Unitary Beamforming under Constant Modulus Constraint in MIMO Broadcast Channels

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***Abstract*—In this paper we analyse unitary beamforming in MIMO broadcast channels where the entries of the beamforming matrix are of constant modulus (CUBF). We provide a general formal description for the beamforming matrices. We show that this description encompasses currently applied constructions such as those based on the Householder transformation. Among other properties the CUBF proves to be particularly robust to channel feedback errors. We propose an iterative construction of the CUBF which maximizes the sum-rate of the system. Furthermore we provide numerical results that show significant gains of the CUBF compared to existing techniques.**

I. Introduction

Multiple-Input Multiple-Output (MIMO) systems have the potential to significantly increase the achievable capacity of a radio link to a Single-User (SU) [1], [2]. Although high individual data rates are often a compelling marketing argument in emerging wireless standards, most of the network operators are interested in increasing their cell throughput or to distribute data rates more uniformly among the users in a cell. This can be achieved if the transmitter employs its antennas to communicate to Multiple non-cooperative Users (MU-MIMO) on the same time-frequency resource. The resulting MIMO Broadcast Channel (BC) has been extensively studied in the past years and many efforts have been made to find transmission schemes that achieve the capacity of the MIMO-BC. In contrast to SU-MIMO, accessing the MIMO-BC capacity is a inherently more difficult problem but it turns out that sharing the MIMO channel between multiple users utilizes the system resources more efficiently [3].

To reduce the interference caused by the imperfect spatial separation of the receivers the transmitter spatially encodes the signal prior to the transmission. This precoding operation and consequently the sum-rate are highly dependent on the Channel State Information available at the Transmitter (CSIT). The full capacity region of the MIMO-BC is achieved by Dirty-Paper Coding (DPC) [4]. This optimal technique is still too complex to be implemented in current wireless systems. It has been shown that suboptimal linear precoding schemes of moderate complexity such as Zero-Forcing Beamforming (ZFBF), Regularized ZFBF (R-ZFBF) [5] or Unitary Beamforming (UBF) [6] achieve a large portion of the MIMO-BC capacity.

In this paper we study UBF where the entries of the beamforming matrix are further constrained to be of equal modulus. This kind of beamformer has several advantages in the case of uniform user power. As all beamforming vectors are orthogonal the receiver can compute the Signal-to-Interference and Noise Ratio (SINR) solely from its channel estimate and beamforming vector. Furthermore, the (average) powers on the different transmit antennas are equal, independent of the UBF setting. This is advantageous when power amplifier nonlinearities and efficiency are taken into account. Moreover UBF is more robust to errors in the CSIT [6]. For these reasons unitary matrices have been adopted in 3GPP LTE [7] for SU-MIMO precoding and are the current assumption for MU-MIMO beamforming. To further reduce the parameterization of the UBF matrices and hence reduce the amount of feedback required in the uplink, people often further constrain the UBF matrices to have entries of equal magnitude, leading to CUBF (Constrained or Constant modulus UBF). In fact, 3GPP LTE defines a finite set of CUBF precoding matrices referred to as a codebook. Among other results we show that the quantization represented by this codebook leads to high suboptimality if applied to MU-MIMO beamforming. Although CUBF represents further constraining on already constrained UBF matrices, apart from the parameterization parsimony, it possesses the advantage of preserving equal transmit antenna powers even under unequal (e.g. optimized) user powers.

This work studies UBF where all entries have a constant modulus. Our analysis is based on complex Hadamard matrices [8], [9] and we study their application to beamforming in the MIMO-BC. To develop a generic construction of the CUBF we use an equivalence relation that allows to structure the problem. One key contribution of this paper is the proposition of a new CUBF design which is based on CSIT. We show that the CUBF generated by the Householder transformation, used in 3GPP LTE, is only one particular subset of the complete set of CUBF matrices. Therefore this restricted set leads to a significant performance loss. In order to evaluate the sum-rate performance we propose an iterative algorithm for the parametrization of the CUBF. Moreover, the impact of imperfect CSIT on the sum-rate of the system is investigated.

Notation: In the following boldface lower-case and uppercase characters denote vectors and matrices, respectively. The operators $(\cdot)^\mathsf{T}$, $(\cdot)^\mathsf{H}$ and $\mathrm{tr}(\cdot)$ denote transpose, conjugate transpose and the trace operator, respectively. The expectation is $\mathsf{E}[\cdot]$ and $\mathrm{diag}(\mathbf{x})$ is a diagonal matrix with vector $\mathbf{x}$ on the main diagonal. The $N \times N$ identity matrix is $\mathbf{I}_N = [\mathbf{e}_1, \ldots, \mathbf{e}_N]$.

II. System Model

Consider the scenario where one transmitter with M antennas communicates to $N \geq M$ single-antenna receivers. We consider random user scheduling, thus there are always $K = M$ users selected for transmission. A beamforming vector $\mathbf{v}_k$ is assigned to each of the K users. We define the beamforming matrix as $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_K] \in \mathbb{C}^{M\times K}$. The transmit signal is formed as

$$\mathbf{x} = \sum_{k=1}^{K} \sqrt{p_k}\mathbf{v}_k s_k \tag{1}$$

where p_k and s_k ($|s_k|^2 = 1$) are the power and the information symbol of user k, respectively. Denote $\mathbf{R}_\mathbf{x} = \mathsf{E}[\mathbf{x}\mathbf{x}^\mathsf{H}]$ the transmit signal covariance matrix and $\mathbf{W} = \mathrm{diag}([p_1, \ldots, p_K])$. Thus the sum-power constraint imposes

$$\mathrm{tr}(\mathbf{R}_\mathbf{x}) = \mathsf{E}[\mathrm{tr}(\mathbf{W}\mathbf{V}^\mathsf{H}\mathbf{V})] \leq P \tag{2}$$

where P is the total available transmit power. We consider narrow-band transmission. Hence for every channel use the received symbol vector reads

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{3}$$

where $\mathbf{H}$ is the channel matrix $\mathbf{H} = [\mathbf{h}_1, \ldots, \mathbf{h}_K]^\mathsf{H} \in \mathbb{C}^{K\times M}$ and $\mathbf{h}_k \in \mathbb{C}^{M\times 1}$ ($k = 1, 2, \ldots, K$) models the channel from the transmitter to user k. The noise vector is Gaussian distributed with $\mathbf{n} \sim \mathcal{CN}(0, \sigma_\mathrm{n}^2\mathbf{I})$ and thus we define the signal-to-noise ratio (SNR) as P/σ_n^2. In particular the received signal of user k is given by

$$y_k = \sqrt{p_k}\mathbf{h}_k^\mathsf{H}\mathbf{v}_k s_k + \sum_{j=1, j\neq k}^{K} \sqrt{p_j}\mathbf{h}_k^\mathsf{H}\mathbf{v}_j s_j + n_k \tag{4}$$

where the first term on the right-hand side is the useful signal of user k. The second term is the inter-user interference resulting from the residual correlation between the users' beamforming vectors $\mathbf{v}_j$ and channel $\mathbf{h}_k$. The last term is the additive noise. As a result the SINR for user k is given by

$$\gamma_k = \frac{p_k|\mathbf{h}_k^\mathsf{H}\mathbf{v}_k|^2}{\sum_{\substack{j=1\\j\neq k}}^{M} p_j|\mathbf{h}_k^\mathsf{H}\mathbf{v}_j|^2 + \sigma_\mathrm{n}^2} \tag{5}$$

The instantaneous sum of the user rates is

$$\mathcal{R} = \sum_{k=1}^{K} \log_2(1 + \gamma_k) \tag{6}$$

The long-term average of the instantaneous sum-rate over the channel realizations of a given distribution.

$$\bar{\mathcal{R}} = \mathsf{E}_\mathbf{H}[\mathcal{R}] \tag{7}$$

In Section VI we measure the performance in terms of ergodic sum-rate.

Note that the $\{\gamma_k\}$ in (5) are invariant to the following transformation

$$\tilde{\mathbf{v}} = e^{j\theta}\mathbf{v} \quad \text{with} \quad \theta \in [0, 2\pi) \tag{8}$$

Consequently the sum-rate (6) does not change when multiplying each beamforming vector with $e^{j\theta}$. Thus, the optimal beamforming vectors $\mathbf{V}$ are not unique. This implies that the first row of $\mathbf{V}$ can be *dephased* i.e. the first row contains only real values.

III. Unitary Beamforming

Linear beamforming techniques are attractive because they offer a good trade-off between performance and complexity. Among them ZFBF and R-ZFBF achieve the full multiplexing gain [5], [10]. In this Section we briefly introduce unitary beamforming to lay the basis for the CUBF in Section IV.

Consider the group of unitary matrices $\mathcal{U}(M)$. Thus, a unitary beamforming (UBF) matrix $\mathbf{V}_\mathrm{u} \in \mathcal{U}(M)$ satisfies

$$\mathbf{V}_\mathrm{u}\mathbf{V}_\mathrm{u}^\mathsf{H} = \mathbf{V}_\mathrm{u}^\mathsf{H}\mathbf{V}_\mathrm{u} = \mathbf{I}_M \tag{9}$$

i.e. all beamforming vectors are mutually orthogonal and of unit norm. Hence the UBF is never able to cancel all inter-user interference except if the user channels $\mathbf{H}$ form itself a unitary matrix. Furthermore with equal power allocation $p_k = P/M$, (5) simplifies to [11]

$$\gamma_k = \frac{\|\mathbf{h}_k\|^2\rho_k^2}{\|\mathbf{h}_k\|^2(1-\rho_k^2) + \frac{M\sigma_\mathrm{n}^2}{P}} \tag{10}$$

with $\rho_k^2 = |\bar{\mathbf{h}}_k^\mathsf{H}\mathbf{v}_k|^2$, $\bar{\mathbf{h}}_k = \frac{\mathbf{h}_k}{\|\mathbf{h}_k\|}$. Here, ρ_k^2 can be interpreted as the *alignment* of a users' beamforming vector with its channel direction.

Note that γ_k in (10) solely depends on user k. The optimization of the UBF with respect to (6) is a non-convex problem and to the authors' knowledge no closed-form solution exists. As a consequence of (8) there remain $M(M-1)$ free parameters for the construction of a UBF.

In [6] an iterative optimization method based on successive Givens rotations was presented. The idea is that every unitary matrix can be represented as a product of Givens rotations and every Givens rotation matrix can be optimized separately. In case of CUBF matrices this kind of optimization is impossible since the multiplication of two CUBF matrices does not maintain the constant modulus property.

IV. UBF With Constant Modulus Elements

In this section we provide the mathematical framework for the construction of unitary beamforming matrices with constant modulus entries $\mathbf{V}_\mathrm{cu}$.

A. Description of Hadamard Matrices

We first have to introduce various definitions that we will use later to parametrize the CUBF.

Definition 1: A square matrix $\mathbf{A}$ of size M where the entries are of equal modulus $|a_{ij}|^2 = \frac{1}{M};\ i,j = \{1,\ldots,M\}$, is called *normalized Hadamard matrix* if

$$\mathbf{A}\mathbf{A}^{\mathsf{H}} = \mathbf{I}_M \tag{11}$$

The set of normalized complex Hadamard matrices of size M is denoted $\mathcal{H}_M$. In the unnormalized case: $\mathbf{A}\mathbf{A}^{\mathsf{H}} = M\,\mathbf{I}_M$.

Definition 2: [8], The complex Hadamard matrices $\{\mathbf{A}, \tilde{\mathbf{A}}\} \in \mathcal{H}_M$ are *equivalent*, written $\mathbf{A} \cong \tilde{\mathbf{A}}$, iff there exist diagonal unitary matrices $\mathbf{D}_{\mathrm{r}}, \mathbf{D}_{\mathrm{c}}$ and permutation matrices $\mathbf{P}_{\mathrm{r}}, \mathbf{P}_{\mathrm{c}}$ such that[1]

$$\mathbf{A} = \mathbf{D}_{\mathrm{r}}\mathbf{P}_{\mathrm{r}}\tilde{\mathbf{A}}\mathbf{P}_{\mathrm{c}}\mathbf{D}_{\mathrm{c}} \tag{12}$$

There are $M!$ row and column permutations. The *equivalence class* of $\mathbf{A} \in \mathcal{H}_M$ is

$$\mathbf{Q}_M(\mathbf{A}) = \{\mathbf{B} \in \mathcal{H}_M | \mathbf{A} \cong \mathbf{B}\} \tag{13}$$

The *set of equivalence classes* $\mathcal{G}_M$ is $\mathcal{G}_M = \mathcal{H}_M/_{\cong}$.

B. Equivalence Classes

Interestingly, the complete set of equivalence classes $\mathcal{G}_M$ is only known for $M<6$. The problem of finding *all* equivalence classes for dimensions $M \geq 6$ remains unsolved and a catalog of known equivalence classes can be found in [8]. In the following we give a short overview of the (unnormalized) equivalence classes for $M = \{2,\ldots,5\}$.

1) $M = 2$: There is only one equivalence class $\mathcal{G}_2 = \mathbf{Q}_2(\mathbf{F}_2)$ with

$$\mathbf{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{14}$$

The real Hadamard matrix coincides with the discrete Fourier transform (DFT) matrix $\mathbf{F}_2$, where $\mathbf{F}_M$ of size M

$$\mathbf{F}_M(m,n) = e^{-j\frac{2\pi}{M}(m-1)(n-1)}\ ;\ m,n = \{1,\ldots,M\} \tag{15}$$

2) $M = 3$: There exists only one equivalence class equal to the DFT matrix $\mathcal{G}_3 = \mathbf{Q}_3(\mathbf{F}_3)$.

3) $M = 4$: Here, there exists a *continuous* family of equivalence classes with one free parameter $\mathcal{G}_4 = \{\mathbf{Q}_4(\mathbf{Q}_4^o(\theta));\ \theta \in [\frac{\pi}{2}, \frac{3}{2}\pi)\}$.

$$\mathbf{Q}_4^o(\theta) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & e^{j\theta} & -e^{j\theta} \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -e^{j\theta} & e^{j\theta} \end{bmatrix} \tag{16}$$

Note that the real Hadamard matrix $\mathbf{Q}_4^o(\pi)$ and the DFT matrix $\mathbf{F}_4 \cong \mathbf{Q}_4^o(\frac{\pi}{2})$ are special cases of (16).

4) $M = 5$: All complex Hadamard matrices are equivalent to the DFT matrix $\mathcal{G}_5 = \mathbf{Q}_5(\mathbf{F}_5)$.

[1]In this definition transposition and complex conjugate are excluded since they are meaningless in the application of beamforming

C. Parametrization of CUBF in MIMO BC

In general, the set of CUBF matrices is equal to the set of normalized complex Hadamard matrices $\mathcal{H}_M$. The description of $\mathcal{H}_M$ is solely given by the equivalence relation (12) and the equivalence classes (13) and can be used to parametrize the CUBF. However, depending on the objective function, some parameters in the general description become obsolete. If the beamforming matrix $\mathbf{V}_{\mathrm{cu}}$ is intended to modify the SINR of each user (and hence the sum-rate) the diagonal unitary matrix $\mathbf{D}_{\mathrm{c}}$ in (12) can be omitted due to the invariance to the transformation in (8). Consequently the diagonal unitary matrix $\mathbf{D}_{\mathrm{r}}$ in (12) takes the form $\mathbf{D}_{\mathrm{r}} = \mathrm{diag}([1, e^{j\varphi_1}, \ldots, e^{j\varphi_{M-1}}])$ with $\varphi_i \in [0, 2\pi),\ i = \{1, ..., M-1\}$.

One may remark that the equivalence relations in (12) involve continuous parameters (phases in the diagonals) and discrete parameters (permutations). One may think of counting the number of continuous parameters by subtracting from the $2M^2$ real entries the number of real constraints imposed by CUBF: M^2 due to unitarity, $(M-1)^2$ for the constant element magnitudes (suffices to apply to a $(M-1)\times(M-1)$ submatrix), and M (for a first row of all 1's). One ends up with $M-1$, which correspond to the $\mathbf{D}_{\mathrm{r}}$ just mentioned. The mystery is then the appearance of θ in $\mathbf{Q}_4^o(\theta)$. The explanation is that counting the obvious constraints must lead to redundancies. The appearance of the additional free parameters can be explained as follows. (Unnormalized) complex Hadamard matrices can in fact be constructed recursively as follows: [9] $\mathbf{V}(\mathbf{A},\mathbf{B}) = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{A} & -\mathbf{B} \end{bmatrix}$ where $\mathbf{A}$ and $\mathbf{B}$ are itself complex Hadamard and hence allow equivalence transformations as in (12). Now, for $\mathbf{A}$ they do not need to be applied since they can equivalently be applied to $\mathbf{V}$. However, since $\mathbf{B}$ appears both as $\mathbf{B}$ and $-\mathbf{B}$, not all equivalences on $\mathbf{B}$ show up in $\mathbf{V}$. At $M = 4$, we can take $\mathbf{A} = \mathbf{B} = \mathbf{F}_2$, but the one such equivalence that needs to be allowed at the level of $\mathbf{B}$ is $\mathbf{D}\mathbf{B}$ with $\mathbf{D} = \mathrm{diag}([1\ e^{j\theta}])$. So we get for $M = 4$: $\mathbf{V}(\mathbf{F}_2, \mathbf{D}\mathbf{F}_2)$.

If $M = 4$ another construction of CUBF matrices via the Householder transformation exists which is used in current practical systems [7]. The set of all CUBF matrices generated by the Householder transformation is

$$\mathcal{V} = \left\{ \mathbf{V} = \mathbf{I}_M - 2\frac{\mathbf{u}\mathbf{u}^{\mathsf{H}}}{\mathbf{u}^{\mathsf{H}}\mathbf{u}} \ \middle|\ \mathbf{u} \in \mathbb{C}^{M\times 1};\ |u_i| = 1;\ u_1 = 1 \right\}. \tag{17}$$

The construction of a CUBF via the Householder transformation describes only a subset of all possible CUBF i.e. $\mathcal{V} \subset \mathcal{H}_4$. To prove $\mathcal{V} \subset \mathcal{H}_4$ observe that $\mathbf{V} \cong \mathbf{Q}_4^o(\pi)$ as $\mathbf{Q}_4^o(\pi) = 2\mathbf{P}_{\mathrm{r}}\mathbf{D}_1\mathbf{D}^{\mathsf{H}}\mathbf{V}\mathbf{D}\mathbf{D}_1\mathbf{P}_{\mathrm{c}}$ with $\mathbf{D} = \mathrm{diag}(\mathbf{u})$, $\mathbf{D}_1 = \mathrm{diag}([1,-1,-1,-1])$, $\mathbf{P}_{\mathrm{c}} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_3]$ and $\mathbf{P}_{\mathrm{r}} = [\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_4]$. Hence $\mathcal{V}$ is the subset of $\mathcal{H}_4$ that stems from the unique real equivalence class $\mathbf{Q}_4^o(\pi)$. Thus restricting $\mathbf{V}_{\mathrm{cu}} \in \mathcal{V}$ leads to a significant performance loss as we show by simulation in Section VI.

V. Optimization of the CUBF

We choose to maximize the sum-rate in (6). In general the beamforming vectors $\{\mathbf{v}_k\}$ and the user powers $\{p_k\}$ have to

$d_0(k,m) = \|a_{km}\|^2 + \|b_{km}\|^2$
$d_1(k,m) = 2\|a_{km}\| \cdot \|b_{km}\|$
$\delta_{km} = \angle b_{km} - \angle a_{km}$
$d_2(k,m) = d_1(k,m)\cos\delta_{km}$
$d_3(k,m) = d_1(k,m)\sin\delta_{km}$

TABLE I
AUXILIARY VARIABLES

be computed according to the following optimization problem

$$\{\mathbf{v}_k^\star, p_k^\star\} = \underset{\{\mathbf{v}_k\},\{p_k\}}{\arg\max} \left\{ \sum_{k=1}^{K} \log(1+\gamma_k) \right\} \tag{18}$$
$$\text{s.t.}: \ \mathrm{tr}(\mathbf{R_x}) \le P;\ \mathbf{V}^{\mathsf{H}}\mathbf{V} = \mathbf{I}_M;\ |v_{ij}|^2 = 1/M\ \forall i,j$$

where γ_k is defined in (5). The problem above is non-convex in $\{\mathbf{v}_k\}$ and $\{p_k\}$ and difficult to solve. However, the description of the CUBF introduced earlier allows us to tackle the problem in (18).

In the following we will assume equal power allocation i.e. $p_k = P/M$. Some aspects of the optimal power allocation strategy are discussed in Section VII.

A. Optimal Parametrization of the CUBF

Under the assumption that there are always $K = M$ users available for transmission and that the transmit power is equally divided among them we can formulate the optimization criterion as follows

$$\{\mathbf{D}_{\mathrm{r}}^\star, \mathcal{G}_M^\star, \mathbf{P}_c^\star, \mathbf{P}_r^\star\} = \underset{\mathbf{D}_{\mathrm{r}},\mathcal{G}_M,\mathbf{P}_c,\mathbf{P}_r}{\arg\max} \left\{ \sum_{k=1}^{K} \log(1+\gamma_k) \right\} \tag{19}$$

where γ_k is defined in (10). The diagonal unitary matrix $\mathbf{D}_{\mathrm{r}}$ contains $M-1$ angles. The optimal permutation matrices $\mathbf{P}_{\mathrm{r}}, \mathbf{P}_{\mathrm{c}}$ have to be found by exhaustive search. Denote $\mathcal{A} = \{\varphi_1, \ldots, \varphi_{M-1}, \theta\}$ the set of angles to be optimized. Note that only for $M = 4$ the set $\mathcal{A}$ contains the additional angle θ. After some algebraic manipulation (19) takes the form

$$\{\mathbf{D}_{\mathrm{r}}^\star, \mathcal{G}_M^\star, \mathbf{P}_c^\star, \mathbf{P}_r^\star\} = \underset{\mathbf{D}_{\mathrm{r}},\mathcal{G}_M,\mathbf{P}_c,\mathbf{P}_r}{\arg\min} \left\{ \prod_{k=1}^{K} \left(1+\beta_k - \rho_k^2\right) \right\} \tag{20}$$

where $\beta_k = \frac{\sigma_{\mathrm{n}}^2 M}{\|\mathbf{h}_k\|^2 P}$. This is still a non-convex optimization and the global optimum can only be found by exhaustive search. Subsequently we present an iterative algorithm to calculate the optimal set of angles $\mathcal{A}$. However, this algorithm can not be guaranteed to converge to the global optimum.

B. Iterative Optimization Algorithm

A joint optimization of the angles in $\mathcal{A}$ is too involved, therefore we optimize the angles one by one while the others are fixed. We can write

$$M\rho_k^2(\varphi_m) = |a_{km} + b_{km}e^{j\varphi_m}|^2 \tag{21}$$

where $\varphi_m \in \mathcal{A}$ and a_{km}, b_{km} are constants. With the substitution

$$s_m = \tan\frac{\varphi_m}{2} \tag{22}$$

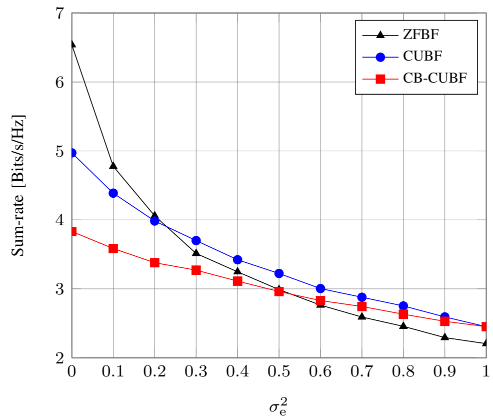

Fig. 1. 2×2 MIMO, sensitivity to errors in CSIT, SNR = 15 dB

and with the auxiliary variables in Table I, we obtain from (21)

$$M\rho_k^2(s_m) = d_0(k,m) - d_2(k,m) - 2\frac{d_3(k,m)s - d_2(k,m)}{1+s^2} \tag{23}$$

From (20) we have the objective function

$$F_m(s_m) = \prod_{k=1}^{M} \left(1+\beta_k - \rho_k^2(s_m)\right) \tag{24}$$

By setting $\frac{\mathrm{d}F_m(s_m)}{\mathrm{d}s_m} = 0$ we have

$$\frac{\mathrm{d}F_m(s_m)}{\mathrm{d}s_m} = -2Ms_mG_m + (1+s_m^2)\frac{\mathrm{d}G_m}{\mathrm{d}s_m} = 0 \tag{25}$$

where $G_m = \prod_{k=1}^{M} \left(c_2(k,m)s_m^2 + c_1(k,m)s_m + c_0(k,m)\right)$ with

$$c_2(k,m) = d_2(k,m) - d_0(k,m) + \beta_k M \tag{26}$$
$$c_1(k,m) = 2d_3(k,m) \tag{27}$$
$$c_0(k,m) = -d_0(k,m) - d_2(k,m) + \beta_k M \tag{28}$$

To solve (25) we have to find the real roots of a polynomial of degree $2M$. Once the roots have been found we undo the substitution in (22) and evaluate (20) to obtain the optimal solution $\varphi_m^\star$. The same approach is used to find $\theta^\star$.

VI. SIMULATION AND RESULTS

In this section we compare the CUBF with the codebooks of CUBF matrices (CB-CUBF) defined in 3GPP LTE [7]. In case of $M = 2$ the codebook contains the identity matrix and two rotations of the DFT matrix according to (12) with $\varphi_1 = \{0, \frac{\pi}{2}\}$ and $\mathbf{P}_{\mathrm{r}} = \mathbf{P}_{\mathrm{c}} = \mathbf{I}_2$. The codebook for 4 transmit antennas is a subset of $\mathcal{V}$ defined in (17) generated by 16 vectors $\mathbf{u}$ where the elements of $\mathbf{u}$ are taken from a 8-PSK constellation and $u_1 = 1$. The optimal CB-CUBF is computed at the transmitter by exhaustive search based on the available CSIT. The performance metric is the achievable ergodic sum-rate (7). Throughout this section we average our results over 10.000 independent Rayleigh fading channel realizations.

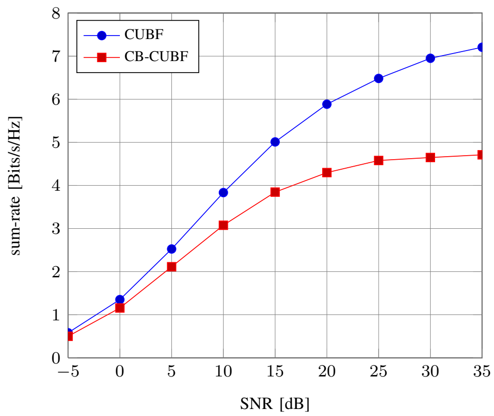

Fig. 2. 2×2 MIMO, Sum-rate vs. SNR

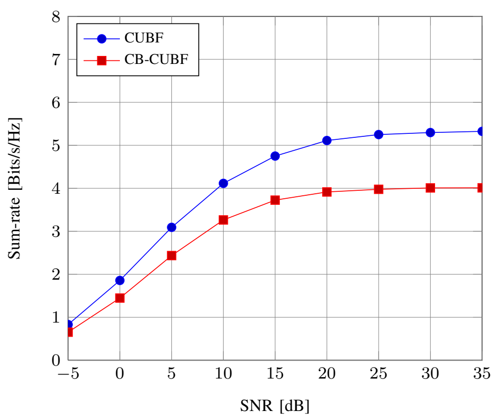

Fig. 3. 4×4 MIMO, Sum-rate vs. SNR

Figure 1 shows the sensitivity of CUBF, CB-CUBF and ZFBF to erroneous CSIT $\mathbf{H}_\mathrm{e}$ which is modeled as $\mathbf{H}_\mathrm{e} = \sqrt{1-\sigma_\mathrm{e}^2}\mathbf{H} + \sigma_\mathrm{e}\mathbf{N}$ where the entries of $\mathbf{H}$ and $\mathbf{N}$ are i.i.d. Gaussian with zero mean and unit variance. From Figure 1 it can be observed that CUBF and CB-CUBF outperform ZFBF starting from $\sigma_\mathrm{e}^2 = 0.22$ and $\sigma_\mathrm{e}^2 = 0.5$, respectively. ZFBF, that achieves high sum-rates under the assumption of perfect CSIT, experiences a severe performance loss as soon as the CSIT is erroneous. In practical systems such a scheme is not attractive since it requires highly accurate CSIT which entails an enormous feedback overhead.

Figures 2 and 3 present the sum-rate performance for a 2×2 and 4×4 MIMO system, respectively. We observe that the CUBF significantly outperforms the CB-CUBF in both MIMO configurations. At an SNR of 20 dB the gain is about 40 % and 30 %, respectively.

VII. Discussion

Though joint CUBF and power allocation optimization is a subject of ongoing research, it has two simple limiting cases, the low and the high SNR regimes.

For BC at low SNR it has been shown in [12] that TDMA is optimal, i.e. TDMA achieves asymptotically the same sum-rate as DPC. Therefore, it is not surprising that optimized CUBF with power allocation assigns all available power to the strongest user (single stream transmission is optimal at low SNR) and that this achieves a rate close to the sum-capacity. The optimal constant modulus beamforming vector is given by Equal Gain Transmission (EGT) [13] to the user whose channel $\mathbf{h} = [h_1, \ldots, h_M]^\mathsf{T}$ has the largest 1-norm. Thus $\mathbf{v} = 1/\sqrt{M}[1, e^{-j(\angle h_2 - \angle h_1)}, \ldots, e^{-j(\angle h_{M-1} - \angle h_1)}]^\mathsf{H}$ where $\angle x$ denotes the phase of x.

In the high SNR regime CUBF is interference limited. The optimization problem in (20) becomes $\min \prod_{k=1}^{K}(1-\rho_k^2)$. The optimal solution is clearly again to put all power on one user, since in this case the inter-user interference is zero and rate saturation is avoided, and the optimal CUBF corresponds again to EGT to the user with largest 1-norm.

VIII. Conclusion

In this paper we present the unitary beamformer with constant modulus elements and relate its construction to the problem of parametrizing complex Hadamard matrices. We show that the construction by the Householder transformation covers only a small subset of all possible CUBF matrices and therefore leads to a performance loss. Furthermore we show that CUBF is superior to ZFBF beamforming techniques under imperfect CSIT.

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