

# Diversity Order of Linear Equalizers for Doubly Selective Channels

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**Abstract**—Diversity order of linear equalization (LE) is investigated for the case of linearly precoded block transmission in doubly selective channels. A connection between the orthogonal deficiency  $od(\cdot)$  [1] of the effective channel matrix at the receiver and an earlier proof that LE achieves full diversity in frequency selective channels [2] is highlighted. For doubly selective channels it is argued that  $od$  of the channel matrix at the receiver shares the same upper bound as that of a closely related matrix for which an  $od$  bound is derived. Finally, simulation results are provided to substantiate all arguments made in the paper.

## I. INTRODUCTION

Fading channels pose a major challenge to reliable communications, particularly over wireless channels. At the receiver, an equalization technique that optimally exploits the inherent diversity in fading channels is a convenient countermeasure against fading channels. Frequency selective fading provides multipath diversity due to the presence of multiple independently fading components. In block transmission systems, when the channel coherence time is shorter than the transmit block length, temporal variations of the channel give rise to time-selectivity. However, this same time-selectivity of the channel also provides Doppler diversity [3] which can be exploited by the receiver. Linear Equalization (LE) is a low-complexity albeit sub-optimal alternative to optimal maximum-likelihood equalization (MLE). Recent research has concentrated on quantifying the performance of diversity order of LE in fading channels. While the diversity order of LE for transmission over frequency selective channels has been studied in [4], the diversity order of LE in time-selective and doubly selective channels is less understood. In [5], the authors used the Complex-Exponential Basis Expansion Model (CE-BEM) [6] with  $Q + 1$  basis functions to model the doubly selective channel of memory  $L$ . The authors showed that by employing linear precoded block transmission, the maximum diversity in the channel is upper bounded by  $(Q + 1)(L + 1)$  and can be achieved when maximum-likelihood decoding is used at the receiver. However, ML incurs a huge computational complexity therefore it is of interest to investigate the diversity order achieved by linear equalization for block transmission over doubly selective channels. In this paper, we study the performance of linear minimum mean squared error zero forcing (MMSE-ZF) receivers for linearly precoded block transmission in doubly selective channels and show that LE

also achieves maximal diversity offered by doubly selective channels with the same precoder that enables MLE to achieve multiplicative multipath-Doppler diversity.

## II. SIGNAL MODEL

In Fig. 1 we show the block diagram of the transmission model for block transmission over fading channels.

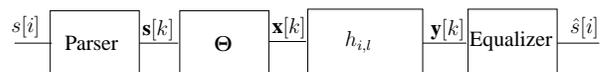


Fig. 1. Block diagram of transmission model.

At the transmitter, complex data symbols  $s[i]$  are first parsed into length- $N$  blocks. The  $n$ -th symbol in the  $k$ -th block is given by  $[s[k]]_n = s[kN + n]$  with  $n \in (0, 1, \dots, N - 1)$ . Each block  $s[k]$  is precoded by a  $M \times N$  matrix  $\Theta$  where  $M \geq N$  and the resultant block  $x[k]$  is transmitted over the block fading channel. In the signal model, we consider the case of doubly-selective channels of order  $L$ . Frequency-selective-only and time-selective-only channels can be represented as special cases of doubly selective channels. It is well known that the temporal variation of the channel taps in doubly selective channels with a finite Doppler spread over a finite duration can be captured by finite Fourier bases. We therefore use CE-BEM [6] with  $Q + 1$  basis functions to model the time variation of each tap in a block duration. The basis coefficients remain constant for the block duration but are allowed to vary with every block. The time-varying channel for each block transmission is thus completely described by the  $Q + 1$  Fourier basis functions and  $(Q + 1)(L + 1)$  coefficients. In general  $Q$  is chosen such that  $Q \geq 2 \lceil f_{max} M T_s \rceil$  where  $1/T_s$  is the sampling frequency and  $f_{max}$  is the Doppler spread of the channel. The coefficients themselves are assumed to be zero-mean complex *i.i.d* Gaussian random variables. This is a reasonable assumption for a rich scattering environment with non-line-of-sight reception. Using  $i$  as the discrete time (sample) index, we can represent the  $l$ -th tap of the channel in the  $k$ -th block

$$h_{i,l} = \sum_{q=0}^Q h_q(k,l) e^{j2\pi f_q i}, \quad (1)$$

$l \in [0, L]$ ,  $f_q = (q - Q/2)/M$ . The corresponding receive signal is formed by collecting  $M$  samples at the receiver to form  $\mathbf{y}[k] = [y(kM+0), y(kM+1), \dots, y(kM+M-1)]^T$ . When  $M \geq L$ , this block transmission system can be represented in matrix-vector notation as [5]

$$\mathbf{y}[k] = \mathbf{H}[k; 0]\mathbf{\Theta}\mathbf{s}[k] + \mathbf{H}[k; 1]\mathbf{\Theta}\mathbf{s}[k-1] + \mathbf{v}[k], \quad (2)$$

where  $\mathbf{v}[k]$  is a AWGN vector whose entries have zero-mean and variance  $\sigma_v^2$  and is defined in the same way as  $\mathbf{y}[k]$ .  $\mathbf{H}[k; 0]$  and  $\mathbf{H}[k; 1]$  are  $M \times M$  matrices whose entries are given by  $[\mathbf{H}[k; t]]_{r,s} = h_{(kM+r, tM+r-s)}$  with  $t \in (0, 1)$  and  $r, s \in (0, \dots, M-1)$ . Defining  $\mathbf{D}[f_q]$  as a diagonal matrix whose diagonal entries are given by  $[\mathbf{D}[f_q]]_{m,m} = e^{j2\pi f_q m}$ ,  $m \in (0, 1, \dots, M-1)$ , and further defining  $[\mathbf{H}_q[k; t]]_{r,s} = h_q(k, tM+r-s)$  as Toeplitz matrices formed of BEM coefficients, it is straightforward to represent Eq. (2) as

$$\mathbf{y}[k] = \sum_{t=0}^1 \sum_{q=0}^Q \mathbf{D}[f_q] \mathbf{H}_q[k; t] \mathbf{\Theta}\mathbf{s}[k-t] + \mathbf{v}[k] \quad (3)$$

### III. DIVERSITY ORDER OF LINEAR EQUALIZERS

#### A. Frequency selective channel

Consider the case of zero-padded (ZP) block transmission of time-domain symbol vector  $\mathbf{s}[k]$  in a frequency selective channel of order  $L$ . Such a scheme involves padding  $\mathbf{s}[k]$  with  $M - N \geq L$  zero symbols before transmission over the frequency selective channel. In other words, the precoding matrix  $\mathbf{\Theta} = [\mathbf{I}_N \mathbf{0}_{N \times (M-N)}]^T$ . Since the frequency selective channel is a special case of a doubly selective channel corresponding to  $Q = 0$ , we can drop the subscript  $q$  in the received signal representation and rewrite Eq. (3) as

$$\mathbf{y}[k] = \mathbf{H}_0[k; 0]\mathbf{\Theta}\mathbf{s}[k] + \mathbf{H}_0[k; 1]\mathbf{\Theta}\mathbf{s}[k-1] + \mathbf{v}[k] \quad (4)$$

In general, since  $M > L$ , the delay spread of the channel introduces inter-block-interference (IBI) at the receiver and is represented by the second term on the RHS of Eq. (4).  $\mathbf{H}_0[k; 1]$  is a strictly upper-triangular matrix with non-zero elements in only the last  $M - L$  columns of the matrix. Zero-padding has the desirable effect of setting IBI to zero since  $\mathbf{H}_0[k; 1]\mathbf{\Theta} = \mathbf{0}$  and the received signal can therefore be expressed as

$$\mathbf{y}[k] = \mathcal{H}[k]\mathbf{s}[k] + \mathbf{v}[k]. \quad (5)$$

where  $\mathcal{H}[k] = \mathbf{H}_0[k; 0]\mathbf{\Theta}$ , the effective channel seen at the receiver due to zero-padding (in general, precoding) at the transmitter, is a  $M \times N$  Toeplitz matrix with  $[h_0(k, 0), h_0(k, 1), \dots, h_0(k, L), \mathbf{0}_{1 \times (M-L-1)}]^T$  as its first column. The linear estimate for the symbols of the  $k^{\text{th}}$  received block is then given by the MMSE-ZF equalizer

$$\mathbf{g}^{\text{MMSE-ZF}} = (\mathcal{H}^H[k]\mathcal{H}[k])^{-1}\mathcal{H}^H[k]. \quad (6)$$

In what follows, we shall simplify the notation of  $\mathcal{H}[k] = \mathcal{H}$ .

1) *Diversity order of LE*: In [1] the authors introduce a metric namely the orthogonality deficiency (*od*) of the equivalent channel matrix  $\mathcal{H}$  at the receiver and prove that LE can achieve the same diversity as MLE if there exists  $0 < \nu < 1$  such that

$$od(\mathcal{H}) < 1 - \nu \quad (7)$$

For the case of ZP transmission in frequency selective channels, the Toeplitz structure of  $\mathcal{H}$  ensures that  $\det(\mathcal{H}^H\mathcal{H}) > 0$ . Our interest lies in finding a lower bound for this value. Here we use well known concepts from linear prediction theory to provide an upper bound for  $od(\mathcal{H})$  and prove that  $od(\mathcal{H})$  is bounded strictly below 1. Consider the linear prediction problem of a stationary process with covariance matrix  $(\mathcal{H}^H\mathcal{H})$  and spectrum  $|H(f)|^2$  given by

$$H(f) = \sum_{l=0}^L h(l)e^{-j2\pi fl}, \quad (8)$$

$$\|\mathbf{h}\|_2^2 = \int_{-1/2}^{+1/2} |H(f)|^2 df. \quad (9)$$

Then,  $(\mathcal{H}^H\mathcal{H})$  can be factorized as  $\mathbf{L}\mathbf{D}\mathbf{L}^H$ , where  $\mathbf{L}$  is a lower-triangular matrix with unit diagonal and  $\mathbf{D}$  is a diagonal matrix whose  $n^{\text{th}}$  diagonal element, denoted by  $\sigma_n^2$  corresponds to the  $(n-1)^{\text{th}}$  order prediction error variance of this process. In the limiting case, we have

$$\lim_{N \rightarrow \infty} (\det(\mathcal{H}^H\mathcal{H}))^{1/N} = \left( \prod_{n=0}^{N-1} \sigma_n^2 \right)^{1/N} \rightarrow \sigma_\infty^2, \quad (10)$$

where the infinite order prediction error variance  $\sigma_\infty^2$  is related to the spectrum  $|H(f)|^2$  [7] as

$$\sigma_\infty^2 = \exp \left( \int_{-1/2}^{+1/2} \ln |H(f)|^2 df \right) = \frac{|H(f)|^2}{|P(f)|^2}, \quad (11)$$

where  $P(f)$  is the monic minimum phase equivalent of  $H(f)$  and is given by

$$P(f) = 1 + \sum_{l=1}^L p_l e^{-j2\pi fl} \quad (12)$$

Due to the fact that minimum phase filter coefficients are bounded, it was shown in [8] that

$$\|\mathbf{p}\|_2^2 = \int_{-1/2}^{+1/2} |P(f)|^2 df = \left( 1 + \sum_{l=1}^L p_l^2 \right) \leq c_L = \sum_{l=0}^L \binom{l}{L}^2. \quad (13)$$

From Eq. (11) and Eq. (13), we have

$$\sigma_\infty^2 = \frac{\|\mathbf{h}\|_2^2}{\|\mathbf{p}\|_2^2} \geq \frac{\|\mathbf{h}\|_2^2}{c_L}. \quad (14)$$

From Eq. (14), Eq. (10) and the definition of orthogonality deficiency in [1] we have

$$od(\mathcal{H}) = 1 - \frac{\det(\mathcal{H}^H\mathcal{H})}{\det(\text{diag}(\mathcal{H}^H\mathcal{H}))} < 1 - \left( \frac{1}{c_L} \right)^N, \quad (15)$$

which concludes our proof.

2) *Discussion:* In [2] it was shown for the first time that the MMSE-ZF equalizer collects full diversity for linearly precoded transmission in frequency selective channels by showing that

$$\|(\mathcal{H}^H \mathcal{H})^{-1} \mathcal{H}^H\|^{-2} > C \|\mathbf{h}\|^2, \quad (16)$$

where  $\|\cdot\|$  on the LHS of Eq. (16) is the Frobenius norm,  $\mathbf{h}$  is the channel impulse response,  $C$  is a constant independent of the channel and is given by  $C = \lambda_{\min}/(C_1 N(R+1))$  (cf. Eq. (18) in [2]) and  $R = M - N$ . We show here that Eq. (16) in fact implies Eq. (7). To demonstrate this, let  $X = (\mathcal{H}^H \mathcal{H})$ . Since  $X$  is positive definite for  $\|\mathbf{h}\|^2$ ; so is  $X^{-1}$ . From a straightforward application of the arithmetic-geometric mean inequality for positive numbers, we have

$$\det(X^{-1}) \leq \left(\frac{1}{N} \text{tr}(X^{-1})\right)^N, \quad (17)$$

$$\det(X) \geq \left(\frac{N}{\text{tr}(X^{-1})}\right)^N \quad (18)$$

Since  $\|(\mathcal{H}^H \mathcal{H})^{-1} \mathcal{H}^H\|^2 = \text{tr}(X^{-1})$ , we have from Eq. (18) and Eq. (16)

$$\det(X) \geq \left(\frac{\lambda_{\min} \|\mathbf{h}\|^2}{C_1(R+1)}\right)^N \quad (19)$$

Substituting this in the definition of orthogonal deficiency we have

$$od(\mathcal{H}) < 1 - \left(\frac{\lambda_{\min}}{C_1(R+1)}\right)^N. \quad (20)$$

### B. Doubly selective channels

We now look at the case of block transmission in doubly selective channels. The channel is assumed to be of order  $L$  and the time-variation of each channel tap within a block is captured by  $Q + 1$  complex-exponential basis functions. The  $k$ -th receive block is then represented as in Eq. (3). The precoding matrix  $\Theta$  that we consider here is known to enable diversity order of  $(Q+1)(L+1)$  for ML receivers in doubly selective channels [5] and is given by

$$\Theta = \mathbf{F}_{P+Q}^H \mathbf{T}_1 \otimes \mathbf{T}_2, \quad (21)$$

where  $\mathbf{F}_{P+Q}$  is a  $(P+Q)$ -point DFT matrix,  $\mathbf{T}_1 = [\mathbf{I}_P \ \mathbf{0}_{P \times Q}]^T$ ,  $\mathbf{T}_2 = [\mathbf{I}_K \ \mathbf{0}_{K \times L}]^T$ .  $P$  and  $K$  are chosen such that  $M = (P+Q)(K+L)$  and  $N = PK$ . The  $PK$ -length symbol vector is defined in the frequency domain. The zero-padding matrix  $\mathbf{T}_2$  nulls the inter-block-interference component in the received signal, i.e.,  $\mathbf{H}_q[k; 1] \Theta \mathbf{s}[k-1] = 0$ . As a result, the received block can now be represented as

$$\mathbf{y}[k] = \sum_{q=0}^Q \mathbf{D}[f_q] \mathbf{H}_q[k; 0] \Theta \mathbf{s}[k] + \mathbf{v}[k], \quad (22)$$

Using standard Kronecker product identities, one can show that

$$\mathbf{H}_q[k; 0] \Theta = \mathbf{F}_{P+Q}^H \mathbf{T}_1 \otimes \tilde{\mathbf{H}}_q[k; 0] \mathbf{T}_2 \quad (23)$$

where  $\tilde{\mathbf{H}}_q[k; 0]$  is a  $K+L \times K+L$  Toeplitz matrix formed by the first  $K+L$  rows and columns of  $\mathbf{H}_q[k; 0]$ . Eq.(22) can then be re-written as

$$\mathbf{y}[k] = \sum_{q=0}^Q \mathbf{D}[f_q] \left( \mathbf{F}_{P+Q}^H \mathbf{T}_1 \otimes \tilde{\mathbf{H}}_q[k; 0] \mathbf{T}_2 \right) \mathbf{s}[k] + \mathbf{v}[k] \quad (24)$$

Note that

$$\mathbf{D}[f_q] = \mathbf{D}_{P+Q}[f_q(K+L)] \otimes \mathbf{D}_{K+L}[f_q] \quad (25)$$

Eq. (25) represents  $\mathbf{D}[f_q]$  as Kronecker product of time-variation over two scales.  $\mathbf{D}_{P+Q}[f_q(K+L)]$  is a size  $P+Q$  diagonal matrix with  $\exp(j2\pi p f_q(K+L))$ ,  $p \in (0, 1, \dots, P+Q-1)$  on its diagonals and represents time-variation on a coarse scale (complex-exponentials sampled at sub-sampling interval of  $(K+L)T_s$ ) and  $\mathbf{D}_{K+L}[f_q]$  is a diagonal matrix of size  $K+L$  with  $\exp(j2\pi k f_q(K+L))$ ,  $k \in (0, 1, \dots, K+L-1)$ . Using Eq. (25) and standard matrix identities, we can decompose the received signal as in Eq. (26) where  $\mathbf{J}[q] = \mathbf{J}^{(q-Q/2)}$  and  $\mathbf{J}$  is a circulant matrix with  $[0, 1, \mathbf{0}_{1 \times P+Q-2}]^T$  as the first column. Since the matrix  $(\mathbf{F}_{P+Q}^H \otimes \mathbf{I}_{K+L})$  has no effect on the diversity of the doubly selective channel, for the analysis of the diversity order of MMSE-ZF receiver, the effective channel matrix can be represented as

$$\mathbf{H}_{ds}[k] = \sum_{q=0}^Q (\mathbf{J}_{P+Q}[q] \mathbf{T}_1) \otimes (\mathbf{D}_{K+L}[f_q] \tilde{\mathbf{H}}_q[k; 0] \mathbf{T}_2) \quad (28)$$

Fig. 2 illustrates the structure of the equivalent channel matrix  $\mathbf{H}_{ds}[k]$  for this case due to precoding.  $\tilde{\mathbf{H}}_q$  represents the product matrix  $\mathbf{D}_{K+L}[f_q] \tilde{\mathbf{H}}_q[k; 0]$ . In particular, it is a block-Toeplitz matrix with constituent blocks which are in turn formed by the product of a diagonal matrix  $\mathbf{D}_{K+L}[f_q]$  and a Toeplitz matrix formed by the corresponding BEM coefficients of the  $q$ -th basis function.

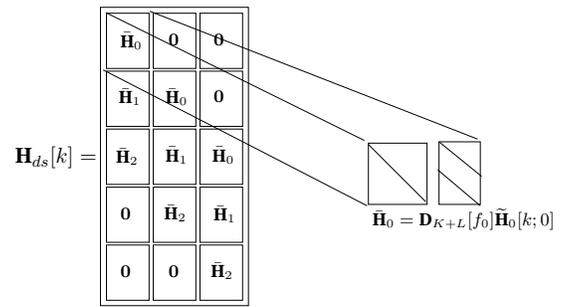


Fig. 2. Equivalent channel matrix for doubly selective channel.

1) *Diversity order of LE in doubly selective channels:* We first consider a closely related matrix  $\tilde{\mathbf{H}}_{ds}[k]$  which is a block Toeplitz matrix where the constituent blocks are formed by  $\tilde{\mathbf{H}}_q[k; 0]$  (In Fig. 2 this corresponds to  $\tilde{\mathbf{H}}_q = \tilde{\mathbf{H}}_q[k; 0]$ ) Then,

$$\tilde{\mathbf{H}}_{ds}[k] = \sum_{q=0}^Q (\mathbf{J}_{P+Q}[q] \otimes \mathbf{H}_q^c[k; 0]) (\mathbf{T}_1 \otimes \mathbf{T}_2), \quad (29)$$

$$= \tilde{\mathcal{H}}[k] \mathcal{T} \quad (30)$$

$$\mathbf{y}[k] = \sum_{q=0}^Q \left( (\mathbf{D}_{P+Q}[f_q(K+L)]\mathbf{F}_{P+Q}^H \mathbf{T}_1) \otimes (\mathbf{D}_{K+L}[f_q]\tilde{\mathbf{H}}_q[k;0]\mathbf{T}_2) \right) \mathbf{s}[k] + \mathbf{v}[k], \quad (26)$$

$$\mathbf{y}[k] = (\mathbf{F}_{P+Q}^H \otimes \mathbf{I}_{K+L}) \sum_{q=0}^Q \left( (\mathbf{J}_{P+Q}[q]\mathbf{T}_1) \otimes (\mathbf{D}_{K+L}[f_q]\tilde{\mathbf{H}}_q[k;0]\mathbf{T}_2) \right) \mathbf{s}[k] + \mathbf{v}[k], \quad (27)$$

where  $\mathbf{H}_q^c[k;0]$  is a circulant matrix whose first column is the same as the first column of  $\tilde{\mathbf{H}}_q[k;0]$ ,  $\mathcal{T} = (\mathbf{T}_1 \otimes \mathbf{T}_2)$ . Note that  $\tilde{\mathbf{H}}_{ds}[k]$  is block Toeplitz with Toeplitz blocks (BTTB) and  $\tilde{\mathcal{H}}[k]$  is block circulant with circulant blocks (BCCB) and is therefore diagonalizable, i.e.,

$$\tilde{\mathcal{H}}[k] = (\mathbf{F}_{P+Q}^H \otimes \mathbf{F}_{K+L}^H) \mathcal{D} (\mathbf{F}_{P+Q} \otimes \mathbf{F}_{K+L}), \quad (31)$$

where  $\mathbf{F}_{K+L}$  is a  $(K+L)$ -point DFT matrix,  $\mathcal{D}$  is a block diagonal matrix with diagonal matrices  $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_{P+Q-1}$  on its diagonals. The entries in  $\mathcal{D}_p$  are given by

$$[\mathcal{D}_p]_{i,i} = \sum_{q=0}^Q \sum_{l=0}^L h(q,l) e^{-\frac{j2\pi il}{K+L}} e^{-\frac{j2\pi pq}{P+Q}}, \quad (32)$$

with  $p \in (0, 1, \dots, P+Q-1)$  and  $i \in (0, 1, \dots, K+L-1)$ . We now introduce the vector of stacked channel coefficients  $\mathbf{h}[k] = [\mathbf{h}_0^T, \mathbf{h}_1^T, \dots, \mathbf{h}_Q^T]$  with  $\mathbf{h}_q = [h_q(k,0), h_q(k,1), \dots, h_q(k,L)]^T$  and

$$\mathbf{V} = \tilde{\mathbf{F}}_{P+Q} \otimes \tilde{\mathbf{F}}_{K+L}, \quad (33)$$

where  $\tilde{\mathbf{F}}_{P+Q}$  corresponds to the first  $1+Q/2$  and last  $Q/2$  columns of  $\mathbf{F}_{P+Q}$  and  $\tilde{\mathbf{F}}_{K+L}$  corresponds to the first  $L+1$  columns of  $\mathbf{F}_{K+L}$ . Then  $\mathcal{D} = \text{diag}(\mathbf{d})$  contains the two dimensional (2-D) DFT of  $\mathbf{h}[k]$ , i.e.  $\mathbf{d} := \mathbf{V}\mathbf{h}[k]$ . Thereafter, by defining  $\mathcal{F} = (\mathbf{F}_{P+Q} \otimes \mathbf{F}_{K+L})\mathcal{T}$ , and  $\varphi := (\phi_0, \phi_1, \dots, \phi_{R-1})$  as the indices of the  $R$  smallest diagonal elements of  $\mathbf{d}$ , we can extend the *Lemma* in [2] for the 2-D case and show that

$$\|(\tilde{\mathbf{H}}_{ds}^H[k](\tilde{\mathbf{H}}_{ds}[k])^{-1}(\tilde{\mathbf{H}}_{ds}^H[k]))\|^{-2} > \mathcal{C} \|\mathbf{h}[k]\|^2, \quad (34)$$

Extending the results in Sec. III-A2 above, we can show that

$$od(\tilde{\mathbf{H}}_{ds}[k]) < 1 - \left( \frac{\Delta_{min}}{\mathcal{C}_1(R+1)} \right)^N, \quad (35)$$

$\Delta_{min}$  and  $\mathcal{C}_1$  are both obtained by a straightforward extension of [2] and  $R = M - N$ . In our case, if we collect the rows of  $\mathbf{V}$  corresponding to the indices in  $\varphi$  in  $\mathbf{V}_\varphi$ , then  $\Delta_{min} = \min_{\text{varphi}}(\lambda_{min}(\mathbf{V}_\varphi^H \mathbf{V}_\varphi))$ , with  $\lambda_{min}(\cdot)$  denoting the minimum eigenvalue of a matrix and the outer minimization is over all subsets of  $(0, 1, \dots, M-1)$ ,  $\mathcal{C} = \Delta_{min}/(\mathcal{C}_1 N(R+1))$ ,  $\mathcal{C}_1 := \max_\varphi \|\Theta_\varphi\|^2$  with  $R$  elements,  $\Theta_\varphi$  being the  $N \times M$  matrix obtained by first removing all rows of  $\mathcal{F}$  with indices belonging to  $\varphi$ , computing the inverse of this square matrix, and inserting  $R$  zero columns at column indices corresponding to the indices in  $\varphi$ . The maximization to obtain  $\mathcal{C}_1$  is done over all subsets of  $(0, 1, \dots, M-1)$  with  $R$  elements.

Furthermore, we can also consider the  $\mathbf{LDL}^H$  decomposition of  $(\tilde{\mathbf{H}}_{ds}^H[k]\tilde{\mathbf{H}}_{ds}[k])$  with the  $n^{\text{th}}$  diagonal element of  $\mathbf{D}$

corresponding this time to the  $(n-1)^{\text{th}}$  order prediction error variance of a 2-D stationary process with covariance matrix  $(\tilde{\mathbf{H}}_{ds}^H[k]\tilde{\mathbf{H}}_{ds}[k])$  and spectrum  $|H(f_1, f_2)|^2$  given by

$$H(f_1, f_2) = \sum_{q=0}^Q \sum_{l=0}^L h(q,l) e^{-j2\pi f_1 l} e^{-j2\pi f_2 q}, \quad (36)$$

then as  $N \rightarrow \infty$ ,

$$(\det(\tilde{\mathbf{H}}_{ds}^H[k](\tilde{\mathbf{H}}_{ds}[k]))^{1/N} \rightarrow \sigma_{\infty,2-D}^2, \quad (37)$$

with

$$\sigma_{\infty,2-D}^2 = \exp \left( \int_{-1/2}^{+1/2} \int_{-1/2}^{+1/2} \ln |H(f_1, f_2)|^2 df_1 df_2 \right) \quad (38)$$

Considering  $H(f_1, f_2)$  as a 1-variable polynomial in  $f_2$  with coefficients being polynomials in  $f_1$ , the following inequalities in Eq. (39) - Eq. (42) leads us to the expression

$$od(\tilde{\mathbf{H}}_{ds}[k]) < 1 - \left( \frac{1}{c_L c_Q} \right)^N \quad (43)$$

The proof above shows that orthogonal deficiency of the BTTB matrix related to the effective channel matrix  $\mathbf{H}_{ds}[k]$  is indeed bounded above by a value strictly less than 1. Note that due to its dependency on  $\det(\mathbf{H}^H \mathbf{H})$ , is related to the degree of predictability of any of its column based on the observations of its previous columns. In the case of  $\tilde{\mathbf{H}}_{ds}[k]$ , this amounts to the prediction error variance of a 2-D stationary process while in the case of  $\mathbf{H}_{ds}[k]$ , this results in the prediction error variance of a 2-D non-stationary process since the stationarity is destroyed by pre-multiplication of the Toeplitz blocks of  $\mathbf{H}_{ds}[k]$  by the diagonal matrix  $\mathbf{D}_{K+L}[f_q]$  (consequently  $\mathbf{H}_{ds}^H[k]\mathbf{H}_{ds}[k]$  is only a block Toeplitz matrix and not BTTB). Since stationary processes are known to be more predictable than non-stationary processes, the orthogonal deficiency of  $od(\mathbf{H}_{ds}[k])$  will always be lower than  $od(\tilde{\mathbf{H}}_{ds}[k])$  and will therefore share the same upper bound. In Table. I we present a comparison of empirical results for maximum value of orthogonal deficiency of  $\mathbf{H}_{ds}[k]$  and  $\tilde{\mathbf{H}}_{ds}[k]$  to corroborate this argument. The column at the extreme right corresponds to an analytical upper-bound for  $od(\tilde{\mathbf{H}}_{ds}[k])$  as in Eq.(43). The other two columns correspond to the maximum value of orthogonal deficiency of  $od(\mathbf{H}_{ds}[k])$  and  $od(\tilde{\mathbf{H}}_{ds}[k])$  over  $10^8$  Monte-Carlo realizations of a doubly selective channel with  $Q = 2, L = 1$  for two different values of  $P, K$ . It is observed that indeed  $od(\mathbf{H}_{ds}[k])$  is slightly lower than  $od(\tilde{\mathbf{H}}_{ds}[k])$ .

Test Case	$od(\mathbf{H}_{ds}[k])$	$od(\tilde{\mathbf{H}}_{ds}[k])$	$1 - (\frac{1}{c_L c_Q})^N$
$K = 2, P = 2$	0.844300451	0.851578361	0.978938817
$K = 4, P = 3$	0.999946896	0.999955760	0.999990658

TABLE I  
COMPARISON OF ORTHOGONAL DEFICIENCY OF  $\mathbf{H}_{ds}$  AND  $\tilde{\mathbf{H}}_{ds}$ .

#### IV. NUMERICAL RESULTS

In this section we provide simulation results to show that MMSE-ZF receiver achieves full diversity in doubly selective channels. The diversity order of MMSE-ZF receiver is estimated based on the slope of the outage probability curve. Monte-Carlo simulations were carried out for a fixed transmission rate for different SNR points. The decision-point SNR for a fixed arbitrary symbol index  $n$  in the  $k$ -th symbol block  $\mathbf{s}[k]$  was computed as

$$\text{SNR}_n = \frac{\rho}{[\mathcal{H}[k]^H \mathcal{H}[k]]_{n,n}^{-1}}, \quad (44)$$

where  $\mathcal{H}[k]$  represents the equivalent channel matrix for the doubly selective channel and  $\rho$  is the SNR. When the decision point SNR was below the SNR required to support the fixed transmission rate, the channel was declared to be in outage. The slope of the outage probability curve was then used as an estimate of the diversity order. In addition to this, we compare the slope of the MMSE-ZF receiver to that of the matched filter bound (MFB) which is known to collect all the available diversity in the channel. In Fig. 3 we plot the performance of LE for linearly precoded transmission in doubly selective channel with  $Q = 2, L = 1, P = 3, K = 4$ . The outage probability curve exhibits a slope of  $(Q+1)(L+1)$  which leads us to conclude that LE achieves full diversity in doubly selective channel when an appropriate diversity enabling precoder is used at the transmitter.

#### V. CONCLUSIONS

In this contribution, we showed that linear equalizers collect full diversity offered by doubly selective channels in the presence of appropriate precoding. For this case, the maximum value of orthogonality deficiency of a matrix closely related to the effective channel matrix at the receiver was shown to be strictly bounded away from 1. It was argued that orthogonality deficiency for the channel matrix also shares the same upper bound and hence achieves full diversity with linear

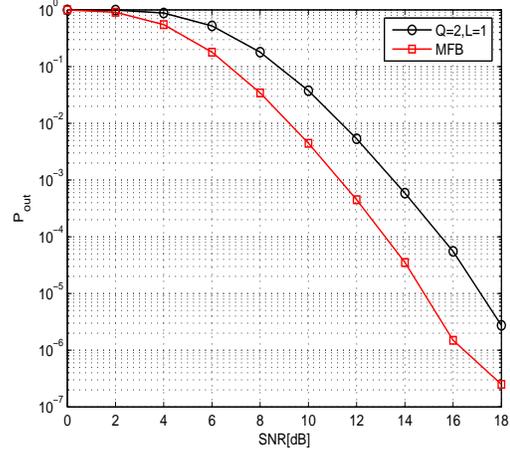


Fig. 3. Diversity order of LE in doubly selective channel.

equalization. Simulation results were provided and shown to sustain the arguments made in this paper.

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$$\int_{-1/2}^{+1/2} \int_{-1/2}^{+1/2} \ln |H(f_1, f_2)|^2 df_1 df_2 = \int_{-1/2}^{+1/2} df_2 \int_{-1/2}^{+1/2} \ln |H(f_1, f_2)|^2 df_1, \quad (39)$$

$$\geq \int_{-1/2}^{+1/2} df_2 \frac{1}{c_L} \ln \int_{-1/2}^{+1/2} |H(f_1, f_2)|^2 df_1, \quad (40)$$

$$\geq \frac{1}{c_L c_Q} \ln \int_{-1/2}^{+1/2} df_2 \int_{-1/2}^{+1/2} |H(f_1, f_2)|^2 df_1, \quad (41)$$

$$\Rightarrow (\det(\tilde{\mathbf{H}}_{ds}^H[k]) (\tilde{\mathbf{H}}_{ds}[k]))^{1/N} \geq \frac{1}{c_L c_Q} \int_{-1/2}^{+1/2} \int_{-1/2}^{+1/2} |H(f_1, f_2)|^2 df_1 df_2. \quad (42)$$