Construction criteria and existence results for approximately universal linear space-time codes with reduced decoding complexity

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Abstract

This work presents new eigenvalue bounds, necessary conditions and existence results for approximately universal linear (lattice) codes that can be drawn from lattices of reduced dimension, and can thus incur reduced decoding complexity. Currently for the $n \times n_r$ MIMO channel, all known $n \times T$ approximately universal codes, except for the Alamouti code for $n = 2, n_r = 1$, draw from lattices of dimension equal to or larger than nT, irrespective of n_r. Motivated by the case where $n_r < n$, the work describes construction criteria for lattice codes that maintain their approximate universality even when they are drawn from lattices of reduced dimensionality.

1. Introduction

Consider the quasi-static, fading, space-time (ST) MIMO channel with n transmit and n_r receive antennas. The $(n_r \times T)$ received signal matrix Y is given by

$$Y \;=\; \theta H X + W \tag{1}$$

where X is an $(n \times T)$ code matrix drawn from a ST code $\mathscr{X}$, T is the duration of transmission, W is the additive noise matrix, H is the $(n_r \times n)$ channel matrix, and the scalar θ is such that

$$\theta^2 ||X||_F^2 \;\le\; T\ \text{SNR}, \quad \text{all } X \in \mathscr{X}. \tag{2}$$

The entries of W are assumed to be i.i.d., circularly symmetric, complex Gaussian $\mathbb{C}\mathscr{N}(0,1)$ random variables, and the entries of H are drawn randomly from an arbitrary distribution.

1.1. Approximate universality over the $n \times n_r$ MIMO channel

Let $\mathscr{X}$ have cardinality $|\mathscr{X}| = 2^{RT}$, corresponding to transmission rate R. The high SNR fundamental error-performance limit over the MIMO channel is defined by the channel's outage region

$$\mathscr{O} := \{H : \max_{p_x} I(x;y|H) < R\} \subset \mathbb{C}^{n_r \times n},$$

where $\max_{p_x} I(x;y|H)$ describes the instantaneous capacity of the channel. This limit was captured in the form of the *diversity multiplexing tradeoff* (DMT) [1], which describes the high-SNR (SNR is here denoted as ρ) approximation of the optimal *diversity-gain*

$$d(r) := -\lim_{\rho\to\infty} \frac{P(H \in \mathscr{O})}{\log \rho} = -\lim_{\rho\to\infty} \frac{P(\text{error})}{\log \rho} \tag{3}$$

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as a function of the *multiplexing gain*

$$r := R/\log\rho.$$

The work in [2, 3, 5] shows that there exist ST designs that perform DMT optimally, irrespective of the fading statistics. Such codes were defined to be *approximately universal* in [2] as follows.

Definition 1 *([2]) A code $\mathscr{X}$ is said to be* approximately universal *over the $n \times n_r$ MIMO channel if*

$$\lim_{\rho\to\infty} \frac{Pr(error|X, H \notin \mathscr{O})}{\log\rho} = -\infty,\ \forall X \in \mathscr{X}. \quad (4)$$

For a code matrix X, we denote the k smallest eigenvalues of $XX^\dagger$ as

$$\lambda_1(X) \leq \lambda_2(X) \leq \cdots \leq \lambda_k(X).$$

The work in [2] provides necessary and sufficient conditions for a code to be approximately universal.

Lemma 1 *([2]) A code is approximately universal over the $n \times n_r$ MIMO channel if and only if*

$$\prod_{i=1}^{\min(n,n_r)} \lambda_i(X) \dot{\geq} \rho^{\min(n,n_r)-r},\ \forall X \in \mathscr{X}. \quad (5)$$

In the above we used the symbol $\doteq$ to denote exponential equality, i.e., the relation

$$\lim_{\rho\to\infty} \frac{\log g(\rho)}{\log\rho} = c$$

is denoted by $g(\rho) \doteq \rho^c$ and $\dot{\geq}, \dot{\leq}$ are defined similarly.

1.2. q-dimensional linear ST codes: Fully dimensional v.s. reduced-dimensional designs

Let $\mathscr{X}$ be a lattice ST code mapping q information elements from a discrete constellation $\mathscr{A}_\rho$, via a lattice generator matrix $G \in \mathbb{C}^{nT\times q}$

$$G = \begin{vmatrix} G_{1(n\times q)} \\ G_{2(n\times q)} \\ \vdots \\ G_{T(n\times q)} \end{vmatrix}.$$

Each codematrix is constructed via column-by-column stacking of an nT-length lattice vector $G\underline{f}$, where

$$\underline{f} \in \mathscr{A}_\rho^q \in \mathbb{C}^q, \text{ and } \|\underline{f}\|^2 \dot{\leq} \rho.$$

The corresponding code is then given by[1]

$$\begin{aligned} \mathscr{X} &= \left\{ matr(G\cdot\underline{f}),\ \underline{f} \in \mathscr{A}_\rho^q \right\}, \\ &= \left\{ \left[\begin{array}{cccc} G_1\underline{f} & G_2\underline{f} & \cdots & G_T\underline{f} \end{array} \right],\ \underline{f} \in \mathscr{A}_\rho^q \right\}. \end{aligned}$$

We call $\mathscr{X}$ a *q-dimensional* lattice code. When $q \geq nT$, the code will be labeled as *fully-dimensional*, whereas when $q < nT$, as *reduced-dimensional*.

2. Existence of approximately universal linear codes

All known approximately universal ST code designs, except for the Alamouti code for $n = 2, n_r = 1$, are lattice code designs of dimension equal to or greater than nT. Such fully-dimensional optimal designs ([5]) achieve approximate universality for all channel dimensions, i.e., for all n_r. For the case where $n_r < n$, the question is raised whether approximate universality can be achieved by reduced-dimensional codes. The existence of such codes would establish the ability to communicate in a provably optimal manner over all MIMO fading channels, irrespective of fading statistics, and do so with reduced decoding complexity whenever $n_r < n$.

In relation to this issue, we provide new bounds on code eigenvalues and then apply these bounds to give criteria for constructing reduced-dimensional approximately universal codes. The criteria will also establish that, with the exception of the Alamouti code for $n = 2, n_r = 1$, currently the only known approximately universal designs are fully-dimensional irrespective of n_r.

[1] Without loss of generality we consider that the complex and real parts of $\underline{f}$ are transformed by the same lattice generator matrix.

For $n_r \geq n$, the result in [5] establishes the fact that approximate universality can be achieved in minimum delay ($T = n$), utilizing the minimum possible lattice dimensionality of $q = nT = n^2$. We henceforth restrict our attention to the case where $n_r < n$, and $T \geq n$, which is necessary for achieving full diversity over the Rayleigh fading channel and consequently is necessary for approximate universality.

Towards establishing the conditions we proceed with bounds on the code eigenvalues. Henceforth all the results refer to q-dimensional $n \times T$ lattice ST codes that are generated by a lattice generator matrix G.

Lemma 2 *For $k \in [1,2,...,n]$ and for*

$$t_k := \min_{U_k \in GL(k,n)} \{rank[(I_T \otimes U_k)G]\}, \qquad (6)$$

there exists a matrix $X'_k \in \mathscr{X}$ such that

$$\prod_{i=1}^{k} \lambda_i(X'_k) \dot{\leq} \rho^{k-krT/t_k}. \qquad (7)$$

In the above we use $\otimes$ to denote the Kronecker product of matrices, and $GL(k,n)$ to denote the set of $k \times n$ matrices of rank $k \leq n$.

For each k, the eigenvalues of $\mathscr{X}$ are bounded as a function of the minimum rank t_k of all possible $kT \times q$ lattice generator matrices

$$A(U_k) := (I_T \otimes U_k)G = \begin{bmatrix} U_k G_1 \\ U_k G_2 \\ \vdots \\ U_k G_T \end{bmatrix} \qquad (8)$$

that generate $k \times T$ codes $U_k \mathscr{X}$, which are row deleted versions of faithful mappings of $\mathscr{X}$.

Using the above bounds on the eigenvalues of the code, we proceed with the necessary conditions.

Theorem 3 *A necessary condition for a code to be approximately universal over the $n \times n_r$ MIMO channel is that*

$$\min_{k \in [1,2,..n]} \frac{1}{k} \min_{U_k \in GL(k,n)} rank\big((I_T \otimes U_k)G\big) \geq T. \qquad (9)$$

We proceed with the proofs of Lemma 2 and Theorem 3.

Proof of Lemma 2: With

$$A(U_k) = (I_T \otimes U_k)G,$$

let

$$U'_k \in GL(k,n)$$

be such that

$$rank(A(U'_k)) = t_k.$$

Using a standard packing argument, we then have that

$$\exists \underline{f}'_k \in \mathscr{A}_\rho^q \text{ such that } \|A(U'_k)\underline{f}'_k\|_F^2 \dot{\leq} \rho^{1-rT/t_k}. \quad (10)$$

Let X'_k be the $n \times T$ codematrix of $\mathscr{X}$ corresponding to vector $\underline{f}'_k$, i.e., let

$$X'_k := matr(G\underline{f}'_k)$$

where the column stacking here is done every n elements. Now note that

$$A(U'_k) = \begin{bmatrix} U'_k G_1 \\ \vdots \\ U'_k G_T \end{bmatrix}$$

and observe that

$$matr(A(U'_k)\underline{f}'_k) = U'_k X'_k,$$

where this particular column stacking is done every k elements, resulting in the $k \times T$ matrix $U'_k X'_k$. Let

$$U' =: \begin{bmatrix} U'_k \\ U'' \end{bmatrix} \in GL(n,n)$$

be an $n \times n$ matrix having U'_k as its first k rows and note that $U'_k X'_k$ are the first k rows of $U' X'_k$. Then

$$\begin{aligned} (\prod_{i=1}^{k} \lambda_i(X'_k))^{1/k} &\doteq \prod_{i=1}^{k} \lambda_i^{1/k}(U' X'_k) \\ &\dot{\leq} \prod_{i=1}^{k} \lambda_i^{1/k}(U'_k X'_k) \\ &\dot{\leq} \sum_{i=1}^{k} \lambda_i(U'_k X'_k) \\ &= \|U'_k X'_k\|_F^2 = \|A(U'_k)\underline{f}'_k\|_F^2 \\ &\dot{\leq} \rho^{1-rT/t_k} \end{aligned}$$

where the first asymptotic equality follows from the fact that $\lambda_{\min}(U') \doteq \lambda_{\max}(U') \doteq \rho^0$, the first inequality is due to the eigenvalue interlacing property, the second inequality due to the arithmetic-mean geometric-mean inequality, and the last inequality comes from (10). $\square$

Proof of Theorem 3: Recall from Lemma 2 that

$$\exists X'_k \in \mathscr{X} \text{ such that } \prod_{i=1}^{k} \lambda_i(X'_k) \leq \rho^{k-krT/t_k}, \quad k = 1, 2, .., n.$$

First let $k \geq n_r$ and rewrite the above to get

$$\prod_{i=1}^{n_r} \lambda_i(X'_k) \prod_{i=n_r+1}^{k} \lambda_i(X'_k) \leq \rho^{k-krT/t_k}$$

which implies that there exists $X'_k \in \mathscr{X}$ such that

$$\prod_{i=1}^{n_r} \lambda_i(X'_k) \dot{\leq} \frac{\rho^{k-krT/t_k}}{(\rho^{1-rT/t_k})^{k-n_r}} \doteq \rho^{n_r - n_r rT/t_k}, \quad k = n_r, n_r+1, ..., n,$$

where we have used the fact that

$$\prod_{i=n_r+1}^{k} \lambda_i(X'_k) = (\rho^{1-rT/t_k})^{k-n_r}$$

corresponding to the case where all eigenvalues are equal. It is easy to see that $t_k \leq t_{k+1}$, which tells us that the above bound for $k = n_r$ implies the bounds for the other cases of $k > n_r$. As a result the bound states that

$$\exists X'_{n_r} \in \mathscr{X} \text{ such that } \prod_{i=1}^{n_r} \lambda_i(X'_{n_r}) \dot{\leq} \rho^{n_r - n_r rT/t_{n_r}}. \tag{11}$$

Now consider the case where $k \leq n_r$ and note that there exists $X'_k \in \mathscr{X}$ such that

$$\prod_{i=1}^{k} \lambda_i(X'_k) \prod_{i=k+1}^{n_r} \lambda_i(X'_k) \leq \rho^{k-krT/t_k} (\|X'_k\|_F^2)^{n_r-k} \dot{\leq} \rho^{n_r - krT/t_{n_r}}$$

where we used the upper bound

$$\prod_{i=1}^{k} \lambda_i(X'_k) \dot{\leq} \rho^{k-krT/t_k}$$

from Lemma 2, and also used that

$$\prod_{i=k+1}^{n_r} \lambda_i(X'_k) \dot{\leq} (\|X'_k\|_F^2)^{n_r-k} \dot{\leq} \rho^{n_r-k}.$$

Consequently for $k \leq n_r$ we get that

$$\exists X'_k \in \mathscr{X} \text{ such that } \prod_{i=1}^{n_r} \lambda_i(X'_k) \dot{\leq} \rho^{n_r - krT/t_k}.$$

Combining the two cases $k \geq n_r$ and $k \leq n_r$ gives that

$$\exists X'_k \in \mathscr{X} \text{ such that } \textstyle\prod_{i=1}^{n_r} \lambda_i(X'_k) \dot{\leq} \rho^{n_r - krT/t_k}, \quad k = 1, 2, .., n$$

which in turn is combined with the necessary condition for approximate universality in (5) to give that

$$t_k/k \geq T, \quad k = 1, 2, ..., n.$$

$\square$

The necessary conditions in the Theorem also imply the following, more specific necessary conditions.

Corollary 4 *For a q-dimensional code to be approximately universal over the $n \times n_r$ MIMO channel, it is necessary that $q \geq n_r T$, that $rank(G_i) = n, \ i = 1, 2, ..., T$, and that there exists no solution to the generalized eigenvalue problem*

$$\alpha \underline{u}^\dagger G_i = \underline{u}^\dagger G_j, \ \alpha \in \mathbb{C}, \ i \neq j. \tag{12}$$

We here note that if the second and third conditions are not met, then the code fails to be approximately universal for any q and n_r.

Proof: If $q < n_r T$ and $k = n_r$ then $A(U_k)$ is of dimension $n_r T \times q$, and $rank(A(U_k)) \leq q < n_r T$ which is a violation of the condition from Theorem 3.

If $rank(G_i) < n$ for some $i \leq T$, then setting the first row $\underline{u}^\dagger$ of U_k to be in the null space of G_i, results in $rank(A(U_k)) < kT$ which again violates the condition in Theorem 3.

Finally setting $\underline{u}^\dagger$ to be the solution to the generalized eigenvalue problem

$$\alpha \underline{u}^\dagger G_i = \underline{u}^\dagger G_j$$

results in having two linearly dependent rows of $A(U_k)$ and in $rank(A(U_k)) < kT$. $\square$

Towards characterizing some properties of layered and perfect codes [11, 4, 5], we proceed with the following necessary condition.

Proposition 5 *A necessary condition for a code $\mathscr{X}$ to be approximately universal is that there should exist no $n \times n$ full rank matrix U, such that $U\mathscr{X}$ has a fixed entry, i.e., it should not be the case that*

$$UX(i,j) = c, \forall X \in \mathscr{X},$$

for some fixed $i \in [1,n], j \in [1,T]$, and some fixed constant c. If this condition is not met, then the code fails to be approximately universal, for any q and n_r.

Proof: Consider that there exists an $n \times n$ full rank matrix U, such that $U\mathscr{X}$ has a fixed entry, i.e., that

$$UX(i,j) = c, \forall X \in \mathscr{X},$$

for some fixed $i \in [1,n], j \in [1,T]$, and some fixed constant c. Without loss of generality, set $c = 0$. Consider the code $U\mathscr{X}$ operating over an $n \times n_r$ channel where all fading coefficients are constantly zero except the fading vector $\underline{h}_i^\dagger$ corresponding to the ith transmit antenna. The optimal error performance over such a channel is equal to the optimal performance over the $1 \times n_r$ SIMO channel described by $\underline{h}_i$. At the same time, the error performance provided by $U\mathscr{X}$ over the above $n \times n_r$ channel, is identical to the performance, over the corresponding SIMO channel, provided by the time-only $1 \times T$ code $U\mathscr{X}_1$ given by the ith row of the codematrices of $U\mathscr{X}$. Let $r_{\max}$ denote the maximum multiplexing gain achievable in this SIMO channel. It is straightforward to see that due to the fixed zero in its codevectors (no information transmitted), this time-only code can only achieve maximum multiplexing gain of

$$r_{\max}(T-1)/T < r_{\max}$$

and thus cannot be optimal over the SIMO channel, which in turn means that the original $n \times T$ code $U\mathscr{X}$ cannot be optimal over the corresponding $n \times n_r$ MIMO channel. As a result $U\mathscr{X}$ does not meet the coding gain requirement of Lemma 1, and neither does $\mathscr{X}$. $\square$

The following is a direct result from Proposition 5.

Corollary 6 *For $z < n$ and for any n_r, z-layered codes (including z-layered perfect codes) fail to be approximately universal.*

This is seen directly from the fact that for $z < n$, z layered codes have $T - z > 0$ zero elements per row.

Finally, drawing from Corollary 4 we have the following.

Corollary 7 *There is no n-dimensional linear space-time code which is approximately universal over the $n \times n_r$ MIMO channel.*

Proof: Set $k = n_r = 1$ and $q = kT = n$, and first consider minimum delay $(n \times n)$ codes over the $n \times 1$ MISO channel. Let

$$A(U_k) = A(\underline{u}^\dagger) = (I_n \otimes \underline{u}^\dagger)G = \begin{bmatrix} \underline{u}^\dagger G_1 \\ \vdots \\ \underline{u}^\dagger G_T \end{bmatrix}.$$

Having a G_i of rank less than n violates the conditions in Corollary 4. On the other hand if $rank(G_i) = n, \forall i$ then a guaranteed to exist eigenvector $\underline{u}^\dagger$ of $G_j G_i^{-1}$ solves

$$\alpha \underline{u}^\dagger G_i = \underline{u}^\dagger G_j \qquad (13)$$

causing a violation of the conditions in Corollary 4. The extension to the general case of $n \times T$ codes and the general $n \times n_r$ MIMO channel follows by noting that if $T > n$ or if $n_r > 1$ then $q < n_r T$ which is a violation of Corollary 4.

$\square$

3. Conclusion

The work provided new bounds on the eigenvalues of ST codes as a function of the minimum rank over all possible lattice generator matrices that generate codes which are row deleted versions of faithful mappings of the original code. Based on these bounds, the work also presented existence results and new construction criteria for codes that perform in a provably optimal manner over all MIMO fading channels, irrespective of fading statistics, and do so by drawing from lattices of reduced dimension whenever $n_r < n$. Reduced dimensional codes incur reduced decoding and signaling complexities.

This setting is particularly relevant to cooperative communications in wireless networks [15][16] where the fading statistics can be arbitrary, the number of receive antennas is usually small, the number of transmitting nodes can be large and the decoding complexity must be kept small due to the small size of the nodes.

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