ON THE PRE-LOG OF THE ASYMPTOTIC CAPACITY OF STATIONARY FREQUENCY-FLAT MULTI-ANTENNA CHANNELS

Umer SALIM, Dirk SLOCK

Eurecom Institute , 2229 route des Crêtes, B.P. 193,
06904 Sophia Antipolis Cedex, FRANCE
e-mail:{umer.salim,dirk.slock}@eurecom.fr

ABSTRACT

We continue previous work [1] in which we analyzed the Mutual Information of Frequency-Flat MIMO Channels with a Block Fading model. Absence of Channel State Information at Transmitter or Receiver (no CSIT/CSIR) was considered there. It was shown that implicitly a (semi-blind) channel estimate needs to be constructed that, in decision-feedback style, depends on the past (detected) inputs and outputs (training part) and on the future outputs (blind part).

On the other hand, for peak-power limited SISO frequency-flat channels with stationary Gaussian fading, it has been shown by Lapidoth [2] that at high SNR, the capacity is determined by a pre-log factor that is equal to the bandwidth of frequencies where the channel Doppler spectrum is zero (the complementary part of the Doppler bandwidth).

In this paper we extend Lapidoth's result to MIMO channels with bandlimited stationary fading. At high SNR, the absence of CSIR decreases the pre-log with a factor equal to 1 minus the average number of parameters per symbol period that parameterize the channel. This reduction term is proportional to the Doppler bandwidth and the number of transmit antennas. We introduce channel parameterizations that naturally induce a split in the transmitted symbols between "learning" symbols (that carry loglog(SNR) information) and "data" symbols (that carry log(SNR) information). The capacity pre-log factor is the proportion of "data" symbols.

1. INTRODUCTION

Information theoretic bounds for different types of channels have got utmost importance since the explosion of research in MIMO promised new dimensions for data communication. Such capacity bounds are very important in the sense that they give the theoretical limits and motivate researchers to achieve them in practical systems literally or asymptotically. The area of capacity analysis for non-coherent (no CSIR and no CSIT) fading channels has received considerable attention in recent years since the usual assumption of perfect CSIR is not true in practical systems and channel realizations need to be estimated for correct decoding of data.

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Usually block fading models are assumed for obtaining capacity bounds in the no CSIR (non-coherent) case. In the standard version of this model [3], the fading remains constant over blocks consisting of T symbol periods, and changes independently from block to block. Capacity bounds are obtained by introducing training segments in an ad hoc fashion. For the standard block fading model, the capacity is shown [3], [4] to grow logarithmically with SNR. Later Veeravalli [5] allowed the fading to vary inside the block with a certain correlation matrix characterized by a rank Q and showed for SISO channels that the capacity pre-log is $(1-Q/T)$.

Non-coherent capacity has also been analyzed with the channel fading process being symbol-by-symbol stationary. In this model, fading is not independent but time selective without block structure. Surprisingly at first, this model leads to very different capacity results: contrary to log(SNR) capacity growth in block fading channels, here the capacity grows only double logarithmically with SNR at high SNR [6], [7], [8] when the fading process is non-bandlimited, i.e. the channel prediction error is non-zero.

For symbol-by-symbol stationary Gaussian fading channels, if the Doppler spectrum is band-limited (limited support), then the fading process is called non-regular and the prediction error using the infinite past is zero. Lapidoth [2] studied the SISO case for this kind of fading processes showing that capacity grows logarithmically with SNR and capacity pre-log is the Lebesgue measure of the frequencies where the spectral density of the fading process (Doppler spectrum) has nulls.

Chen and Veeravalli [9] introduce a block-stationary channel model that can encompass both the per-symbol stationary and block fading models. They obtain the SISO capacity pre-log for both cases. They argue that the log(SNR) regime results from the rank deficiency of the correlation matrix of the fading process (though bandlimited fading only leads to rank

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deficiency over a block as the block length goes to infinity).

2. SYSTEM MODEL

We consider a MIMO fading channel whose time-k output $Y_k \in C^N$ is given by

$$Y_k = H_k X_k + Z_k \tag{1}$$

where $X_k \in C^M$ denotes the time-k channel input vector and the fading matrix $H_k \in C^{N\times M}$ represents the time-k fading matrix and $Z_k \in C^N$ denotes the additive gaussian noise vector. Here C denotes the complex field and, M and N represent the number of transmit and receive antennas respectively. We assume that the zero-mean circularly complex Gaussian noise is spatiotemporally white with spatial covariance matrix I_N, which represents the $N \times N$ identity matrix. The channel fading process $\{H_k\}$ is assumed to be stationary, ergodic and with finite second order moment, i.e. $E[||H_k||^2] < \infty$. We take the fading process to be strictly bandlimited, so it is a non-regular stochastic process with limited Doppler spectrum support. Moreover we impose the restriction that the support is of size $1/D$ for each channel entry, where D is an integer (extensions to a rational D are possible). Sometimes we will be working over a block of D symbol times. In that case the joint description of (1) over D symbol periods becomes

$$\mathbf{Y_k} = \mathbf{H_k}\mathbf{X_k} + \mathbf{Z_k} \ . \tag{2}$$

We adopt the convention of representing the variables for D-symbol block as boldface letters. Here $\mathbf{Y_k}$ and and $\mathbf{Z_k}$ have lengths $N \times D$, $\mathbf{X_k}$ is of length $M \times D$ and
$\mathbf{H_k} = blockdiag(H_{kD}, H_{kD+1}, H_{kD+2}, ...H_{kD+D-1})$.

For the input power constraint, we typically choose to work under the peak power constraint as normally communication systems are peak-power limited in practice. Thus power at all transmitting antennas can never exceed SNR, the peak power, thus

$$X_k^H X_k^H \ \leq \ SNR \ . \tag{3}$$

Throughout this paper, $(.)^T$ and $(.)^H$ will denote transpose and Hermitian transpose operators respectively.

The capacity pre-log is normally defined as

$$PreLog = \lim_{SNR\to\infty} \frac{C(SNR)}{log(SNR)} \ . \tag{4}$$

We define a new capacity parameter which may help us better understand the asymptotic capacity reduction when CSIR is not available. It is called Asymptotic Capacity Reduction Factor (ACRF) and is defined as

$$ACRF = \lim_{SNR\to\infty} \frac{C_{NO-CSIR}(SNR)}{C_{CSIR}(SNR)} \ . \tag{5}$$

Thus the ACRF is the ratio of non-coherent capacity to coherent capacity at very high values of SNR.

3. NON-COHERENT CAPACITY BOUNDS

The capacity is calculated from the well-known expression

$$C = \lim_{n\to\infty} \frac{1}{n} \sup_{p_x^n} I(X^{1:n}; Y^{1:n}) \tag{6}$$

where the maximization is done over all input distributions which satisfy the power constraint. The mutual information in the above expression can be decomposed as follows

$$I(X^{1:n}; Y^{1:n}) = I(X_d^{1:n}, X_t^{1:n}; Y_d^{1:n}, Y_t^{1:n})$$

$$= \underbrace{I(X_t^{1:n}; Y_d^{1:n}, Y_t^{1:n})}_{I_1} + \underbrace{I(X_d^{1:n}; Y_d^{1:n}, Y_t^{1:n} | X_t^{1:n})}_{I_2} \tag{7}$$

The subscripts t and d denote "training" and "data" respectively, and superscript $1 : n$ shows that the length of the sequence ranges from 1 to n. Training and data here can be time multiplexed (in which case also the outputs get time multiplexed) or superimposed or a combination of both. Or training and data can more generally live in two complementary subspaces. The term "training" here may be misleading. Indeed, also the "training" symbols carry information. Nevertheless, apart from data transmission they also allow the channel to be estimated, with channel estimates that serve as a basis for the complementary "data" symbols. To diminish the confusion, we shall instead call these "training" symbols "learning" symbols. In suboptimal approaches, these learning symbols may get replaced by classical traning symbols.

3.1. Capacity Lower Bound

For the lower bound on the capacity, we can consider $X_t^{1:n}$ as pure training sequence, so information I_1 goes to zero. But this known training sequence allows channel estimation with finite estimation error covariance so that the effect of this channel estimation error, when the channel estimate gets used in the data part (and no other information gets used for the estimation of the channel), is at worst a finite increase of the effective noise power. Hence the difference of I_2 from the full CSIR case is at most some finite constant. Hence the prelog of I_2 is that of the full CSIR case, which leads to
$ACRF = 1 - \dfrac{training \quad size}{training \quad size + data \quad size}$
where the "size"s should be interpreted as the dimensions of the corresponding subspaces. But for this, the training length should be sufficient to allow deterministic identifiability of the channel, meaning that if the channel is considered as a deterministic signal, it should be identifiable with zero error in the absence of noise. Hence we get for the overall capacity:
$ACRF \ \geq \ 1 - \dfrac{learning \quad size}{learning \quad size + data \quad size}$

3.2. Capacity Upper Bound

For the upper bound on the channel capacity, we cannot ignore the capacity associated to $X_t^{1:n}$. Now, since X_t and Y_t live in corresponding subspaces whereas Y_d lives in an orthognal subspace, $I_1 = I(X_t^{1:n}; Y_d^{1:n}, Y_t^{1:n}) = I(X_t^{1:n}; Y_t^{1:n})$. Now, as long as the the size of the learning part is not more than the smallest possible size that allows channel identifiability, then we are in the "regular" case of Lapidoth [6] and the capacity I_1 grows with SNR at most as $(learning\ size) \log\log(SNR)$, hence its prelog is zero. For an upper bound on I_2, we can just take the full CSIR assumption leading to a capacity growth with a prelog equal to the data size. As a result we get for the overall capacity: $ACRF \geq 1 - \frac{learning \quad size}{learning \quad size + data \quad size}$.
Combining lower and upper bound, we get equality for ACRF.

Finer Analysis of I_2: a decomposition leads to

$$I_2 = \sum_{i=1}^{n} I(X_d^i; Y_d^{1:n}, Y_t^{1:n} | X_t^{1:n}, X_d^{1:i-1}) \tag{8}$$

(this decomposition is not necessarily in time, it can also be along a subspace basis). This sum term indicates that for the detection of each of the data symbols in SISO or vector X_d in MIMO case, we can use the channel estimate from the learning part and use all previously detected input symbols. Furthermore the presence of all output symbols stresses the need to do blind channel estimation to fully exploit the information present in that term as discussed in [1]. This indicates that to get the actual capacity, and in particular the proper constant term at high SNR, one needs to performs semi-blind channel estimation within the data subspace, since based on past input and output and future output. The future output may give important channel information, especially in the multiple receive antenna case. In any case, whether the channel estimate is based on the "learning" input and output only or whether it is based on full semiblind information does not change the prelog factor, but only an additive constant in the asymptotic capacity. So, to summarize, in the no CSIR case, the input can be split into a "learning" subspace and its orthogonal complement, the "data" subspace. The "learning" subspace is of minimal dimension to just allow deterministic identifiability of the channel and hence corresponds to the "regular" case in Lapidoth's terminology and carries information of the order of loglog(SNR). The "data" subspace is the main subspace for transmission of data and its reduced dimension represents the reduced prelog factor.

4. CAPACITY PRE-LOG FOR SISO SYSTEMS

In this section, we give two approaches which show us the capacity limit in high SNR regime and even enable us to achieve the capacity pre-log.

4.1. Sub Sampling Approach

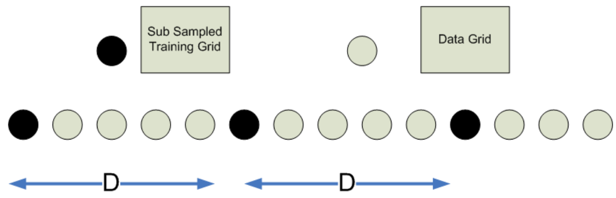

Fig. 1. Subsampling Grid.

As Doppler spectrum is band-limited to 1/D, so we can downsample with the integer downsampling factor D according to Nyquist's theorem. Thus we get a grid as shown in the figure 1. Over the downsampled instants, we transmit learning symbols known to the receiver and rest are the data symbols. So there is one learning symbol after each (D-1) data symbols. Thus over a block of D symbols, we have D prediction problems, (D-1) of which are singular, i.e. the prediction error will go to zero in the absence of noise and D-th prediction error is a white noise at sub-sampled rate (1/D).

$$\sigma^2 = \exp^{\int_{-1/2D}^{1/2D} lnS_{hh}(f)df} \tag{9}$$

Channel estimates on learning grid may be obtained by causal linear prediction and for the data grid they can be obtained by non-causal LMMSE Wiener filtering. Because of the presence of additive noise, prediction will not be perfect. The error in channel estimation has its worst effect when it is white, so in this case it gets added up with the white noise already present. This reduces the effective SNR at the receiver and causes a shift in the curve of capacity versus SNR but the slope of this curve remains unchanged corresponding to capacity pre-log.

About these learning symbols in the grid, we don't specify them to be perfectly known to receiver before transmission. They may be learning symbols in the true sense that they are known to receiver before transmission or they may be coming from a low rate stream which allows decoding even in the absence of CSIR. In this case, this data stream would have its capacity growth like loglog(SNR), but once detected, these symbols act as training symbols for the pure data symbols. Either way, whether it is pure training or low rate stream, it will give capacity pre-log of zero. Thus in a straight forward manner, the fraction $(1 - 1/D)$ is left for data transmission where we will have non-causal channel estimates from this learning grid. So this number gives us pre-log of SISO system and $ACRF^{SISO} = (1 - 1/D)$.

This subsampling approach makes causal estimates over the learning grid and for data grid, channel estimate corresponding to each data symbol is obtained by non-causal estimates over the learning grid and causal estimates over the previously detected symbols.

4.2. Learning and Data Subspaces

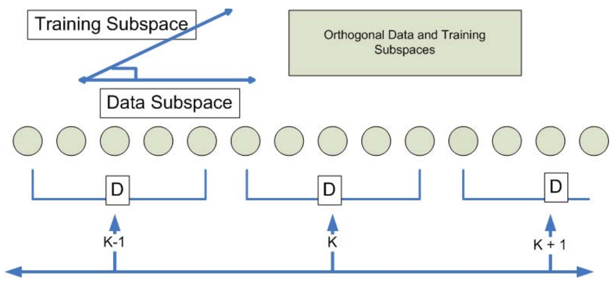

Fig. 2. Learning and Data Subspaces

We can vectorize our channel with D elements in each vector. Corresponding to this vector channel, input vector X may have D dimensioinal subspace. We could make an arrangement so as to use one dimension for learning and rest of (D-1) dimensions as data. The one dimensional subspace used by the learning should have its projection orthogonal to the projection of (D-1) dimensional subspace used by data. If we put power constraint over input vector X, then we need to optimize the power between data and learning part but we put the constraint separately over both so X is also peak power contrained.

$$X = [A_t \quad A_d] \begin{bmatrix} X_t \\ X_d \end{bmatrix} \tag{10}$$

where A_t and A_d are special matrices such that their projections are orthogonal, i.e. $[A_t \quad A_d]$ is unitary.

$$P_{A_t} = P_{A_d}^{\perp} \tag{11}$$

where $P_A = A(A^H A)^{\#} A^H$, and $(.)^{\#}$ denotes Moore-Penrose pseudo-inverse. So the received signal is

$$Y = HA_tX_t + HA_dX_d + V \tag{12}$$

Receiver can recover learning symbols as the subspaces spanned by learning and data are orthogonal. Here again, the learning sub-space may have true training symbols or they may be coming from a low rate stream. But the situation is same as it was in the subsampled scheme. Continuously we have $1/D$ resource usage as a training or low rate data transmission which gives us capacity pre-log of zero, but for the rest $1 - 1/D$ resource, channel estimates are available from the learning sub-space so communication becomes coherent for this resource and we get capacity increase of log(SNR), and hence capacity pre-log and ACRF both are equal to $1 - 1/D$. There is again reduction in effective SNR because of noisy channel estimation causing shift in the capacity curve but leaving the capacity pre-log unharmed.

For this scheme, channel estimation is purely causal. Channel estimate at each symbol instant is obtained by previously received learning symbols and previously detected data symbols. Thus it differs from subsampling approach due to its causal functionality.

5. CAPACITY PRE-LOG FOR MIMO SYSTEMS

In this section, we give the MIMO extensions to the schemes we proposed for SISO case in the previous section.

5.1. Sub Sampling Approach

With the same reasoning as in the SISO case, as Doppler spectrum is band-limited to 1/D for each channel entry, we downsample with the integer downsampling factor D according to Nyquist's theorem. But as it was shown by Hassibi [10], to properly estimate the MIMO channel matrix, we need learning length in symbol periods equal to the number of transmit antennas. Therefore in a group of D transmissions, we need to transmit learning for M symbol times. So in the very beginning, there is a multiplicative factor of $(1 - M/D)$ with the capacity as in a group of D transmissions, M carry only the learning symbols. But these transmissions allow us to make channel estimates. Channel estimates on this learning grid may be obtained by causal linear prediction and for the data grid they can be obtained by non-causal LMMSE Wiener filtering. In fact capacity decomposition done in section 3 shows us that this channel estimation should be based upon all learning symbols and previously detected data symbols. Thus for the rest of the symbol intervals in the block, we have coherent scenario which helps us achieve the capacity pre-log of $min(M, N)$. Thus capacity pre-log for this MIMO system turns out to be $min(M, N) * (1 - M/D)$ and $ACRF^{MIMO} = (1 - M/D)$. ACRF clearly shows that Doppler bandwidth gets multiplied by the number of antennas, causing greater capacity reduction as compared to SISO systems.

5.2. Learning and Data Subspaces

For this approach, we consider a block of D symbol periods. First we vectorize our channel matrix in a vector of length $M \times N$ and then combine D of them in a long vector. So now the length of this vector channel is $M \times N \times D$.

$$\begin{aligned} h_k &= vec(H_k) \\ \underline{h_{k'}} &= [h_{k'D}, h_{k'D+1}, h_{k'D+2}, ...h_{k'D+D-1}]^T \end{aligned} \tag{13}$$

$$\mathbf{X} = [\mathbf{A_t} \quad \mathbf{A_d}] \begin{bmatrix} X_t \\ X_d \end{bmatrix} \tag{14}$$

where bold faced letters show that we are working over block of D-symbol period and A_t and A_d are special matrices such that their projections are orthogonal, i.e. $[A_t \quad A_d]$ is unitary. The received signal can be expressed as

$$\mathbf{Y} = \mathbf{HA_tX_t} + \mathbf{HA_dX_d} + \mathbf{V} \tag{15}$$

The matrices A_t and A_d can further be expressed as

$$A_t = \underbrace{[I_M \quad 0]^T}_{B_t} \otimes I_M \qquad A_d = \underbrace{[0 \quad I_{D-M}]^T}_{B_d} \otimes I_M \tag{16}$$

Furthermore these can be expressed as

$$C_t = B_t \otimes I_N \qquad C_d = B_d \otimes I_N \tag{17}$$

Then at the receiver side, because of orthogonal projections learning and data dimensions can be separated.

$$Y_t = C_t^H Y = C_t^H H A_t X_t + C_t^H V \tag{18}$$

$$Y_d = C_d^H Y = C_d^H H A_d X_d + C_d^H V \tag{19}$$

Input vector X for D symbol times will have length $M \times D$ and will span $M \times D$ dimensional subspace as we mentioned in our D-symbol model in equation 2. Now out of these $M \times D$ dimensions available, we use $M \times M$ dimensional subspace for learning which will enable us to have sufficiently accurate channel estimation because at receiver from Y_t channel estimates can be obtained and rest of $M \times (D - M)$ dimensional subspace will carry data.

For the vector of channel coefficients over D-symbol period, if we consider a MIMO auto-regressive (AR) model of infinite order

$$\underline{h_{k'}} = \sum_{i=1}^{\infty} U_i \underline{h_{k'-i}} + \underline{\tilde{h}_{k'}} \tag{20}$$

The covariance matrix R of prediction error $\underline{\tilde{h}_{k'}}$ will have its rank $M \times M$.

The $M \times M$ dimensional subspace used by learning will be orthogonal to $M \times (D - M)$ dimensional subspace used by data. In case of no noise, it will allow perfect estimation of channel, but noise presence gives error in estimation. But transmission over the data subspace behaves like coherent communication. But learning subspace will introduce a capcity reduction factor $ACRF^{MIMO} = (1 - M/D)$ and we will get the capacity pre-log of $min(M, N) * (1 - M/D)$ from the data subspace.

We have seen in MIMO case that $ACRF^{MIMO} = (1 - M/D)$. Thus we remark that Doppler bandwidth gets multiplied with the number of transmitting antennas causing more capcity reduction.

6. CONCLUDING REMARKS

We have shown the capcity pre-log for SISO and MIMO channels. Moreover our presented two schemes help achieve this pre-log in practical systems. A striking observation is that the systems with M transmit antennas should be varying M times more slowly as compared to the systems with only one transmit antenna in order to have comparable capacity reduction in case of no CSIR.

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