Asymptotic Design and Analysis of Multistage Detectors for Asynchronous CDMA Systems

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Abstract

In random symbol asynchronous but chip synchronous CDMA systems, the linear MMSE detector suffers from performance degradation compared to its version in synchronous scenarios due to the finite observation window. The performance degradation is known only for a symbol centered in an observation window of length equal to the symbol interval. We propose an algorithm to determine the asymptotic performance of the LMMSE detector for any finite observation window and for any symbol impinging the observed signal. Additionally, a multistage detector that does not suffer from windowing effects and performs as well as the correspondent detector in synchronous systems is presented. In contrast to the synchronous case, in which the full rank LMMSE detector always outperforms the reduced rank linear MMSE detector, we show that, with a sufficient large delay, the proposed multistage detector can even outperform the full rank linear MMSE detector with finite fixed observation window. The output signal-to-interference and noise ratio (SINR) of the reduced rank detector is shown to be constant for all the symbols.

I. Introduction

There has been an increasing interest in the asymptotic analysis of linear detectors under the assumption of random spreading sequences [1], [2], [3]. However, the literature is mainly focused on synchronous CDMA systems and only few works analyze linear detectors in asynchronous scenarios [4] [5] [6]. In [5] the linear MMSE detectors for symbol asynchronous but chip synchronous systems are shown to reach the performance of the linear MMSE detector for synchronous systems as the observation window size tends to infinity. Additionally, the exact performance degradation due to the windowing is provided for a symbol centered in an observation window of length equal to the symbol interval T_s. A loose lower bound of the SINR is also known for any linear MMSE detector with observation windows length multiple of T_s. However, the mismatch between the lower bound in [5] and the simulation results is quite large. In this work we focus our attention on chip synchronous but symbol asynchronous CDMA systems with multistage detectors. By allowing for a detection delay equal to the number of stages we propose a structure of multistage detectors that does not suffer from windowing effects and performs as well as the multistage detector for synchronous systems. We propose also an algorithm to determine a lower bound of the LMMSE detector SINR arbitrarily close to its true value in scenarios with equal received powers for all the users. This algorithm allows an arbitrarily close approximation of the asymptotic LMMSE detector SINR for any symbol impinging the received signal. We show that the proposed multistage detector can outperform the linear MMSE detector. The rationale behind this fact is that the observation window of the proposed reduced rank LMMSE detector increases automatically with the number of stages while being fixed for the full rank LMMSE detector. Therefore, although the reduced rank LMMSE detector does not fully exploit the available statistic in comparison to the full rank LMMSE detector, it considers a wider statistic closer to the infinite observations, which is sufficient and leads to the synchronous performance. The observation window shift performed in the reduced rank LMMSE detector implementation allows constant performance on all the transmitted symbols in contrast to the full rank LMMSE detector, whose performance depends on the detected symbol position in the observation window.

II. System Model and Notations

Let us consider a direct-sequence CDMA system with K users and spreading factor N. We focus on a symbol asynchronous system. However, to make the analysis tractable we will assume the system to be chip-synchronous as in [5]. The results in [6] can be directly applied to this system to extend the analysis carried out in this paper to the general case, removing the assumption of synchronicity on the chips. Let us consider user 1 as the reference user. Without loss of generality we can assume that the time shift between any user and user 1 is, at most, one symbol and the users are ranked in ascending order of time shift with respect to the reference user. Let $\mathbf{y}(n) \in \mathbb{C}^N$ and $\mathbf{b}(n) \in \mathbb{C}^K$ be, respectively, the observed vector synchronized to the reference user and the vector of the transmitted user modulation symbols at time n. $\mathbf{S}(n) \in \mathbb{C}^{2N \times K}$ is the spreading matrix containing the users' spreading sequences at time n, opportunely shifted and zero elsewhere. For notation reasons we split the matrix $\mathbf{S}(n)$ in two matrices $\mathbf{S}_u(n), \mathbf{S}_d(n) \in \mathbb{C}^{N \times K}$ such that $\mathbf{S}(n) = [\mathbf{S}_u^T(n), \mathbf{S}_d^T(n)]^T$.

The asynchronous system is then described by

$$\mathcal{Y} = \mathcal{SAB} + \mathcal{N} \tag{1}$$

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where $\mathcal{Y} = [\ldots, \mathbf{y}^T(n-1), \mathbf{y}^T(n), \mathbf{y}^T(n+1)\ldots]^T$, $\mathcal{B} = [\ldots, \mathbf{b}^T(n-1), \mathbf{b}^T(n), \mathbf{b}^T(n+1)\ldots]^T$, $\mathcal{A}$ is a block diagonal matrix with all blocks equal to $\mathbf{A}$ and $\mathbf{A} = \mathrm{diag}(a_1, a_2, \ldots a_K)$ is a matrix of complex amplitudes. $\mathcal{N}$ is the additive white gaussian noise with variance σ^2. The matrix $\mathcal{S}$ is a bi-diagonal block matrix built as follows:

$$\mathcal{S} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{S}_d(n-1) & \mathbf{S}_u(n) & \mathbf{0} & \ddots \\ \ddots & \mathbf{0} & \mathbf{S}_d(n) & \mathbf{S}_u(n+1) & \ddots \\ \ddots & \mathbf{0} & \mathbf{0} & \mathbf{S}_d(n+1) & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{2}$$

In all the following we assume that:

A-1 the nonzero elements of the matrix $\mathcal{S}$ are independent and identically distributed (i.i.d.);

A-2 $|s_{ij}| \leq \frac{\log N}{\sqrt{N}}$;

A-3 $E\{s_{ij}\} = 0$, $E\{|s_{ij}|^2\} = \frac{1}{N}$.

The sequence of the empirical eigenvalue distribution of $\mathbf{A}\mathbf{A}^H$ converges almost surely, as $K \to \infty$, to a non random distribution function with upper bounded support. $\beta = \frac{K}{N}$ is the system load or the number of physical users per chip. The time shifts are i.i.d. distributed among the users. The time shift normalized to the symbol interval T_s, τ, has probability mass function (p.m.f) $\mathrm{P}_N(\tau)$. The support of $\mathrm{P}_N(\tau)$ is $[0, \gamma]$, with $\gamma \leq 1$. As $N \to \infty$ the sequence $\{\mathrm{P}_N(\tau)\}$ converges to the p.d.f $p_\tau(\tau)$. We will also consider the system corresponding to a finite observation window of length T symbols centered in the n-th transmitted symbol of the reference user. In order to keep the notation simple we assume T to be integer and odd. However, the result will hold for any $T \in \mathbb{R}$. In this case, the model has the following form:

$$\underbrace{\begin{bmatrix} \mathbf{y}(n-\frac{T-1}{2}) \\ \vdots \\ \mathbf{y}(n) \\ \vdots \\ \mathbf{y}(n+\frac{T-1}{2}) \end{bmatrix}}_{\mathcal{Y}_{N,T}(n)} = \underbrace{\begin{bmatrix} \mathbf{S}_d(n-\frac{T-1}{2}) & \mathbf{S}_u(n-\frac{T+1}{2}) & \mathbf{0} \\ \ddots & \ddots & \mathbf{0} \\ & \ddots & \ddots \\ \mathbf{0} & & \\ \mathbf{0} & \mathbf{S}_d(n+\frac{T-1}{2}) & \mathbf{S}_u(n+\frac{T+1}{2}) \end{bmatrix}}_{\mathcal{S}_{N,T}(n)} \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{0} & \cdots \\ \cdots & \ddots & \cdots \\ \cdots & \mathbf{0} & \mathbf{A} \end{bmatrix}}_{\mathcal{A}_{K,T}(n)} \underbrace{\begin{bmatrix} \mathbf{b}(n-\frac{T-1}{2}) \\ \vdots \\ \mathbf{b}(n) \\ \vdots \\ \mathbf{b}(n+\frac{T-1}{2}) \end{bmatrix}}_{\mathcal{B}_{N,T}(n)} + \underbrace{\begin{bmatrix} \mathbf{n}(n-\frac{T-1}{2}) \\ \vdots \\ \mathbf{n}(n) \\ \vdots \\ \mathbf{n}(n+\frac{T-1}{2}) \end{bmatrix}}_{\mathcal{N}_{N,T}(n)} \tag{3}$$

III. Multistage Detector

Let $\chi_M(\mathcal{S}\mathcal{A}) = \mathrm{span}\left\{\mathcal{R}^m\mathcal{A}^H\mathcal{S}^H\right\}_{m=0}^{M-1}$ where $\mathcal{R}$ denotes the covariance matrix $\mathcal{A}^H\mathcal{S}^H\mathcal{S}\mathcal{A}$. Let us also define the matrices $\mathbf{W}_m(n) = \mathrm{diag}(w_{m1}(n), w_{m2}(n), \ldots, w_{mk}(n))$ and $\mathcal{W}_m = \mathrm{diag}\{\ldots, \mathbf{W}_m(n), \mathbf{W}_m(n+1), \ldots\}$. The individual LMMSE detector in $\chi_M(\mathcal{S}\mathcal{A})$ is the linear detector $\mathcal{M} = \sum_{m=0}^{M-1} \mathcal{W}_m\mathcal{R}^m\mathcal{A}^H\mathcal{S}^H$ such that $\mathrm{E}\{\|\mathcal{M}\mathcal{Y} - \mathcal{B}\|^2\}$ is minimum. This is equivalent to the minimization of the mean square error (MSE) for each component $\mathbf{b}_k(n)$ of $\mathbf{B}$ in the correspondent subspace $\chi_{M,k,n}(\mathcal{S}\mathcal{A}) = \mathrm{span}\left\{\mathrm{row}(\mathcal{R}^m\mathcal{A}^H\mathcal{S}^H), n, k\right\}_{m=0}^{M-1}$, where $\mathrm{row}(\mathbf{X}, n, k)$ is the row of the matrix $\mathbf{X}$ corresponding to the user k at time instant n. Because of the bi-diagonal block structure of $\mathcal{S}$, the matrix $\mathcal{R}$ is a tri-diagonal block matrix and its power $\mathcal{R}^m$ is a $(2m+1)$-diagonal matrix. Therefore, the row vector $\mathrm{row}(\mathcal{R}^m\mathcal{A}^H\mathcal{S}^H, n, k)$ has, at most, $(2m+1)K$ nonzero elements and the M-stage detector for the unlimited system model can be implemented with a finite delay equal to MT_s. Figure 1 shows its structure.

The j-th stage consists of a re-spreading block that multiplies the input vector by the matrix $\mathbf{S}(n-j+1)\mathbf{A}(n-j+1)$ and a filter matched to the transmitted vector at time $n-j$, $\mathbf{S}(n-j)\mathbf{A}$. It receives as input vector $\mathrm{row}(\mathcal{R}^j\mathcal{A}^H\mathcal{S}^H, n-j, 1:K)\mathcal{Y}$, where $\mathrm{row}(\mathcal{X}, n, r : s)$ denotes the $s-r+1$ rows of the matrix $\mathcal{X}$ corresponding to the users $r, r+1, \ldots s$ at time instant n. The re-spreading block provides two output vectors, the upper part vector $\mathbf{S}_u(n-j)\mathbf{A}\mathrm{row}(\mathcal{R}^{j-1}\mathcal{A}^H\mathcal{S}^H, n-j, 1:K)$ and $\mathbf{S}_d(n-j+1)\mathbf{A}\mathrm{row}(\mathcal{R}^{j-1}\mathcal{A}^H\mathcal{S}^H, n-j, 1:K)$. The input to the following matched filter is given by

$$\begin{bmatrix} \mathbf{S}_u(n-j)\mathbf{A}\mathrm{row}(\mathcal{R}^{j-1}\mathcal{A}^H\mathcal{S}^H, n-j-1, 1:K) + \mathbf{S}_d(n-j-1)\mathbf{A}\mathrm{row}(\mathbf{R}^{j-1}\mathbf{A}^H\mathbf{S}^H, n-j-2, 1:K) \\ \mathbf{S}_u(n-j+1)\mathbf{A}\mathrm{row}(\mathcal{R}^{j-1}\mathcal{A}^H\mathcal{S}^H, n-j, 1:K) + \mathbf{S}_d(n-j)\mathbf{A}\mathrm{row}(\mathbf{R}^{j-1}\mathbf{A}^H\mathbf{S}^H, n-j-1, 1:K) \end{bmatrix}. \tag{4}$$

The output of the j-th stage is delayed by $(M-1-J)T_s$ before being used as input of the filter with weight matrix $\mathbf{W}_j$ that provides the soft estimate of $\mathbf{b}(n-M+1)$. The weight matrices $\mathbf{W}_m(n)$ can be derived by the following equation:

$$\mathbf{w}_k(n) = (\mathbf{\Phi}_k(n))^{-1}\mathbf{c}_k(n) \tag{5}$$

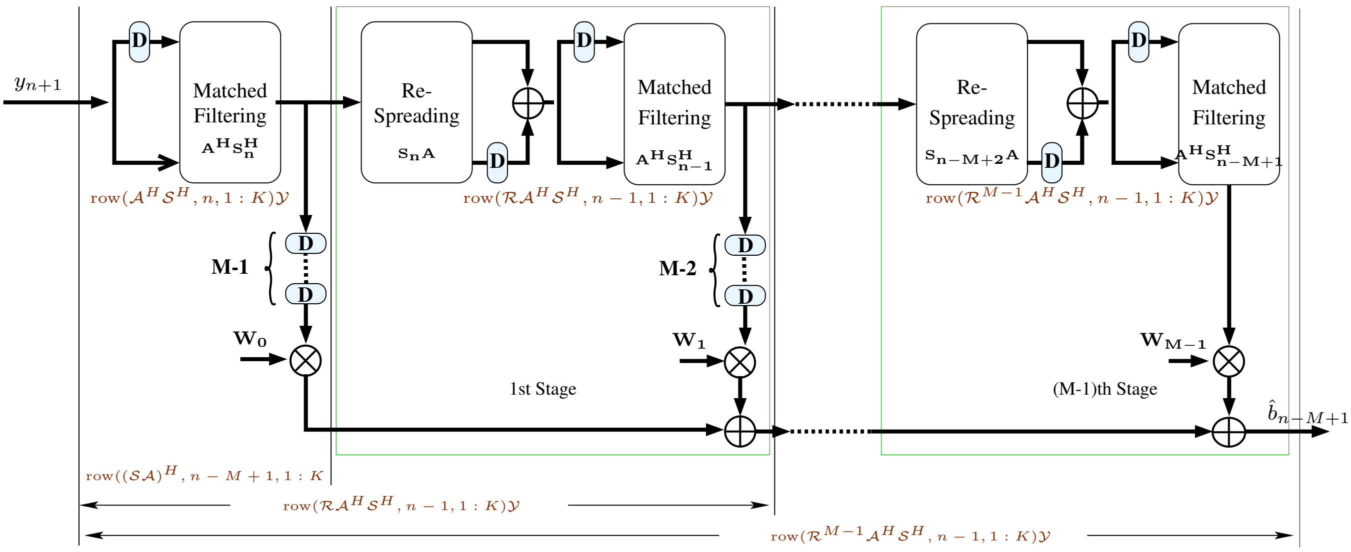

Fig. 1. LMMSE detector in $\chi(\mathcal{SA})$ for asynchronous systems

where $(\mathbf{w}_k(n))_m = (\mathbf{W}_m(n))_{kk}$ [1], $\mathbf{c}_k(n)$ is an M-dimensional vector , $\mathbf{\Phi}_k(n) \in \mathbb{R}^{M\times M}$, $(\mathbf{c}_k(n))_m = (\mathcal{R}^{m+1}(n))_{kk}$, $(\mathbf{\Phi}_k(n))_{lm} = (\mathcal{R}^{l+m}(n))_{kk} + \sigma^2(\mathcal{R}^{l+m-1}(n))_{kk}$ and $(\mathcal{R}^m(n))_{kk}$ denotes the diagonal element of the matrix $\mathcal{R}^m$ corresponding to the user k at time instant n. The output signal-to-interference and noise ratio (SINR) of user k is given by

$$\mathrm{SINR}_k(n) \quad = \quad \frac{\mathbf{c}_k^T(n)(\mathbf{\Phi}_k(n))^{-1}\mathbf{c}_k(n)}{1-\mathbf{c}_k^T(n)(\mathbf{\Phi}_k(n))^{-1}\mathbf{c}_k(n)} \tag{6}$$

In the asymptotic case, as $N, K \to \infty$ with $\frac{K}{N} = \beta$ the expression of the individual LMMSE detector in $\chi_M(\mathcal{SA})$ requires the existence and the expression of the limits $\lim_{K=\beta N\to\infty} \mathcal{R}^m(n))_{kk} = \mathcal{R}^m_\infty(n))_{kk}$ $\quad k \in [1, K]$ and $1 \le m \le M^2$. It is known [4], [5] that for $K = \beta N$, $T \to \infty$ the eigenvalue distribution of $\mathcal{R}$ converges to the same eigenvalue distribution of the matrix $\overline{\mathbf{R}} = \mathbf{A}^H\mathbf{S}^H(n)\mathbf{S}(n)\mathbf{A}$ for synchronous systems, i.e. $p(\tau) = \delta(\tau)$, up to some eigenvalues in zero. In the following section it will shown that the same property holds also for the diagonal elements $(\mathcal{R}^m_\infty(n))_{kk}$ and

$$\lim_{K=\beta N\to\infty} (\mathcal{R}^m(n))_{kk} = \overline{\mathbf{R}}^m_{kk,\infty} \qquad \forall 1 \le m \le M^2. \tag{7}$$

Recursive and close form expressions for $\overline{\mathbf{R}}^m_{kk,\infty}$ can be found in [7].

IV. Asymptotic Performance of the LMMSE Detector

In this section we propose a method to determine a family of lower bounds for the asymptotic (i.e. $K = \beta N \to \infty$) SINR of an LMMSE detector, whose supremum coincides with $\mathrm{SINR}^\infty_{\mathrm{LMMSE}}$. It is well known that for a finite system with K users the individual K-stage detector coincides with the LMMSE detector [8]. Therefore, the SINR of the individual LMMSE detectors in $\chi_M(\mathcal{A}_{K,T}\mathcal{S}_{N,T})$ for $M < K$ provides a family of lower bounds for the $\mathrm{SINR}_{\mathrm{LMMSE}}$ of the full rank LMMSE detector. Additionally, it has been shown that for moderate to heavy load an 8-stage detector for synchronous system essentially achieves full rank performance. It was established in [9] that the reduced rank multistage filter output SINR converges exponentially in the filter rank toward to the full rank LMMSE filter output SINR. We will verify numerically that the same property holds for asynchronous systems.

Let us consider an asynchronous system with finite observation window T and equal powers:

$$\mathcal{Y}_T(n) = \mathcal{S}_T(n)\mathcal{B}_T(n) + \mathcal{N}_T(n). \tag{8}$$

Making use of (6), the problem of determining the family of lower bounds of $\mathrm{SINR}_{\mathrm{LMMSE}}$ reduces into determining the diagonal elements of the matrix $\mathcal{R}_T(n) = \mathcal{S}_T^H(n)\mathcal{S}_T(n)$. A recursive algorithm to determine them is provided by Theorem 1. In order to prove Theorem 1 we conjecture that for N sufficient large the spectrum of the matrix $\mathcal{R}_{N,T}$ is upper bounded [2].

[1] $(.)_m$ denotes the m-th component of the vector argument and $(.)_{mn}$ denotes the element ij of the matrix argument.

[2] This property is verified for the matrices $\mathbf{S}(n)\mathbf{A}\mathbf{A}^H\mathbf{S}^H(n)$ and $\mathbf{S}(n)\mathbf{S}^H(n)$ for synchronous systems. In fact, extensive computer simulations were performed in order to verify this property [10] and several papers proved it [11], [12]. However, no analogous result for the matrix $\mathcal{R}_{N,T}$ is known to the authors.

Theorem 1 *Let $\mathcal{S}_T$ be an $TN \times (T+1)K$ bi-diagonal block matrix with blocks $S(j) = [S_u^T(j), S_d^T(j)]^T \in \mathbb{C}^{2N\times K}$, and $S_u(j), S_d(j) \in \mathbb{C}^{N\times K}$, as follows:*

$$\mathcal{S}_T = \begin{bmatrix} S_d(1) & S_u(2) & \mathbf{0} & \dots & \dots & \dots \\ \mathbf{0} & S_d(2) & S_u(3) & \mathbf{0} & \dots & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & \dots & \mathbf{0} & S_d(T-1) & S_u(T) & \mathbf{0} \\ \dots & \dots & \dots & \mathbf{0} & S_d(T) & S_u(T+1) \end{bmatrix}. \tag{9}$$

Let $\widetilde{S}(k)$ for $k = 1, \dots T+1$ be independent matrices in $\mathbb{C}^{2N\times K}$ with elements $\widetilde{s}_{ij}(k)$, $1 \le i \le N$, $1 \le j \le K$, satisfying properties A-1, A-2 and A-3 and the remaining elements equal to zero. The matrix $\mathcal{S}(k)$ is obtained from $\widetilde{S}(k)$ circularly shifting each column by τN positions independently of all the others and according to a p.m.f. $P_N(\tau)$, and then, sorting the column vectors by ascending order of τ.

Let the sequence of p.m.f $\{P_N(\tau)\}$ converge to a p.d.f. $p_\tau(\tau)$ with support $[0,\gamma]$ and $\gamma \le 1$, distribution function $F_\tau(\tau)$. Define for each N $v_N : [0,T] \times [0,(T+1)\beta] \to \mathbb{R}$ the limiting joint distribution of the variance:

$$v_N(x,y) = N\mathrm{E}\{|s_{ij}|^2\} \qquad \text{for } i,j \text{ satisfying} \tag{10}$$

$$\frac{i}{N} \le x \le \frac{i+1}{N} \qquad \frac{j}{N} \le y \le \frac{j+1}{N} \tag{11}$$

then $v_N(x,y)$ converges uniformly to a limited bounded function v such that $v(x,y) = 1$ in the region whose border is defined by the two curves $r(x)$ and $c(y)$ with

$$r(x) = \begin{cases} \frac{\beta}{\gamma} F_\tau^{-1}\left(\frac{x-i}{\gamma}\right) + i\beta & i \le x \le i+\gamma \\ (i+1)\beta & i+\gamma < x < i+1 \end{cases} \qquad 0 \le i \le T-1 \tag{12}$$

and

$$c(y) = \begin{cases} 0 & 0 \le y \le \beta \\ (i-1) + i\beta F_\tau\left(\frac{\gamma(y-i\beta)}{\beta}\right) & i\beta < y < (i+1)\beta \end{cases} \qquad 1 \le i \le T \tag{13}$$

and $v_N(x,y) = 0$ elsewhere. Moreover, let the function $l(y) \in \mathbb{R}$ be defined as

$$l(y) = \begin{cases} \frac{\beta}{\gamma} F_\tau^{-1}\left(\frac{y}{\gamma}\right) & 0 \le y \le \beta \\ 1 & \beta < y < T\beta \\ \frac{\beta}{\gamma} F_\tau^{-1}\left(\frac{(T+1)\beta - y}{\gamma}\right) + (1-\gamma) & \beta T \le y \le \beta(T+1) \end{cases}. \tag{14}$$

Then

$$(\mathcal{T}_T^m(N))_{kk} = ((\mathcal{S}_T(N)\mathcal{S}_T^H(N))^m)_{kk} \;\overset{\mathcal{P}}{\to}\; \mathcal{T}_T^m(x) \qquad \text{and} \qquad x = \lim_{N\to\infty} \frac{k(N)}{N} \tag{15}$$

$$(\mathcal{R}_T^m(N))_{kk} \;\overset{\mathcal{P}}{\to}\; \mathcal{R}_T^m(y) \qquad \text{and} \qquad y = \lim_{N\to\infty} \frac{k(N)}{N} \tag{16}$$

with $\mathcal{R}_T^m(y)$ and $\mathcal{T}_T^m(x)$ determined by the following recursion:

$$f(\mathcal{R}_T^n, x) = \frac{1}{\beta}\int_{r(x)}^{r(x)+\beta} \mathcal{R}_T^n(y)\,\mathrm{d}y \qquad 0 \le x \le T \tag{17}$$

$$g(\mathcal{T}_T^n, y) = \frac{1}{l(y)}\int_{y}^{y+l(y)} \mathcal{T}_T^n(x)\,\mathrm{d}x \qquad 0 \le y \le (T+1)\beta \tag{18}$$

$$\mathcal{T}_T^{n+1}(x) = \beta\sum_{s=0}^{n} \mathcal{T}_T^s(x) f(\mathcal{R}_T^{n-s}, x) \qquad 0 \le x \le T \tag{19}$$

$$\mathcal{R}_T^{n+1}(y) = l(y)\sum_{s=0}^{n} \mathcal{R}_T^s(y) g(\mathcal{T}_T^{n-s}, y) \qquad 0 \le y \le (T+1)\beta \tag{20}$$

with $\mathrm{T}_T^1(x) = \beta$ and $\mathrm{R}_T^1(y) = l(y)$.

The proof of the theorem is omitted for space reasons. Figure 3 illustrates the meaning of the functions $v(x,y)$, $r(x)$, $c(x)$,

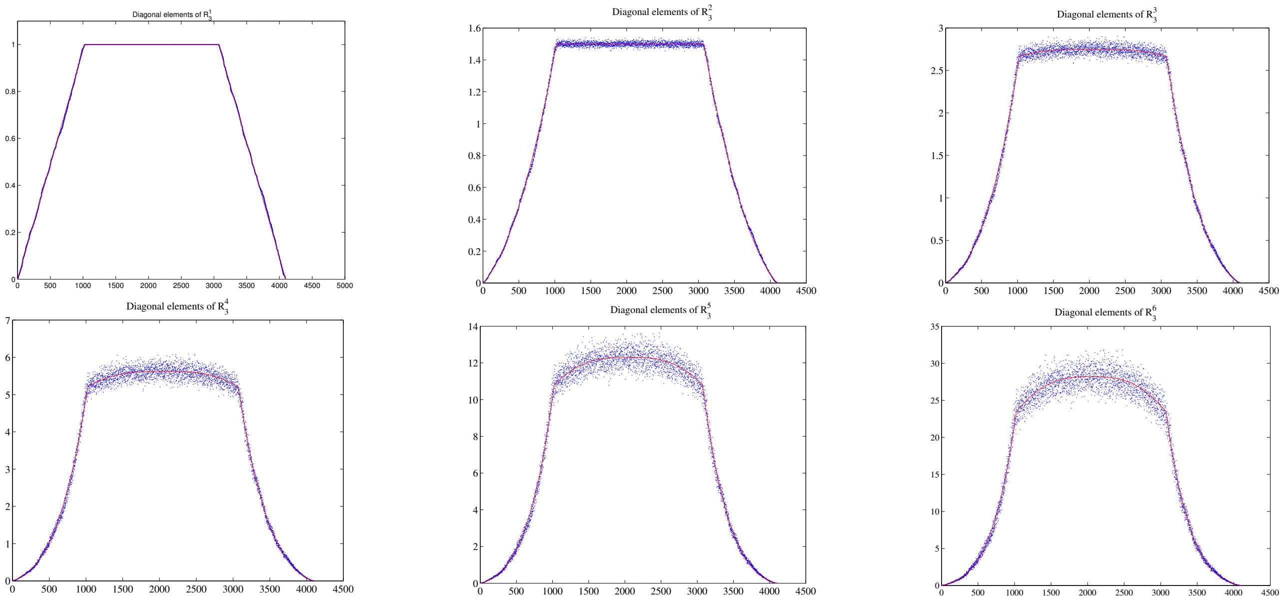

Fig. 2. Comparison between the theoretical $\mathcal{R}_3^n(y)$ for $n = 1 \dots 6$ and $\mathcal{R}_{3,kk}^n(2048)$, $\beta = \frac{1}{2}$

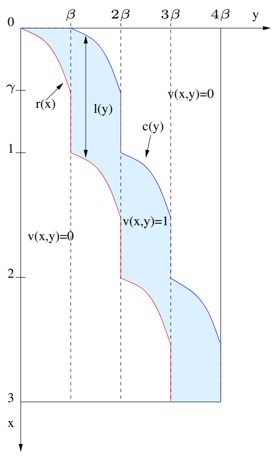

Fig. 3. Meaning of the functions $v(x,y)$, $r(x)$, $c(y)$ and $l(y)$

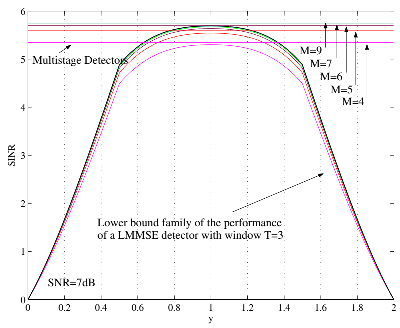

Fig. 4. Asymptotic $\mathrm{SINR}_{\mathrm{LMMSE}}$ for $T = 3$, $\frac{E_b}{N_0} = 7$ dB and performance of the LMMSE detector in $\chi_M(\mathcal{AS})$ for varying M

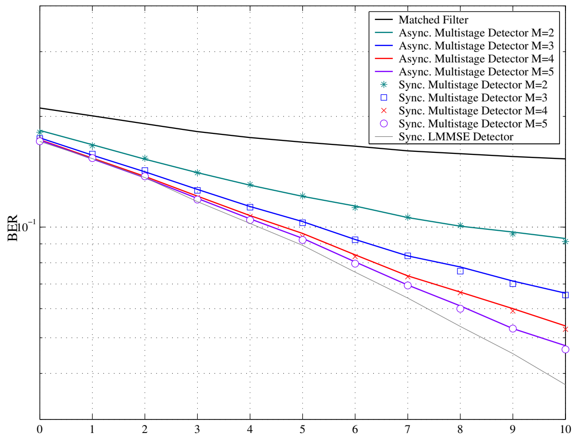

Fig. 5. BER versus $\frac{E_b}{N_0}$ for $\beta = \frac{1}{2}$ and varying number of stages

and $l(y)$. Let us provide, in the following, an application example of the theorem. Let us assume $T = 3$, $\gamma = 1$ and the delay uniformly distributed in the interval $[0, T_s]$, then $F_\tau(\tau) = \tau$, $r(x) = \beta x \ \forall x \in [0, 3]$,

$$c(y) = \begin{cases} 0 & 0 \le y \le \beta \\ \frac{y-\beta}{\beta} & \beta \le y \le 4\beta \end{cases} \qquad \text{and} \qquad l(x) = \begin{cases} \frac{y}{\beta} & 0 \le y \le \beta \\ 1 & \beta \le y \le 3\beta \\ 4\beta - \frac{y}{\beta} & \beta \le y \le 4\beta \end{cases}. \tag{21}$$

Therefore, $\mathrm{T}_T^1(x) = \beta$ and $\mathrm{R}_T^1(y) = l(y)$,

$$f(\mathcal{R}_T^n, x) = \begin{cases} \frac{1}{\beta}\left[\int\limits_{\beta x}^{\beta} \frac{y}{\beta}\mathrm{d}y + \int\limits_{\beta}^{\beta x+\beta} \mathrm{d}y\right] & 0 \le x \le 1 \\ \frac{1}{\beta}\int\limits_{\beta x}^{\beta x+\beta} \mathrm{d}y & 1 \le x \le 2 \\ \frac{1}{\beta}\left[\int\limits_{\beta x}^{3\beta} \mathrm{d}y + \int\limits_{3\beta}^{\beta x+\beta}\left(4\beta - \frac{y}{\beta}\right)\mathrm{d}y\right] & 0 \le x \le 1 \end{cases} \tag{22}$$

and $g(\mathcal{T}_T^1, y) = \beta$ $0 \leq y \leq 4\beta$. We can then apply (19) and (20). In Figure 2 the asymptotic values of $\mathcal{R}_3^n(y)$ for $n = 1 \dots 6$ are compared to the values $\mathcal{R}_{3,kk}^n(N)$, for $N = 2048$ and $\beta = \frac{1}{2}$, of a single realization. Simulations with various distributions of the elements s_{ij} show that the diagonal elements of finite large matrices match very well the asymptotic values determined by (20).

The difficulty in extending the previous theorem to a system with unbalanced powers ($\mathbf{A} \neq \mathbf{I}$) is due to the difficulty in determining $\mathcal{T}_T^m(x)$. However, for $T \to \infty$ no truncation effects occur and, as for synchronous systems, $\mathcal{T}_T^m(x)$ is independent of x and is equal to the normalized trace of $\mathcal{T}_T^m$. For $T \to \infty$ it is known [4], [5] that the asymptotic eigenvalue distribution of $\mathcal{T}$ coincides with the eigenvalue distribution for synchronous systems. Hence, with an approach analogous to the one applied to derive Theorem 1, we can derive an equation equivalent to equation (20) for systems with unbalanced powers. This leads to the same results as in the synchronous systems.

V. Numerical Results

Throughout this section, we consider linear MMSE detectors with observation window $T = 3$. Figure 4 shows the family of lower bounds of the output $\mathrm{SINR}_{\mathrm{LMMSE}}$ for a system with $\beta = \frac{1}{2}$ and $\frac{E_b}{N_0} = 7$ dB. As for the synchronous case, the convergence of these bounds toward to the supremum is very fast and the lower bound corresponding to $M = 8$ matches completely the one obtained for $M = 9$. The SINR reaches its maximum for the transmitted symbol centered in the observation window and decreases smoothly for the transmitted symbols whose spreading is still completely observed ($y \in [\beta, 3\beta]$). The performance degrades rapidly for symbols only partially included in the observation window. In contrast to the synchronous case, in the asynchronous case the LMMSE detector in $\chi_M(\mathcal{SA})$, with M sufficiently large, can outperform the full rank LMMSE detector with finite observation window T. This is due to the fact that both the detectors use only a subset of a sufficient statistic, but the LMMSE detector in $\chi_M(\mathcal{SA})$ can intrinsically exploit a wide subset, while the full rank LMMSE detector requires an increment of the window size. The performance of the LMMSE detector in $\chi_M(\mathcal{SA})$ were assessed by simulations. We assumed flat Rayleigh block fading channels with unitary variance, $\frac{\pi}{4}-$QPSK modulation and perfect knowledge of the channels. Figure 5 shows the performance improvements of the LMMSE detector in $\chi_M(\mathcal{SA})$ for increasing number of stages. The curves obtained for the asynchronous system match completely the ones obtained in the synchronous case.

VI. Summary of Results and Conclusion

We proposed a scheme for the LMMSE detector in $\chi_M(\mathcal{SA})$ that does not suffer from windowing effects in asynchronous systems, in contrast to the full rank LMMSE detector. We also provided an algorithm to determine the performance of the LMMSE detector with finite observation window for all the transmitted symbols that impinge the received signal. In contrast to the synchronous systems, the LMMSE detector in $\chi_M(\mathcal{SA})$ for asynchronous systems can outperform the full rank LMMSE detector with finite window T when choosing a sufficient large rank M.

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