = \max(g_{m1}p_{m1}, g_{m2}p_{m2})$ and $g_{m\ell}p_{m\ell} = \min(g_{m1}p_{m1}, g_{m2}p_{m2})$. In other words, index $g$ and $\ell$ denote the ***greatest*** and the ***lowest*** interference, respectively.

When we aim at maximizing $\mathcal{W}_{kh}$, it is straightforward to recognize that the decision variables $p_{kh}$ and $R_{kh}$ are not independent, but for a certain value $p_{kh}$ of the transmitted power, $\mathcal{W}_{kh}$ is maximized for $R_{kh} = \log(1+\frac{p_{kh}g_{kh}}{N0+p_{m*}g_{m*}})$, being $* \in \{g;\ell\}$. Therefore, our problem reduces to consider the following functions in $\boldsymbol{p}$ and $\boldsymbol{\lambda}$

$$\mathcal{W}_{kh}(\boldsymbol{p},\boldsymbol{\lambda}) = \begin{cases} \text{(I)} & \log(1+\frac{g_{kh}p_{kh}}{g_{mg}p_{mg}+\sigma^2}) - \lambda_k p_{kh}. \\ \text{(II)} & \gamma_{m\ell}\log(1+\frac{g_{kh}p_{kh}}{g_{m\ell}p_{m\ell}+\sigma^2}) - \lambda_k p_{kh}. \\ \text{(III)} & -\lambda_k p_{kh}. \end{cases} \tag{28}$$

Taking into account the dependency of the decision variables $R_{kh}$ and $p_{kh}$, the payoff function (25) reduces to

$$q_k(\boldsymbol{p},\boldsymbol{\lambda}) = \gamma_{k1}\mathcal{W}_{k1}(\boldsymbol{p},\boldsymbol{\lambda}) + \gamma_{k2}\mathcal{W}_{k2}(\boldsymbol{p},\boldsymbol{\lambda}) \tag{29}$$

Note that $\mathcal{W}_{kh}(\boldsymbol{p},\boldsymbol{\lambda})$ depends on $p_{kh}$, the power to be allocated in the channel state $g_{kh}$. Now, we assume that the power allocation of the interfering user and consequently the interference pairs $(g_{m1}p_{m1}, g_{m2}p_{m2})$ are known. Therefore, the greatest and the lowest interference can be obtained, i.e. $p_{mg}g_{mg}$ and $p_{m\ell}g_{m\ell}$. Then, the best response of user $k$ to this interference would be given by

$$\tilde{p}_{kh} = \arg\max_{p_{kh}} \mathcal{W}_{kh}(\boldsymbol{p},\boldsymbol{\lambda}), \qquad \forall h \in \{1,2\} \tag{30}$$

and the Nash equilibrium $\overline{\boldsymbol{p}}$ of the per subcarrier game satisfies the following condition

$$q_k(\overline{p}_k,\overline{p}_m,\boldsymbol{\lambda}) \geq q_k(p_k,\overline{p}_m,\boldsymbol{\lambda}) \ \forall k,m \in \{1,2\}, k \neq m, \forall p_k \in \mathcal{R}_+. \tag{31}$$

Let us denote the three branches of the function (28) by $\mathcal{W}_{kh}^{(x)}$ where $x \in \{(I),(II),(III)\}$. In addition, we denote the best response in a specific branch $x$ by $\tilde{p}_{kh}^{(x)}$ and by $\boldsymbol{p}_{-kh}$ the vector obtained from $\boldsymbol{p}$ by suppressing $p_{kh}$. In the following, we define three disjoint regions for the best response $\tilde{p}_{kh}$ corresponding to the three branches of the function, i.e. $\mathcal{R}_{kh}^{(x)}, x \in \{(I),(II),(III)\}$. In other words, if $\tilde{p}_{kh} = \arg\max_{p_{kh}} \mathcal{W}_{kh}(\boldsymbol{p},\boldsymbol{\lambda})$ belongs to the region $\mathcal{R}_{kh}^{(x)}$, it satisfies $\tilde{p}_{kh} = \arg\max_{p_{kh}} \mathcal{W}_{kh}^{(x)}(\boldsymbol{p},\boldsymbol{\lambda})$.

The following disjoint regions for the best response $\tilde{p}_{kh}$ can be defined: (1) $\mathcal{R}_{kh}^{(I)}$ where $\mathcal{W}_{kh}^{(I)}$ is the maximizing function, i.e. $\tilde{p}_{kh} = \arg\max_{p_{kh}} \mathcal{W}_{kh}^{(I)}(\boldsymbol{p},\boldsymbol{\lambda})$. The function $\mathcal{W}_{kh}^{(I)}((\tilde{p}_{kh}^{(I)},\boldsymbol{p}_{-kh}),\boldsymbol{\lambda})$ should be positive and the following inequality should be satisfied: $\mathcal{W}_{kh}^{(I)}((\tilde{p}_{kh}^{(I)},\boldsymbol{p}_{-kh}),\boldsymbol{\lambda}) > \mathcal{W}_{kh}^{(II)}(\tilde{p}_{kh}^{(II)},\boldsymbol{\lambda})$; (2) $\mathcal{R}_{kh}^{(II)}$ where $\mathcal{W}_{kh}^{(II)}$ is the maximizing function, i.e. $\tilde{p}_{kh} \in \arg\max_{p_{kh}} \mathcal{W}_{kh}^{(II)}(\boldsymbol{p},\boldsymbol{\lambda})$. The function $\mathcal{W}_{kh}^{(II)}((\tilde{p}_{kh}^{(II)},\boldsymbol{p}_{-kh}),\boldsymbol{\lambda})$ should be positive and the following inequality should be satisfied: $\mathcal{W}_{kh}^{(II)}((\tilde{p}_{kh}^{(II)},\boldsymbol{p}_{-kh}),\boldsymbol{\lambda}) > \mathcal{W}_{kh}^{(I)}(\tilde{p}_{kh}^{(I)},\boldsymbol{\lambda})$; (3) $\mathcal{R}_{kh}^{(III)}$ where both functions $\mathcal{W}_{kh}^{(I)}((\tilde{p}_{kh}^{(I)},\boldsymbol{p}_{-kh}),\boldsymbol{\lambda})$ and $\mathcal{W}_{kh}^{(II)}((\tilde{p}_{kh}^{(II)},\boldsymbol{p}_{-kh}),\boldsymbol{\lambda})$ are non-positive. The maximum value of $\mathcal{W}_{kh}$ is equal to zero and $\tilde{p}_{kh} = 0$.

Note that, in a single piece, the function $\mathcal{W}_{kh}^{(x)}(\boldsymbol{p},\boldsymbol{\lambda})$ is a concave function of $p_{kh}$. Therefore, the argument $\tilde{p}_{kh}$ which maximizes $\mathcal{W}_{kh}(\boldsymbol{p},\boldsymbol{\lambda})$ in each piece can be obtained directly by the first derivative. The resulting best response of player $k$ is

$$\tilde{p}_{kh} = \begin{cases} \frac{1}{\lambda_k} - \frac{g_{mg}}{g_{kh}}p_{mg} - \frac{\sigma}{g_{kh}} & \in \mathcal{R}_{kh}^{(I)} \\ \frac{\gamma_{m\ell}}{\lambda_k} - \frac{g_{m\ell}}{g_{kh}}p_{m\ell} - \frac{\sigma}{g_{kh}} & \in \mathcal{R}_{kh}^{(II)} \\ 0 & \in \mathcal{R}_{kh}^{(III)} \end{cases} \tag{32}$$

$$\mathcal{W}_{kh}((\tilde{\boldsymbol{p}}_{kh},\boldsymbol{p}_{-kh}),\boldsymbol{\lambda}) = \begin{cases} \log \frac{g_{kh}}{\lambda_k(g_{mg}p_{mg}+\sigma)} - 1 + \frac{\lambda_k}{g_{kh}}(g_{mg}p_{mg}+\sigma) & \boldsymbol{p}_m \in \mathcal{R}_{kh}^{(I)} \\ \log \frac{\gamma_{m\ell} g_{kh}}{\lambda_k(g_{m\ell}p_{m\ell}+\sigma)} - \gamma_{m\ell} + \frac{\lambda_k}{g_{kh}}(g_{m\ell}p_{m\ell}+\sigma) & \boldsymbol{p}_m \in \mathcal{R}_{kh}^{(II)} \\ 0 & \boldsymbol{p}_m \in \mathcal{R}_{kh}^{(III)} \end{cases} \tag{33}$$

$$\mathcal{R}_{kh}^{(I)} \equiv \{\boldsymbol{p}_m | \tilde{p}_{kh} > 0, p_{mg}g_{mg} \leq \frac{\tilde{p}_{kh}g_{kh}}{e^{\lambda_k \tilde{p}_{kh}} - 1} - \sigma, \log(\frac{p_{m\ell}g_{m\ell}+\sigma}{p_{mg}g_{mg}+\sigma}) - 1 - \log\gamma_{m\ell} + \gamma_{m\ell} + \frac{\lambda_k}{g_{kh}}(p_{mg}g_{mg} - p_{m\ell}g_{m\ell}) > 0\}$$

$$\mathcal{R}_{kh}^{(II)} \equiv \{\boldsymbol{p}_m | \tilde{p}_{kh} > 0, p_{m\ell}g_{m\ell} \leq \frac{\tilde{p}_{kh}g_{kh}}{e^{\frac{\lambda_k \tilde{p}_{kh}}{\gamma_{m\ell}}} - 1} - \sigma, \log(\frac{p_{m\ell}g_{m\ell}+\sigma}{p_{mg}g_{mg}+\sigma}) - 1 - \log\gamma_{m\ell} + \gamma_{m\ell} + \frac{\lambda_k}{g_{kh}}(p_{mg}g_{mg} - p_{m\ell}g_{m\ell}) < 0\}$$

$$\mathcal{R}_{kh}^{(III)} \equiv \{\boldsymbol{p}_m | p_{mg}g_{mg} \geq \frac{\tilde{p}_{kh}g_{kh}}{e^{\lambda_k \tilde{p}_{kh}} - 1} - \sigma, p_{m\ell}g_{m\ell} \geq \frac{\tilde{p}_{kh}g_{kh}}{e^{\frac{\lambda_k \tilde{p}_{kh}}{\gamma_{m\ell}}} - 1} - \sigma\} \tag{34}$$

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & \{0,0,\frac{g_{21}}{g_{11}},\frac{g_{21}}{g_{11}},0\} & \{0,\frac{g_{22}}{g_{11}},0,0,\frac{g_{22}}{g_{11}}\} \\ 0 & 1 & \{0,0,\frac{g_{21}}{g_{12}},\frac{g_{21}}{g_{12}},0\} & \{0,\frac{g_{22}}{g_{12}},0,0,\frac{g_{22}}{g_{12}}\} \\ \{0,\frac{g_{11}}{g_{21}},0,0,\frac{g_{11}}{g_{21}}\} & \{0,0,\frac{g_{12}}{g_{21}},\frac{g_{12}}{g_{21}},0\} & 1 & 0 \\ \{0,\frac{g_{11}}{g_{22}},0,0,\frac{g_{11}}{g_{22}}\} & \{0,0,\frac{g_{12}}{g_{22}},\frac{g_{12}}{g_{22}},0\} & 0 & 1 \end{bmatrix} \tag{35}$$

The corresponding utility of user $k$, $\mathcal{W}_{kh}((\tilde{\boldsymbol{p}}_{kh},\boldsymbol{p}_{-kh}),\boldsymbol{\lambda})$, is given by the piecewise function in (33) at the top of next page. Note that the pieces implies constraints on the power value $\tilde{p}_{kh}$ specified in (34) at the top of next page.

The pieces are defined by conditions which are functions of the interfering elements $(p_{m\ell}g_{m\ell}, p_{mg}g_{mg})$. Now, by making use of the best responses we determine the Nash equilibria for the per subcarrier game as the intersections of the best responses.

The following theorem provides the set of all power allocations which jointly maximize $\{\mathcal{W}_{11},\mathcal{W}_{12},\mathcal{W}_{21},\mathcal{W}_{22}\}$. These are the Nash equilibria of the per subcarrier game $\mathcal{G}_{\text{sub}}^n$.

THEOREM 2 *The per subcarrier game for a 2-transmitters network with the best responses defined as (30), has a unique NE if and only if the two following conditions are satisfied: (I) for a pair* $(\lambda_1,\lambda_2)$*, the matrix* $\boldsymbol{M}$ *defined in (35) at the top of next page is full rank and (II) the unique solution* $\overline{\boldsymbol{p}}$ *of the system of equation*

$$\overline{\boldsymbol{p}} = \boldsymbol{M}^{-1}\boldsymbol{b}(\lambda_1,\lambda_2), \tag{36}$$

*with*

$$\boldsymbol{b}(\lambda_1,\lambda_2) = \begin{bmatrix} \{0, \frac{1}{\lambda_1} - \frac{\sigma}{g_{11}}, \frac{\gamma_{21}}{\lambda_1} - \frac{\sigma}{g_{11}}, \frac{1}{\lambda_1} - \frac{\sigma}{g_{11}}, \frac{\gamma_{22}}{\lambda_1} - \frac{\sigma}{g_{11}}\} \\ \{0, \frac{1}{\lambda_1} - \frac{\sigma}{g_{12}}, \frac{\gamma_{21}}{\lambda_{12}} - \frac{\sigma}{f_{12}}, \frac{1}{\lambda_1} - \frac{\sigma}{g_{12}}, \frac{\gamma_{22}}{\lambda_1} - \frac{\sigma}{g_{12}}\} \\ \{0, \frac{1}{\lambda_2} - \frac{\sigma}{g_{21}}, \frac{\gamma_{12}}{\lambda_2} - \frac{\sigma}{g_{21}}, \frac{1}{\lambda_2} - \frac{\sigma}{g_{21}}, \frac{\gamma_{11}}{\lambda_2} - \frac{\sigma}{g_{21}}\} \\ \{0, \frac{1}{\lambda_2} - \frac{\sigma}{g_{22}}, \frac{\gamma_{12}}{\lambda_2} - \frac{\sigma}{g_{22}}, \frac{1}{\lambda_2} - \frac{\sigma}{g_{22}}, \frac{\gamma_{11}}{\lambda_2} - \frac{\sigma}{g_{22}}\} \end{bmatrix}$$

*belongs to the regions defined by*

$$\mathcal{R}_{\boldsymbol{p}} = \begin{bmatrix} \{\mathcal{R}_{11}^{(III)}, (\mathcal{R}_{11}^{(I)}|A2), (\mathcal{R}_{11}^{(II)}|A2), (\mathcal{R}_{11}^{(I)}|\hat{A}2), (\mathcal{R}_{11}^{(II)}|\hat{A}2)\} \\ \{\mathcal{R}_{12}^{(III)}, (\mathcal{R}_{12}^{(I)}|A2), (\mathcal{R}_{12}^{(II)}|A2), (\mathcal{R}_{12}^{(I)}|\hat{A}2), (\mathcal{R}_{12}^{(II)}|\hat{A}2)\} \\ \{\mathcal{R}_{21}^{(III)}, (\mathcal{R}_{21}^{(I)}|A1), (\mathcal{R}_{21}^{(II)}|A1), (\mathcal{R}_{21}^{(I)}|\hat{A}1), (\mathcal{R}_{21}^{(II)}|\hat{A}1)\} \\ \{\mathcal{R}_{22}^{(III)}, (\mathcal{R}_{22}^{(I)}|A1), (\mathcal{R}_{22}^{(II)}|A1), (\mathcal{R}_{22}^{(I)}|\hat{A}1), (\mathcal{R}_{22}^{(II)}|\hat{A}1)\} \end{bmatrix} \tag{37}$$

*Here,* $A1 \equiv \{(\overline{p}_{11},\overline{p}_{12})|\overline{p}_{11}g_{11} > \overline{p}_{12}g_{12}\}$ *and* $A2 \equiv \{(\overline{p}_{21},\overline{p}_{22})|\overline{p}_{22}g_{22} > \overline{p}_{21}g_{21}\}$*. The complementary regions are denoted by* $\hat{A}1$ *and* $\hat{A}2$*. The notation* $\{.\}$ *with several variables suggests that the corresponding element takes one of the values. In addition, the notation* $(.|.)$

*conditions the region whereto the power of user of interest belongs, to an specific order of the interfering signal. In each row, there is a one-to-one correspondence between the values in* $\{.\}$ *of* $\boldsymbol{M}, \boldsymbol{b}$, *and* $\mathcal{R}_{\boldsymbol{p}}$.

Note that (36) provides the intersection of the best responses (32).

*Remark 1:* The condition that matrix $\boldsymbol{M}$ is full rank implies that the matrix $\boldsymbol{M}$ cannot be symmetric. Additionally, if $\text{rank}(\boldsymbol{M}) = \text{rank}(\overline{\boldsymbol{M}}) < 4$, where $\overline{\boldsymbol{M}}$ is the matrix built by concatenating the two matrices $\boldsymbol{M}$ and $\boldsymbol{b}$, i.e. $\left[\boldsymbol{M}|\boldsymbol{b}\right]$, the system of equations $\boldsymbol{MP} = \boldsymbol{b}(\lambda_1, \lambda_2)$ admits infinite solutions. They are Nash equilibria if they also belong to $\mathcal{R}_{\boldsymbol{p}}$. No NE exists if $\text{rank}(\boldsymbol{M}) \neq \text{rank}(\overline{\boldsymbol{M}})$.

*Remark 2:* Taking into account the structure of vector $\boldsymbol{b}(\lambda_1, \lambda_2)$, in a given channel state $\boldsymbol{g} = \begin{bmatrix} g_{11} & g_{12} & g_{21} & g_{22} \end{bmatrix}^T$, the solution to system (36) can be expressed as a function of the pair $(\lambda_1, \lambda_2)$. Let us denote $\boldsymbol{M}^{-1}$ by $\boldsymbol{A}$. We rewrite (36) as

$$\begin{bmatrix} \overline{p}_{11} \\ \overline{p}_{12} \\ \overline{p}_{21} \\ \overline{p}_{22} \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \hline \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{array}\right] \begin{bmatrix} b_{11}(\frac{1}{\lambda_1}, \sigma) \\ b_{12}(\frac{1}{\lambda_1}, \sigma) \\ b_{21}(\frac{1}{\lambda_2}, \sigma) \\ b_{22}(\frac{1}{\lambda_2}, \sigma) \end{bmatrix} \tag{38}$$

where $\boldsymbol{A}_i, i \in \{1,2,3,4\}$ are $2 \times 2$ matrices. It can be verified that all the non-zero elements of $\boldsymbol{A}_1$ and $\boldsymbol{A}_4$ are positive and all the non-zero elements of $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ are negative.

PROPOSITION 1 *Any non-zero power allocation of transmitter* $k$ *at the Nash equilibrium of* $\mathcal{G}^n_{\boldsymbol{\lambda}}$ *is linear in the pair* $(\frac{1}{\lambda_1}, \frac{1}{\lambda_2})$. *Let us assume that the solution* $\overline{\boldsymbol{p}}$ *satisfies condition (II) of Theorem 2 with* $\mathcal{R}_{\boldsymbol{p}} \equiv \mathcal{R}^{\#}_{\boldsymbol{p}}$, *being* $\mathcal{R}^{\#}_{\boldsymbol{p}}$ *one of the possible regions defined by* $\mathcal{R}_{\boldsymbol{p}}$. *The corresponding region for vector* $\boldsymbol{\lambda} = [\lambda_1, \lambda_2]$ *is referred to as* $\mathcal{R}^{\#}_{\boldsymbol{\lambda}} = \{(\lambda_1, \lambda_2) | \lambda_1 > 0, \lambda_2 > 0, \overline{\boldsymbol{p}}(\lambda_1, \lambda_2) \in \mathcal{R}^{\#}_{\boldsymbol{p}}\}$. *Then*

$$\overline{\boldsymbol{p}}(\lambda_1, \lambda_2) = \begin{bmatrix} \alpha_{11} & -\beta_{11} \\ \alpha_{12} & -\beta_{12} \\ -\beta_{21} & \alpha_{21} \\ -\beta_{22} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} \\ \frac{1}{\lambda_2} \end{bmatrix} - \begin{bmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{bmatrix} \sigma \tag{39}$$

*for* $(\lambda_1, \lambda_2) \in \mathcal{R}^{\#}_{\boldsymbol{\lambda}}$ *and being* $\alpha_{i,j}, \beta_{ij}, c_{ij}$, *with* $i, j \in \{1,2\}$ *positive and depending on* $\boldsymbol{g}, \boldsymbol{\gamma}$, *and* $\mathcal{R}^{\#}_{\boldsymbol{p}}$.

Let us denote the set of all possible regions $\mathcal{R}_p$ defined in (37) by $\mathcal{Y}$. The cardinality of the set is $\mathcal{N}_Y$. We index the regions in an arbitrary order with a number between 1 and $\mathcal{N}_Y$ and denote the index by $y$.

In the rest of this section, in order to simplify the analysis, we concentrate on a single region $\mathcal{R}^y_p, y \in \{1, ..., \mathcal{N}_Y\}$. The following analysis holds in general for any arbitrary region. Hereafter, the region index $y$ is considered as a parameter of the functions whenever a single region is intended, e.g.. $\mathcal{W}_{kh}(y, \overline{\boldsymbol{p}}(\boldsymbol{\lambda}), \boldsymbol{\lambda})$.

Let us assume that $\overline{\boldsymbol{p}}_m \in \mathcal{R}^{(I)}_{kh}$. The value of the corresponding function in (33) is

$$\mathcal{W}_{kh}(y, \overline{\boldsymbol{p}}(\boldsymbol{\lambda}), \boldsymbol{\lambda}) = \log \frac{g_{kh}}{\lambda_k (g_{mg}(\frac{\alpha_{mg}}{\lambda_m} - \frac{\beta_{mg}}{\lambda_k} - c_{mg}\sigma) + \sigma)} - 1 + \frac{\lambda_k}{g_{kh}} (g_{mg}(\frac{\alpha_{mg}}{\lambda_m} - \frac{\beta_{mg}}{\lambda_k} - c_{mg}\sigma) + \sigma))$$

where, based on Proposition 1, the value of $\overline{\boldsymbol{p}}_m$ is replaced by $\frac{\alpha_{mg}}{\lambda_m} - \frac{\beta_{mg}}{\lambda_k} - c_{mg}\sigma$. Two properties of function $\mathcal{W}_{kh}(y, \overline{\boldsymbol{p}}(\boldsymbol{\lambda}), \boldsymbol{\lambda})$, namely submodularity (decreasing differences) and convexity, are presented in the following two lemmas. We will elaborated more on these properties in the next subsection.

LEMMA 1 *The continuous and twice differentiable function $\mathcal{W}_{kh}(\boldsymbol{\lambda})$ has decreasing differences property.*

LEMMA 2 *For a fix $\lambda_m$, the continuous and twice differentiable function $\mathcal{W}_{kh}(y, \overline{\boldsymbol{p}}(\boldsymbol{\lambda}), \boldsymbol{\lambda})$ is concave in $\lambda_k$ when the pair $(\lambda_k, \lambda_m)$ satisfies the following condition*

$$\{(\lambda_k, \lambda_m)|\lambda_k > 0, \lambda_m > 0, \overline{\boldsymbol{p}}(\lambda_k, \lambda_m) \geq 0\}. \tag{40}$$

Note that the condition on the power $\overline{\boldsymbol{p}}(\lambda_k, \lambda_m)$ in (40) is implied by physical reasons and it is not restrictive for our study. In the following section, we consider a global game per each possible region $\mathcal{R}_p$ and we discuss the existence of a Nash equilibrium in that region.

### *4.2 Global Game*

We consider a network wherein all subcarriers have the same channel state ditribution. In such a network the global game utility function (24) boils down to

$$C_k(\boldsymbol{\lambda}) = NL_k(\overline{\boldsymbol{d}}^n(\boldsymbol{\lambda}); \boldsymbol{\lambda}) + \lambda_k \overline{P}_k \qquad k = 1, 2 \tag{41}$$

Let us consider the above problem in a single region $\mathcal{R}_p^y, y \in \{1, ..., \mathcal{N}_Y\}$. In order to specialize all the functions as the ones of this region we add the variable $y$ as a parameter to all the functions, e.g. $L_k(y, \overline{\boldsymbol{d}}^n(\boldsymbol{\lambda}); \boldsymbol{\lambda})$. From (41) and (29) we have

$$C_k(y, \boldsymbol{\lambda}) \;=\; N\Big(\gamma_{k1}\mathcal{W}_{k1}(y, \overline{\boldsymbol{p}}(\boldsymbol{\lambda}), \boldsymbol{\lambda}) + \gamma_{k2}\mathcal{W}_{k2}(y, \overline{\boldsymbol{p}}(\boldsymbol{\lambda}), \boldsymbol{\lambda})\Big) + \lambda_k \overline{P}_k \qquad k \;=\; 1, 2 \tag{42}$$

For further studies, we define the global game per region by $\mathcal{G}_{\text{glob}}^y \equiv (\mathcal{S}, \mathcal{D}_{\text{glob}}^y, \mathcal{U}_{\text{glob}}^y)$ where the cost functions $\mathcal{U}_{\text{glob}}^y$ are defined in (42) and the action set is $\mathcal{D}_{\text{glob}}^y \equiv \Big(\boldsymbol{\lambda}|\overline{\boldsymbol{p}}(\boldsymbol{\lambda}) \in \mathcal{R}_p^y, \lambda_k \in \mathbb{R}_+, k = 1, 2\Big)$.

Let us define a relaxed game $\mathcal{G}_{\text{relaxed}}^y \equiv (\mathcal{S}, \mathcal{D}_{\text{relaxed}}^y, \mathcal{U}_{\text{glob}}^y)$ obtained by relaxing the condition of type $\mathcal{W}_{kh}^{(I)} \leq \mathcal{W}_{kh}^{(II)}$(or $\mathcal{W}_{kh}^{(I)} \geq \mathcal{W}_{kh}^{(II)}$) and the condition of type $\overline{p}_{m1}g_{m1} \geq \overline{p}_{m2}g_{m2}$ (or $\overline{p}_{m1}g_{m1} \leq \overline{p}_{m2}gm2$) from the set $\mathcal{D}_{\text{glob}}^y$. In other words, the action set is $\mathcal{D}_{\text{relaxed}}^y \equiv \Big(\boldsymbol{\lambda}|\overline{p}_k(\boldsymbol{\lambda}) \geq 0, \lambda_k \in \mathbb{R}_+, k = 1, 2\Big)$. In the following, we prove the submodularity of $\mathcal{G}_{\text{relaxed}}^y$. Based on this property the existence of a Nash equilibrium for $\mathcal{G}_{\text{relaxed}}^y$ follows.

THEOREM 3 *The two-player global game $\mathcal{G}_{\text{relaxed}}^y$ is a submodular game when the strategy set $\mathcal{D}_{\text{relaxed}}^y$ is not empty.*

PROPERTY 1 *Nash equilibria in $\overset{\circ}{\mathcal{D}}{}_{\text{relaxed}}^y$, the interior of $\mathcal{D}_{\text{relaxed}}^y$, are all the solutions of the system*

$$\frac{\partial C_k(y, \boldsymbol{\lambda})}{\partial \lambda_k} = 0, \qquad k = 1, 2 \tag{43}$$

*in $\overset{\circ}{\mathcal{D}}{}_{\text{relaxed}}^y$.*

Table I: (Algorithm I) finding the NEs of the global game for $T-PCSI$

Initialize $\mathcal{E}=\emptyset$.
**for** $y \in \{1,...,\mathcal{N}_Y\}$.
Set matrix $\boldsymbol{M}$ for $\mathcal{R}^y_{\boldsymbol{p}}$.
Initialize $\mathcal{E}^y=\emptyset$.
**if** $\text{rank}(\boldsymbol{M})=4$.
compute $\boldsymbol{A}=\boldsymbol{M}^{-1}$.
compute $\overline{\boldsymbol{p}}(\lambda_1,\lambda_2)$.
**else** determine constraints on $\boldsymbol{\lambda}$ such that $\text{Rank}(\boldsymbol{M})=\text{Rank}(\overline{\boldsymbol{M}})$.
determine the infinite solutions of $\boldsymbol{Mp}=\boldsymbol{b}$ parametric
in the unknown $p_{kh}$.
**endif**.
compute $C_k(y,\boldsymbol{\lambda}),\ k=1,2$.
find all the solutions $\boldsymbol{\lambda}$ of
$\frac{\partial C_k(y,\lambda)}{\partial\lambda_k}=0, k=1,2$
and collect them in the set $\mathcal{E}^y$.
set $\mathcal{E}^y=\mathcal{E}^y\bigcap\mathcal{D}^y_{\text{glob}}$.
**for** each $\boldsymbol{\lambda}^*\in\mathcal{E}^y$
check=1
**for** all $\lambda_k$
**for** all regions $\mathcal{R}^z$ such that $\mathcal{R}^z_p\in\mathcal{Y}_{\text{cond}}$
**if** $C_k(y,\lambda^*_k,\lambda^*_m)\le C_k(z,\lambda_k,\lambda^*_m)$
check =1.
**else**
check=0.
**endif**.
**endfor**.
**endfor**.
**if** check=1
$\mathcal{E}=\mathcal{E}\bigcup\{(y,\boldsymbol{\lambda}^*)\}$
**endif**.
set $\mathcal{E}=\mathcal{E}\bigcup\mathcal{E}^y$.
**endfor**
**endfor**

Note that system (43) is a system of rational functions in $\boldsymbol{\lambda}$, and all its solutions can be determined as roots of a polynomial in $\lambda_k$ or $\lambda_m$.

PROPERTY 2 *The Nash equilibria of $\mathcal{G}^y_{\text{relaxed}}$ on the boundary satisfying $\overline{p}_{kh}=0, k,h\in\{1,2\}$ are Nash equilibria of $\mathcal{G}_{\text{glob}}$ only if they are Nash equilibria in $\overset{\circ}{\mathcal{D}}{}^z_{\text{relaxed}}$, the action set interior of the game $\mathcal{G}^z_{\text{relaxed}}$, where the region $\mathcal{R}^z_{\boldsymbol{p}}$ is obtained from the region $\mathcal{R}^y_{\boldsymbol{p}}$ by enforcing $\overline{p}_{kh}=0$.*

Thanks to this property, we do not ignore any NE of the global game if we ignore the equilibrium at the boundary determined by $\tilde{p}_{kh}=0$.

PROPERTY 3 *Let $\boldsymbol{\lambda}^*$ be a NE of $\mathcal{G}^y_{\text{relaxed}}$ corresponding to the per subcarrier game equilibrium $\tilde{\boldsymbol{p}}^y(\boldsymbol{\lambda}^*)$. $\boldsymbol{\lambda}^*$ is a NE of $\mathcal{G}_{\text{glob}}$ if and only if (1) it belongs to the actions set of the global game per region, $\mathcal{D}^y_{\text{glob}}$ and (2) it satisfies the following inequality*

$$C_k(y,\lambda^*_k,\lambda^*_m)\le C_k(z,\lambda_k,\lambda^*_m),\forall\lambda_k\ge 0,k,m=1,2,k\ne m,\mathcal{R}^y_{\boldsymbol{p}}\in\mathcal{Y},\mathcal{R}^z_{\boldsymbol{p}}\in\mathcal{Y}_{\text{cond}} \quad (44)$$

Table II: (Algorithm II) Iterative algorithm for $T-CCSI$

```
initilaize (λ1, λ2)
 repeate
  initilize p = (p11(g11), p12(g12), p^n_21(g21), p^n_22(g22))
   repeate
    for k = 1 : 2
     p_k = arg max E_{g^n_k} Σ_{k=1}^{2} (r_k(g^n, p^n) − λ_k p^n_k(g_k))
    end
   until p converges
  update (λ1, λ2) using subgradient method
 untill (λ1, λ2) converges.
```

*being $\mathcal{Y}_{\text{cond}}$ is a subset of $\mathcal{Y}$ wherein the per subcarrier game strategy of transmitter $m$ is identical to the one's in region $\mathcal{R}^y_{\boldsymbol{p}}$.*

LEMMA 3 *If a NE of $\mathcal{G}^y_{\text{glob}}$ belongs to a boundary corresponding to a condition of type $\mathcal{W}^{(I)}_{kh} \leq \mathcal{W}^{(II)}_{kh}$(or $\mathcal{W}^{(I)}_{kh} \geq \mathcal{W}^{(II)}_{kh}$) or a condition of type $\overline{p}_{m1}g_{m1} \geq \overline{p}_{m2}g_{m2}$ (or $\overline{p}_{m1}g_{m1} \leq \overline{p}_{m2}gm2$) it is not a NE of $\mathcal{G}_{\text{glob}}$.*

Property III and Theorem 3 yield the following theorem.

THEOREM 4 *The NEs of $\mathcal{G}_{\text{glob}}$ are all the NEs of $\mathcal{G}^y_{\text{relaxed}}, \forall y \in \{1, ..., \mathcal{N}_Y\}$ which (1) belong to $\overset{\circ}{\mathcal{D}}{}^y_{\text{glob}}$, and (2) satisfy condition (44).*

## 5. Algorithm

The algorithm to determine the NE of the dual game $\mathcal{G}^D$ consists in determining all the NEs of the $\mathcal{N}$ relaxed games $\mathcal{G}^y_{\text{relaxed}}$ defined over the regions $\mathcal{R}^y_{\boldsymbol{p}} \in \mathcal{Y}$. Then, among them, it selects the ones which satisfy all the conditions for being NE of the global game. Such conditions are expressed in Property I - Property IV. The algorithm is presented in Table I. Note that the NE obtained with this algorithm are not unique. A selection criterion has to be enforced to both transmitters in order to guarantee the convergence of the system toward to an equilibrium. Several criteria can be enforced. As an example we can propose the selection of the NE which maximizes the sum throughput for non symmetric systems, i.e. systems with the same channel statistics for both transmitters.

## 6. Numerical results

We consider a 2-transmitter network in which the transmitters simultaneously communicate with a single receiver over 10 subcarriers. In the first set of results, the system parameters are set as follows. The channel gains for the two transmitters are set to $(g_{11}, g_{12}) = (1/3, 2/3); (g_{21}, g_{22}) = (7/8, 1/8)$ and the corresponding probabilities are $(\gamma_{11}, \gamma_{12}) = (0.3, 0.7); (\gamma_{21}, \gamma_{22}) = (0.1, 0.9)$. Note that the gap between the two gain levels for transmitter 2 is greater than the ones of transmitter 1. Moreover, the values of $\gamma$s indicate that for transmitter 2 the occurrence of the higher channel gain is less probable than the lower. A reversed situation occurs for transmitter 1.

Additionally, we consider two levels of information at the transmitters: (i) $T-CCSI$ : complete channel side information at both transmitters, (ii) $T-PCSI$ :

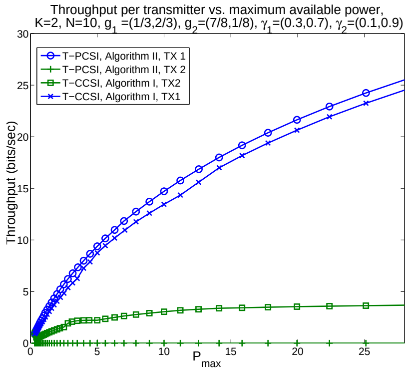


Figure 1: Aggregate throughput vs maximum available power at the transmitter, $K = 2, N = 10, g_1 = (1/3, 2/3), g_2 = (7/8, 1/8), \gamma_1 = (0.3, 0.7), \gamma_2 = (0.1, 0.9)$

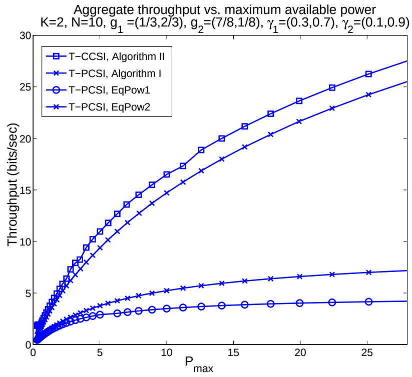


Figure 2: Throughput per transmitter vs maximum available power at the transmitter, $K = 2, N = 10, g_1 = (1/3, 2/3), g_2 = (7/8, 1/8), \gamma_1 = (0.3, 0.7), \gamma_2 = (0.1, 0.9)$

partial channel side information, i.e. each transmitter know its own channel state and the statistics of the other's links.

For $T-CCSI$, the problem is defined in (5). The power allocation algorithm based on the dual method introduced in [4] is implemented. The algorithm is detailed in Table II and assigns an initial value to the powers and the Lagrangian multipliers and iterates until convergence to a local optimum power allocation of the constrained optimization (5). Note that this algorithm converges into a local optimum depending on the initial value. For $T-PCSI$, the distributed joint rate and power allocation is obtained via three different algorithms. The first two algorithms are based on heuristic approaches and the last one is the proposed algorithm in Table I. Note that, unlike Algorithm II, Algorithm I is not iterative and will immediately provide all the NEs of the global game.

In both heuristic approaches, transmitter $k = 1, 2$ divides the maximum available power $\overline{P}_k$ equally among the subcarriers. Let us assume $P_s = P_{\max}/N$. In the first approach, namely $EqPow1$, with the intention to avoid outage, we set the transmission rate on channel $g_{kh}$ to $R_{kh} = \log(1 + \frac{P_s g_{kh}}{P_s g_{mg} + \sigma})$ where $g_{mg} = \max(g_{m1}, g_{m2})$. The value of the average throughput is $\rho_{kh} = R_{kh}$. In the second heuristic approach, namely $EqPow2$, we accept a certain level of outage. We calculate the two rates $R_{kh}^{mg} = \log(1 + \frac{P_S g_{kh}}{P_s g_{mg} + \sigma^2})$ and $R_{kh}^{ml} = \log(1 + \frac{P_S g_{kh}}{P_s g_{ml} + \sigma^2})$ where $g_{ml} = \min(g_{m1}, g_{m2})$. We further calculate the average throughput for both cases, i.e. $\rho_{kh}^{mg} = R_{kh}^{mg}$ and $\rho_{kh}^{mg} = \gamma_{ml} R_{kh}^{ml}$, and we determine the maximum. Finally, we set the rate $R_{kh}$ to the one corresponding to the maximum throughput.

Let us compare the performance of the above four algorithms. We adopt the throughput attained by each algorithm as performance measure and we plot it versus the maximum available power at the transmitter. The throughput here is in bits/sec. For the $T-CCSI$ optimization, the throughput is equal to the sum of the maximum achievable rate over each subcarrier. The maximum available powers at both transmitters are identical, i.e. $\overline{P}_1 = \overline{P}_2 = P_{\max}$. For the first set of simulations the noise power is fixed at $-5db$ and $P_{\max}$ increases linearly from 0.3 W ($-5db$) to 28 W ($15db$).

Figure 1 compares the performance of Algorithm I for $T-PCSI$ and Algorithm II for $T-CCSI$ separately for the two transmitters. Note that the optimization based on Algorithm II does not guarantee the global optimum but only a local optimum. For Algorithm I we adopt the maximum sum throughput as selection criterion of a Nash equilibrium.

Interestingly, the simulations show that all the NE points obtained through Algorithm I are those wherein only one transmitter emits with the full power and the other remains off. This kind of result holds also for all the sets of parameters we consider for simulations. This suggests that Algorithm I can be simplified to finding the NEs in which only one transmitter emits. The set of the NE and/or retained NE after the application of a selection criterion includes the cases where transmitter $k$ allocates the whole power in only one channel state $g_{kh}, h = 1, 2$ and/or when it divides the power optimally among the channel gains $g_{k1}$ and $g_{k2}$ assuming that there is no interference from the other transmitter.

By performing Algorithm II, Transmitters of type $T-CCSI$ have increasing throughput while the power budget increases.

Figure 2 shows the aggregate throughput obtained by the four algorithms. The two heuristic algorithms have a saturating behavior at very low power levels compared to the optimization and the game based algorithm. In other words, these algorithms are not able to exploit the additional available resources. Interestingly, the increase of the throughput for a NE in $T - PCSI$, follows closely the increase of the optimal power allocation in the case of $T - CCSI$.

## 7. Conclusion

The joint power and rate allocation in a two-user OFDM system with a large number of subcarriers and partial channel state information at the transmitters for slow frequency selective fading channel is studied. A total throughput maximization problem is introduced and it is proved that the dual approach yields optimum resource allocation asymptotically as $N \to +\infty$. Although, the dual problem has linear complexity in the number of subcarriers, the complexity is still exponential in he number of users. A suboptimal low complexity approach is introduced in the form of 2-player game. We defined a two-level game, namely per subcarrier games and global game, whose NEs are obtained. The performance of such NE points is compared to the performance of the optimum power allocation for the case of complete channel state information and the uniform power allocation for the case of partial channel side information. Interestingly, the simulations show that all the NEs obtained from the game are those wherein only one transmitter emits with full power and the other remains off. Therefore, finding all NEs of the game can be reduced to finding all optimal power allocations of one transmitter assuming that there is no interference. We further adopt as selection criteria of a NE the maximum sum throughput.

## References

[1] W. Yu and J. Cioffi, “FDMA capacity of Gaussian multiple-access channels with ISI,” *IEEE Trans. on Communications*, Jan. 2002.

[2] G. Song and Y. Li, “Cross-layer optimization for OFDM wireless networks Part I: theoretical framework,” *IEEE Trans. on Wireless Communications*, March 2005.

[3] L. Cottatellucci, G. He, and S. Gault, "Information theoretic capacity of WiMAX," in *RRM in Wimax: from theoretical capacity to system simulations*, Wiley and Iste, 2009.

[4] W. Yu and R. Lui, "Dual methods for nonconvex spectrum optimization of multicarrier system," *IEEE Transactions on Communications*, vol. 54, July 2006.

[5] K. Seong, M. Mohseni, and J. Cioffi, "Optimal resource allocation for OFDMA downlink systems," in *Proc. IEEE International Symposium on Information Theory*, (Seattle, WA), 2006.

[6] S. T. Chung, S. J. Kim, J. Lee, and J. M. Cioffi, "A game-theoretic approach to power allocation in frequency-selective Gaussian interference channels," in *Proc. IEEE Int. Symp. Information Theory (ISIT 2003)*, (Yokohama, Japan), p. 316, June 2003.

[7] X. Lei, K. Avratchenkov, L. Cottatellucci, and A. Garnaev, "Competitive unlicensed spectrum sharing with partial information on slow fading channels," in *WWIC 2010*, June 2010.

[8] G. Scutari, D. R. Palomar, and S. Barbarossa, "Simultaneous iterative water-filling for Gaussian frequency-selective interference channels," in *Proc. IEEE Int. Symp. Information Theory (ISIT 2006)*, pp. 600–604, July 2006.

[9] G. He, S. Gault, M. Debbah, and E. Altman, "Distributed power allocation game for uplink OFDM systems," in *Proc. of 6th International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks and Workshops, 2008 (WiOPT 2008)*, (Berlin, Germany), April 2008.

[10] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York: Cambridge University Press, 2007.

 