The Long-Term Average Capacity Region per Unit Energy with Application to Protocols for Sensor Networks

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ABSTRACT

We consider a wireless sensor network (WSN) where K sensors must deliver a message within a frame of N time slots by spending a given finite energy. If the message is not transmitted within the required delay then it becomes useless and the residual energy is wasted. The channel is block-fading, with independent fades for each sensor and each slot. Sensors know the fading levels up to the current slot but do not know the future fading levels. The receiver collects the signal on all the slots of the frame and performs joint decoding of all the messages.

We characterize the region of long-term average achievable rates and we show that the optimal policy for $N = 1$ coincides with constant power allocation and, as N increases, it tends to waterfilling in time. In this setting, we also characterize the long-term average capacity region per unit energy by showing that the optimal policy is "one-shot", totally decentralized and extremely simple, i.e., every user spends all its available energy on the first slot of the frame whose fading gain is larger than a pre-computed time varying threshold. Since capacity per unit energy is related to the minimum E_b/N_0 required for reliable communication, the "one-shot" policy not only makes the most efficient use of the energy but also reduces to the minimum the interference to other users. These characteristics make the "one-shot" policy appealing for systems with severe energy limitation as WSN.

1. INTRODUCTION

We consider a network where K sensors must deliver a measurement to a mobile node within a frame of N slots by spending a given finite energy. If the message is not transmitted within a frame it becomes useless and the residual energy is wasted. The links between each sensor and the mobile node are modeled as channels with multiplicative independent fading and additive Gaussian noise. We assume that transmission occurs over different slots in time and that sensors know the fading levels up to the current slot but do not know the fading levels on future slots. The receiver collects the signals of all the sensors on all the slots of the frame and performs joint decoding.

This model, admittedly simplified with respect to the reality, fits the characteristics of a wireless sensor network. Briefly, a WSN comprises many stationary nodes and a very small number of mobile nodes. Unlike terminals in conventional wireless networks, i.e., ad-hoc networks or cellular systems, the sensors nodes in a WSN operate under very dynamic/different conditions (take measurements, elaborate the acquired data, discover other nodes to establish links, act as relay for other nodes in case of lack of connectivity or unreachable target, etc.) and work unattended. In order to guarantee network connection and long operational lifetime, energy must managed carefully, especially because sensors run on batteries whose frequent substitution might be impossible and/or impractical. In terms of energy consumption, transmitting data on the air is of much higher cost than non-real time processing, hence source and channel coding are of primary importance in order to lower down the transmission rate. Mobile nodes periodically collect data from the sensors. They send a reference signal that sensors use to detect their presence, to synchronize and to measure their channels. Since the mobile moves, the channel from the mobile to the sensors changes over time, hence sensors must transmit at variable rate/quality in order to deliver in any case a useful message. Transmission must take place within the time the mobile node is reachable otherwise the data will be lost.

As an example, imagine sensors for tele-surveillance located over a large geographical area and a non-geostationary satellite that periodically flies above them so that the sensors are in the coverage of its spot beam antenna. The sensors have solar cells to charge their batteries. At every passage of the satellite, and within the time the satellite is reachable, the sensors send their data, an image or a measurement, by using the energy that they have stored in the batteries. Because of phenomena like tropospheric scattering, rain or physical obstacles, the channel between each sensor and the satellite is slowly time-varying and can be considered frequency-flat. Due to time variation of atmospheric conditions, the rate at which reliable communication is possible is a random variable. Since in this kind of application it is important to deliver some measurement, even if not at the best quality, the sensors encode the data by layered source coding and, depending on the instantaneous channel conditions, transmit the fundamental coarse information and more or less refinement. Notice that source coding need not be in real time, so it costs (virtually) no energy. On the contrary, transmission must be done with the accumulated energy in the battery. A sensible criterion for this setting is to maximize the expected number of transmitted bit per joule.

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In this setting, we characterize the "long-term average capacity region" and the "long-term average capacity region per unit energy" by finding the optimal rate/power allocation policies. We prove that long-term average capacity is achieved for $N = 1$ by constant power allocation, while, as N increases, the optimal causal policy tends to the optimal ergodic policy without delay constraint and non-causal channel state information. Our setting gives the correct trade-off between peak-to-average constrained systems ($N = 1$) and complete freedom in the power allocation ($N \to \infty$) and proves that past and future channel knowledge are immaterial when the delay constraint is not too severe. On the other hand, the optimal policy achieving long-term average capacity per unit energy is "one-shot", i.e., transmission occurs in only one slot of the frame whose selection depends on the fading on the channel, it is decentralized, i.e., every sensor only need knowledge of its own channel state, it does not depend on the number of active sensors but only on the fading statistics. In systems requiring aggressive power control management, transmission at low rate and simple transmission protocols, this strategy seems "ideal". Furthermore, with the "one-shot" policy, transmission occurs at minimum energy per bit needed for reliable communication, which implies not only that the energy is used in the most efficient way but also that interference to other users is reduced to the minimum.

Our work is mainly inspired by Negi and Cioffi [1, 2] who investigated the optimal causal power control law and its implications on average capacity and outage performance in a single user system. They identified the "one-shot" law as an approximation of the long-term average capacity achieving policy in the low Signal-to-Noise-Ratio (SNR) regime. As a matter of fact, their argument can be made rigorous by using the framework of capacity per unit-cost as introduced by Verdú in [3], which is the approach taken in this work. Furthermore, in our work we do not just state a "capacity formula", we give a coding theorem (achievable and converse part) to prove that the quantity maximized in [1, 2] is the long-term average capacity of the channel. We also give a limiting analysis as the delay constraint N is relaxed and we prove the convergence of our long-term average quatities to the corresponding ergodic capacities. Finally, we quantify the loss of the optimal causal strategy with the optimal strategy with non-causal channel knowledge of the channel.

The paper is organized as follow: in Section 2 we describe the system model, in Section 3 we list the results about the long-term average capacity region, the long-term average capacity region per unit energy and their limiting behavior, in Section 4 we give some numerical examples and in Section 5 we poit out our conclusions. All the mathematical derivations can be found in [4].

2. SYSTEM MODEL

We consider a block-fading Gaussian Multi-Access Channel (MAC) where K transmitters must deliver their message within N slots to the receiver by spending a fixed maximum energy. The number of complex symbols/dimensions per slot is $L = \lfloor WT \rfloor$, where T is the slot duration and W is the channel bandwidth. The baseband complex received vector in slot n is

$$\boldsymbol{y}_n = \sum_{k=1}^{K} c_{k,n} \boldsymbol{x}_{k,n} + \boldsymbol{z}_n \tag{1}$$

where $\boldsymbol{z}_n$ is a proper complex Gaussian random vector of dimension L with i.i.d. (independent and identically distributed) components of zero mean and unit variance, $\boldsymbol{x}_{k,n}$ is the complex signal of user k transmitted in slot n, $c_{k,n}$ is the complex fading coefficient for user k with power gain $\alpha_{k,n} = |c_{k,n}|^2$ with cdf (cumulative distribution function) $F_\alpha^{(k)}(x)$ i.i.d. for all $n = 1, \cdots, N$ and mutually independent for $k = 1, \cdots, K$.

The receiver has perfect (*non-causal*) Channel State Information (CSI) while the transmitters have perfect *causal* CSI [1, 2], i.e., in slot n the transmitters know the channel state up to time n, defined by

$$\mathcal{S}_n \triangleq \{c_{k,i} : k = 1, \cdots, K,\ i = 1, \cdots, n\} \tag{2}$$

Each transmitter k is subject to the per-codeword input constraint (referred to as "short-term" power constraint)

$$\frac{1}{NL} \sum_{n=1}^{N} |\boldsymbol{x}_{k,n}|^2 \le \gamma_k \tag{3}$$

where γ_k is the average transmit energy per symbol, and because of the noise variance normalization adopted here it has the meaning of *transmit* SNR. In the following we will use the notation $\beta_{k,n} = |\boldsymbol{x}_{k,n}|^2/L$ for the *instantaneous* SNR of transmitter k in slot n.

For finite N and L no positive rate is achievable. However, we can consider a sequence of channels indexed by the slot length L and study the achievable rates in the limit for $L \to \infty$ and fixed N. This is a standard mathematical abstraction in the study of the limit performance of block-fading channels [5] and it is motivated by the fact that, in many practical applications, the product WT is large and T is much smaller than the fading coherence time. Even in the limit of large L (which essentially means that on every slot the effect of the noise has been averaged out), the rate K-tuple at which reliable communication is possible over a frame of N slots is a random vector, because only a fixed number N of fading coefficients affect each user codeword. We allow *variable rate coding* so that users can coordinate their rates in order to be always inside the fading-dependent capacity region. Variable-rate coding in our setting is essentially different from variable-rate coding in an ergodic setting, such as in [6, 7] (where, actually, capacity can be achieved with constant rate transmission and constant transmit energy per codeword), in what we assume that each transmitter has an infinite "bit-reservoir" and, depending on the fading instantaneous realization, transmits a variable number of bits per frame. Here the transmit energy per codeword is kept fixed while transmission rate fluctuates from frame/codeword to frame/codeword. We consider a sequence of frames, where coding and decoding are performed frame-by-frame according to a variable-rate coding scheme described above, where the channel state sequence $\mathcal{S}_N$ over each frame is generated according to some ergodic and stationary process, and we study the *long-term average rate region* subject to the short-term power constraint (3). Moreover, in the energy limited case investigated here, a meaningful system design criterion is to look for the *long-term average capacity per unit energy* (bit/joule). The formal definition of variable rate coding and the operative definitions of long-term average capacity region and of long-term average capacity region per unit-energy are omitted here for sake of brevity, they can be found in [8] and [4].

3. MAIN RESULTS

In analogy with [6, 3], we characterize the long-term average capacity region and the long-term average capacity per unit energy for our system. We also give limiting analysis for large delay N and we quantify the loss due to the causal knowledge of the channel.

3.1. The long-term average capacity region

We have the following result:

Theorem 1. The long-term average capacity region is

$$C_{K,N}(\boldsymbol{\gamma}) = \bigcup_{\boldsymbol{\beta}\in\Gamma_{K,N)}(\boldsymbol{\gamma})} \Big\{ \boldsymbol{R} \in \mathbb{R}_+^K \ : \ \forall A \subseteq \{1,\cdots,K\}$$
$$\sum_{k\in A} R_k \le \mathrm{E}\left[\frac{1}{N}\sum_{n=1}^{N}\log\left(1+\sum_{k\in A}\alpha_{k,n}\beta_{k,n}\right)\right]\Big\} \quad (4)$$

where expectation is with respect to the channel state $\mathcal{S}_N$ and where $\Gamma_{K,N}(\boldsymbol{\gamma})$ is the set of *feasible* causal short-term power allocation policies $\boldsymbol{\beta} = \{\beta_{k,n} \ : \ k = 1,\cdots,K, n = 1,\cdots,N\}$ defined as

$$\Gamma_{K,N}(\boldsymbol{\gamma}) \triangleq \left\{\boldsymbol{\beta}\in\mathbb{R}_+^{KN} : \frac{1}{N}\sum_{n=1}^{N}\beta_{k,n}(\mathcal{S}_n) \le \gamma_k\right\} \quad (5)$$

where $\beta_{k,n}(\mathcal{S}_n)$ indicates the causality constraint. $\square$

We explicit characterize the boundary surface of $C_{K,N}(\boldsymbol{\gamma})$, following the approach of [6], as the set of all K-tuples $\boldsymbol{R}\in\mathbb{R}_+^K$ that solve

$$\max_{\boldsymbol{R}\in C_{K,N}(\boldsymbol{\gamma})} \sum_{k=1}^{K}\mu_k\, R_k \quad (6)$$

for some $\boldsymbol{\mu} = (\mu_1,\cdots,\mu_K)$. We have the following:

Theorem 2. Define for $n = 1,\cdots,N$ and with initial condition $S_0(P_1,\cdots,P_K;\boldsymbol{\mu}) = 0$ the Dynamic Programming recursion

$$S_n(P_1,\cdots,P_K;\boldsymbol{\mu}) = \mathrm{E}\Bigg[\max_{\forall k: p_k\in[0,P_k]} \sum_{k=1}^{K}\mu_{\pi_k}\log\left(1+\frac{\alpha_{\pi_k}\,p_{\pi_k}}{1+\sum_{j<k}\alpha_{\pi_j}\,p_{\pi_j}}\right) + S_{n-1}(P_1-p_1,\cdots,P_K-p_K;\boldsymbol{\mu})\Bigg] \quad (7)$$

where the expectation is with respect to $\boldsymbol{\alpha} = (\alpha_1,\cdots,\alpha_K)$ and where π is the permutation that orders $\boldsymbol{\mu}$ in decreasing order, i.e., $\mu_{\pi_1} > \cdots > \mu_{\pi_K}$. Let $(\widehat{p}_{1,n}(\boldsymbol{\alpha};\boldsymbol{\mu},\boldsymbol{P}),\cdots,\widehat{p}_{K,n}(\boldsymbol{\alpha};\boldsymbol{\mu},\boldsymbol{P}))$ the value of $(p_1,\cdots,p_N)$ that achieves the maximum in (7). Then, the boundary surface of $C_{K,N}(\boldsymbol{\gamma})$ is the closure of

$$\text{convex-hull}\left\{\widehat{\boldsymbol{R}}_N(\boldsymbol{\mu},\boldsymbol{\gamma}) \ : \ \boldsymbol{\mu}\in\mathbb{R}_+^K, \sum_{k=1}^{K}\mu_k = 1\right\} \quad (8)$$

where the rates $\widehat{\boldsymbol{R}}_N(\boldsymbol{\mu},\boldsymbol{\gamma})$ are given by

$$\widehat{R}_{k,N}(\boldsymbol{\mu},\boldsymbol{\gamma}) = \mathrm{E}\left[\frac{1}{N}\sum_{n=1}^{N}\log\left(1+\frac{\alpha_{k,n}\,\widehat{\beta}_{k,n}}{1+\sum_{j<\pi^{-1}(k)}\alpha_{\pi_j,n}\,\widehat{\beta}_{\pi_j,n}}\right)\right] \quad (9)$$

($\pi^{-1}(k)$ gives the position of index k in the permuted vector $\boldsymbol{\pi}$) and the optimal power policy $\widehat{\boldsymbol{\beta}}$ is given by

$$\widehat{\beta}_{k,n} = \widehat{\beta}_{k,n}(\mathcal{S}_n;\boldsymbol{\mu},\boldsymbol{\gamma}) = \widehat{p}_{k,N-n+1}\left(\boldsymbol{\alpha}_n;\boldsymbol{\mu},N\boldsymbol{\gamma}-\sum_{j=1}^{n-1}\widehat{\boldsymbol{\beta}}_j(\mathcal{S}_j;\boldsymbol{\mu},\boldsymbol{\gamma})\right) \quad (10)$$

for all n and k. $\square$

Although for finite N a closed form solution of (7) seems infeasible, for large N can prove:

Theorem 3. In the limit for large N, the long-term average capacity region $C_{K,N}(\boldsymbol{\gamma})$ tends to the ergodic capacity region [6]

$$C_K^{(\mathrm{erg})}(\boldsymbol{\gamma}) = \bigcup_{\boldsymbol{\beta}\in\Gamma_K^{(\mathrm{erg})}(\boldsymbol{\gamma})}\Big\{\boldsymbol{R}\in\mathbb{R}_+^K \ : \ \forall A\subseteq\{1,\cdots,K\}$$
$$\sum_{k\in A}R_k \le \mathrm{E}\left[\log\left(1+\sum_{k\in A}\alpha_k\beta_k(\boldsymbol{\alpha})\right)\right]\Big\} \quad (11)$$

where $\Gamma_K^{(\mathrm{erg})}(\boldsymbol{\gamma})$ is the set of *feasible* power allocation functions $\boldsymbol{\beta} = \{\beta_k \ : \ k = 1,\cdots,K\}$ defined by

$$\Gamma_K^{(\mathrm{erg})}(\boldsymbol{\gamma}) \triangleq \left\{\boldsymbol{\beta}\in\mathbb{R}_+^K \ : \ \mathrm{E}[\beta_k(\boldsymbol{\alpha})] \le \gamma_k\right\} \quad (12)$$

and the expectation is with respect to the instantaneous channel state $\boldsymbol{\alpha} = (\alpha_1,\cdots,\alpha_K)$. $\square$

Remark 1. From Theorem 2, by solving recursion (7) for $n = 1$, we see that the optimal solution is $\widehat{p}_{k,1} = P_k$ for every $\boldsymbol{\mu}$ and for every $\boldsymbol{\alpha}$. Hence, from (10) with $N = 1$, we have that $\widehat{\beta}_{k,1} = \gamma_k$ for all k, i.e., the optimal solution with $N = 1$ is constant power allocation. From (10), we also see that $\widehat{\beta}_{k,N} = N\gamma_k - \sum_{j=1}^{N-1}\widehat{\beta}_{k,j}$ which means that, on the last available slot, all the remaining energy is used regardless of the fading value, which is sensible since the remaining energy cannot be used on the next frame.

Remark 2. From Theorem 3, we see that as N increases the penalty incurred by the use of a short-term causal power allocation policy with respect to the ergodic power allocation policy decreases. This means that, when the delay is not too strict, the past information is irrelevant and the knowledge of the future is immaterial. In fact, from the proof of Theorem 5, is based on the fact that power policy tends to “uniformize” as N gets larger in the sense that the same law is applied on every slot in an i.i.d. fashion.

3.2. Long-term average capacity per unit energy

A byproduct of the proof of Theorem 1 is that the long-term average capacity region coincides with the standard "ergodic" capacity region of the N-slot extension channel, which is frame-wise memoryless. The following theorem is an immediate consequence of this fact and of the general theory of capacity per unit cost [3]:

Theorem 4. The long-term average capacity region per unit energy is given by

$$U_{K,N} = \bigcup_{\boldsymbol{\gamma} \in \mathbb{R}_+^K} \left\{ \boldsymbol{r} \in \mathbb{R}_+^K \, : \, (\gamma_1 \, r_1, \cdots, \gamma_K \, r_K) \in C_{K,N}(\boldsymbol{\gamma}) \right\} \tag{13}$$

□

In analogy with [3], it is easy to show:

Theorem 5. The long-term average capacity region per unit energy is the hyper-rectangle

$$U_{K,N} = \left\{ \boldsymbol{r} \in \mathbb{R}_+^K : r_k \leq s_N^{(k)} \right\} \tag{14}$$

where $s_N^{(k)}$, defined by

$$s_N^{(k)} = \lim_{\gamma_k \to 0} \frac{1}{\gamma_k} \sup_{\boldsymbol{\beta} \in \Gamma_{1,N}(\gamma_k)} \mathrm{E}\left[\frac{1}{N} \sum_{n=1}^{N} \alpha_{k,n} \beta_{k,n} \right] \tag{15}$$

is the k-th user single-user long-term average capacity per unit energy and is given by the Dynamic Programming recursion

$$s_n^{(k)} = \mathrm{E}[\max\{s_{n-1}^{(k)}, \alpha_k\}] \tag{16}$$

for $n = 1, \ldots, N$ with initial condition $s_0^{(k)} = 0$ and where expectation is with respect to $\alpha_k \sim F_\alpha^{(k)}(x)$. Furthermore, $s_N^{(k)}$ is achieved by the "one-shot" power allocation policy

$$\beta_{k,n}^\star = \left\{ \begin{array}{ll} N\gamma_k & \text{if } n = n_k^\star \\ 0 & \text{otherwise} \end{array} \right. \tag{17}$$

where the random variable $n_k^\star$ is defined as

$$n_k^\star = \min\left\{ n \in \{1, \ldots, N\} : \alpha_{k,n} \geq s_{N-n}^{(k)} \right\} \tag{18}$$

□

The behavior of $s_N^{(k)}$ when N grows to infinity is given by the following:

Theorem 6. For large N, the k-th user single-user long-term average capacity per unit energy $s_N^{(k)}$ tends to the k-th user single-user ergodic capacity per unit energy given explicitly by

$$\lim_{N \to \infty} s_N^{(k)} = \sup\{\alpha_k\} \tag{19}$$

where $\sup\{\alpha_k\} \triangleq \inf\{x \geq 0 \, : \, F_\alpha^{(k)}(x) = 1\}$. □

Remark 3. We have nicknamed the optimal policy $\beta^\star$ "one-shot" because the whole available energy $N\gamma_k$ is spent all at once in a single slot. In fact, in each slot

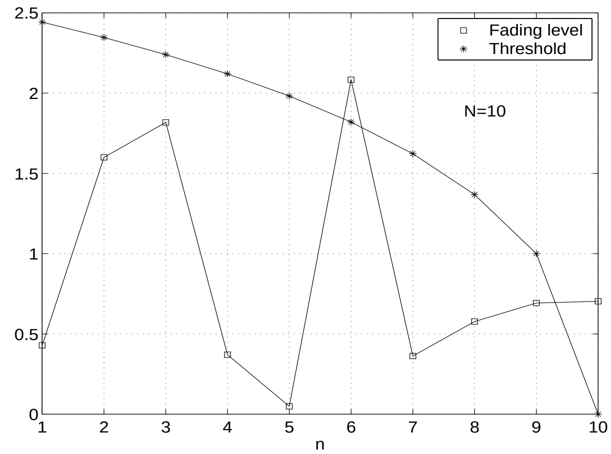

Figure 1: Fading realization over a frame of $N = 10$ slots.

$n \in \{1, \cdots, N\}$, the transmitter compares the instantaneous fading gain $\alpha_{k,n}$ with the time varying threshold $s_{N-n}^{(k)}$, if the fading is above the threshold then it transmits on the current block by using all the available energy otherwise it waits for the next slot. Since the threshold to be used on the last slot is $s_0^{(k)} = 0$, the available energy is used within the required delay of N slots with probability one. These features of optimal policy were already found out in [1], but in this work the authors did not realize that what they called "approximation for low SNR" is actually the general solution to long-term average capacity per unit energy for every finite delay N. Fig. 1 shows a fading realization over a window of $N = 10$ slots. We can this that in this case transmission would have occurred in slot $n = 6$. Thresholds are computed assuming Rayleigh fading.

Another interesting feature of the "one-shot" policy, from an implementation point of view, is that there is no need for the users to store in memory the past values of the fading gains since the only information needed about the past is whether or not transmission has already took place, thus saving memory space.

The threshold sequence $\{s_n^{(k)}\}_{n=0}^\infty$ depends only on the fading statistic, then it can be easily pre-mputed and stored in memory. When varying the delay requirements from N_1 to N_2, the threshold sequence need not to be re-computed, only a different "chunk" $\{s_n^{(k)}\}_{n=0}^{N_2-1}$, instead of $\{s_n^{(k)}\}_{n=0}^{N_1-1}$, has to be used. Notice also that the number of active users K does not affect the value of the thresholds.

Remark 4. The optimality of a coding scheme in the wideband regime is defined and studied in [9]. Let $C(\gamma)$ be the capacity expressed in nat/dimension as a function of γ, and let $\mathsf{C}(E_b/N_0)$ denote the corresponding spectral efficiency in bit/s/Hz as a function of the energy per bit vs. noise power spectral density, E_b/N_0, given implicitly by

$$\left\{ \begin{array}{l} \frac{E_b}{N_0} = \frac{\gamma \log 2}{C(\gamma)} \\ \mathsf{C}\left(\frac{E_b}{N_0}\right) = \frac{C(\gamma)}{\log 2} \end{array} \right. \tag{20}$$

The value $(E_b/N_0)_{\min}$ for which $\mathsf{C}(E_b/N_0) > 0 \Leftrightarrow$

$E_b/N_0 > (E_b/N_0)_{\min}$ is given by [9] $\left(\frac{E_b}{N_0}\right)_{\min} = \frac{\log 2}{\dot{C}(0)}$ where $\dot{C}(0)$ is the first derivative of the capacity function $C(\gamma)$ at $\gamma = 0$. From the proof of Theorem 5, we see immediately that the reciprocal of $(E_b/N_0)_{\min}$ for the k-th user is its capacity per unit energy (expressed in bit/joule), of the channel, i.e.,

$$\left(\frac{E_b}{N_0}\right)_{\min} = \frac{\log 2}{s_N^{(k)}} \tag{21}$$

The "one-shot" policy not only makes the most efficient use of the energy, by maximizing the number of expected correctly received number of bits per joule, but also reduces to the minimum the interference to other users since all the users transmit at minimum E_b/N_0. Notice that as the delay constraint is relaxed, i.e., N grows, the minimum required E_b/N_0 lowers down. Of course nothing is for free: the fact that the system works at the minimum E_b/N_0 is because it uses of a large number of degree of freedom (L) per information bit, i.e., the system works in the so-called " wideband regime".

3.3. The non-causal policy achieving long-term average capacity per unit-energy

Consider the single user case, since we saw that in the multiuser case the long-term average capacity region is the Cartesian product of the single-user long-term average capacities. If we allow the input to depend on the whole CSI $\mathcal{S}_N$ in a non-causal way, it is immediate to show that the optimal policy achieving long-term average capacity per unit energy is "maximum selection"

$$\beta_{k,n}^{\star\,(\mathrm{nc})} = \begin{cases} \frac{N\gamma_k}{|M_k|} & \text{if } n \in M_k \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

where

$$M_k = \{n\,:\,\alpha_{k,n} = \max\{\alpha_{k,1},\cdots,\alpha_{k,N}\}\} \tag{23}$$

and $|M| \in \{1,\cdots,N\}$ denotes the cardinality of M. Power policy (22) equally divides the available energy among the slots whose fading is equal to the maximum. Note that with continuous fading distribution we have $\Pr[|M_k| > 1] = 0$. Hence, the long-term average capacity per unit energy is

$$s_N^{(k,\mathrm{nc})} = \mathrm{E}[\max\{\alpha_{k,1},\cdots,\alpha_{k,N}\}] \tag{24}$$

4. NUMERICAL EXAMPLES

In this section we give numerical values of the long-term average capacity for two types of channel: *discrete two states fading channel* and *continuous Rayleigh fading channel*. In the following we omit the super script "(k)" referring to the user.

The two states fading channel. This fading channel models a communication system that requires a line of sight, as low orbit satellite communication systems. The fading can be either $\alpha = 0$ (bad channel, i.e., no line of sight) or $\alpha = 1$ (good channel). The probability of the good state is $\delta = \Pr[\alpha = 1] = 1 - \Pr[\alpha = 0]$ with $\delta \in [0,1]$. For this channel $\mathrm{E}[\alpha] = \delta$ and $\sup\{\alpha\} = 1$. We have

$$s_N = 1 - (1-\delta)^N \tag{25}$$

Note that for this special channel $s_N = s_N^{(\mathrm{nc})}$ i.e. the anti-causal knowledge of the channels gains does not improve the performance of the system. Fig. 2 shows the value of s_N as a function of N for the two states fading channel.

The Rayleigh fading channel. The channels gain are i.i.d with cdf $F_\alpha(x) = 1 - \mathrm{e}^{-x}$ for $x \geq 0$. For this channel $\mathrm{E}[\alpha] = 1$ and $\sup\{\alpha\} = \infty$. The long-term average capacity can be computed from the recurtion

$$s_N = s_{N-1} + \mathrm{e}^{-s_{N-1}} \tag{26}$$

with initial condition $s_0 = 0$. In this case the anti-causal knowledge of the channels gains would give

$$s_N^{(\mathrm{nc})} = \sum_{n=1}^{N} \binom{N}{n} \frac{(-1)^{n+1}}{n} \tag{27}$$

strictly larger than s_N for all $N > 1$. Fig. 3 shows the value of s_N and $s_N^{(\mathrm{nc})}$ as a function of N for the Rayleigh fading channel.

Notice that the "one-shot" policy is only optimal in the low SNR regime. Consider the single-user long-term average rate that can be achieved by applying policy $\boldsymbol{\beta}^\star$. In order to be always inside the instantaneous fading dependent capacity region, the user must encode at rate $r_n^\star = \log(1 + \alpha_n N\gamma)$ on the slot n for which $n^\star = n$. Hence, on a long-term average, the user gets $C_{1,N}^\star(\gamma)$ given by

$$\begin{aligned} C_{1,N}^\star(\gamma) &= \frac{1}{N}\,\mathrm{E}\left[\sum_{n=1}^{N} \log(1 + \alpha_n N\gamma\, 1\{n^\star = n\})\right] \\ &= \frac{1}{N} S_N^\star(N\gamma) \end{aligned} \tag{28}$$

where $S_N^\star(P)$ is given by the recursion

$$\begin{aligned} S_N^\star(P) &= \Pr[\alpha_j < s_{N-1}] S_{N-1}^\star(P) \\ &\quad + \int_{s_{N-1}}^{\infty} \log(1 + Px)\,\mathrm{d}F_\alpha(x) \end{aligned} \tag{29}$$

for $n = 1,\cdots,N$ and with initial condition $S_0^\star(P) = 0$. Fig. 4 shows $C_{1,N}^\star(\gamma)$ for the Rayleigh fading case for different value of N. Note that for small γ, $C_{1,N}^\star(\gamma)$ increases with N but for higher γ it decreases, proving that $\boldsymbol{\beta}^\star$ is optimal only in the energy limited (low SNR or wideband) regime. As N increases in the high SNR regime, the rate $C_{1,N}^\star(\gamma)$ drops to zero.

5. CONCLUSIONS

In this work we characterized the long-term average capacity region per unit energy of a Gaussian block-fading multiuser channel with causal feedback and delay constraint, we derived the optimal power allocation policy and showed that it is "one-shot", decentralized and makes

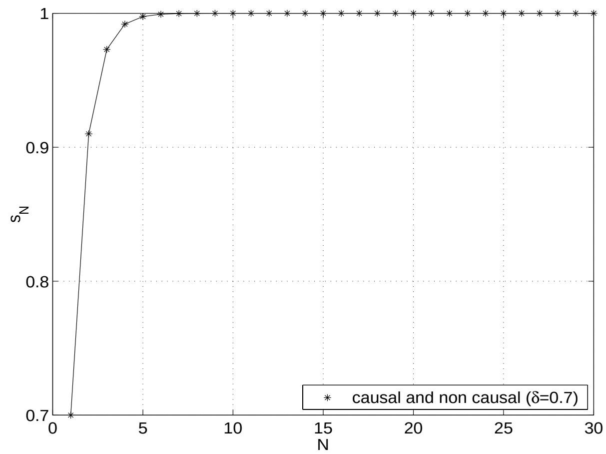

Figure 2: s_N vs. N for the two states channel.

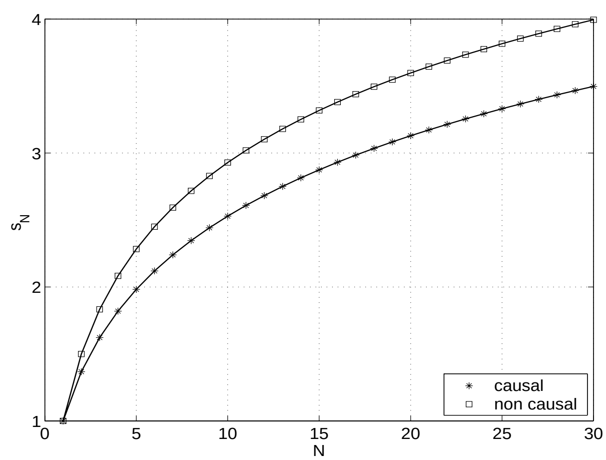

Figure 3: s_N vs. N for the Rayleigh channel.

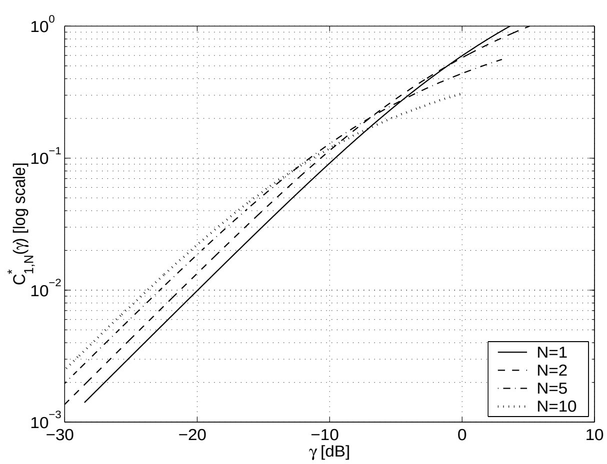

Figure 4: $C^{\star}_{1,N}$ vs. γ for the Rayleigh channel.

the system work at minimum energy per information data bit. We also gave a limiting analysis, in which we showed that for $N = 1$ optimal policy coincides with constant power allocation, while, as N increases, it tends to the ergodic policy.

Throughout the whole paper we insisted that the "one-shot" power policy is completely decentralized, hence very simple, and we listed a number of "practical" advantages of this fact. Actually, in order to achieve rate points on the closure of the long-term average capacity $C_{K,N}(\boldsymbol{\gamma})$ users not only must use their power on the most favorable channel conditions but they also need to coordinate their transmission rate in order to be always inside the instantaneous fading dependent capacity region. To be more clear, let assume that there is only one user in the system and that choses to allocate power according to $\boldsymbol{\beta}^{\star}$, then it achieves rate $C^{\star}_{1,N}(\gamma)$ given in (30). If there is another user in the system, and both allocate power in a decentralized way according to $\boldsymbol{\beta}^{\star}$ but also allocate rates in a decentralized way according to $r^{\star}_n$, then when they happen to transmit on the same slot the receiver cannot jointly decode them, hence the system is in outage. The probability of outage, for two user system, is hence $\mathrm{P}_{\mathrm{out}}(N) = \Pr[n^{\star}_1 = n^{\star}_2]$ and it can be computed with the following recursion

$$\begin{aligned}\mathrm{P}_{\mathrm{out}}(N) &= \left(1 - F^{(1)}_{\alpha}(s^{(1)}_{N-1})\right)\left(1 - F^{(2)}_{\alpha}(s^{(2)}_{N-1})\right) \\ &\quad + F^{(1)}_{\alpha}(s^{(1)}_{N-1})F^{(2)}_{\alpha}(s^{(2)}_{N-1}) \cdot \mathrm{P}_{\mathrm{out}}(N-1) \qquad (30)\end{aligned}$$

with initial condition $\mathrm{P}_{\mathrm{out}}(0) = 0$. Fig. 5 shows the probability of outage in the Rayleigh fading case as a function of N. It can be can seen that at $N = 10$ they are going to collide on 1 frame out of 10 ($\mathrm{P}_{\mathrm{out}}(10) = 0.1$). To avoid outage the system must coordinate the rates.

One would be tempted to say that we can anyway avoid rate coordination since "TDMA is optimal in wideband regime". In fact, form the proof of Theorem 5, long-term capacity per unit energy can be achieved either with superposition coding or with TDMA inside each slot. Actually, the analysis that lead to the derivation of the long-term capacity per unit energy is an analysis of "infinite bandwidth regime" and not of "wideband regime", i.e, small SNR but not vanishing. Recent works [10, 4] have shown that actually TDMA can be heavily suboptimal, in term of achievable rates, especially in a multiuser faded environment so intrinsically rich in diversity. In fact, when many users are active in a faded environment, with high probability the best user enjoys channel gain that is larger than its average, hence the performance is dominated not by the average but by the maximum.

What do we conclude? In order to fully exploit diversity we need joint processing in the form of rate coordination and joint decoding at the receiver, but this is expensive in complexity. When we are constrained by complexity, then performance has to be sacrificed. Anyway, this theoretical work provides some guidelines for the design of practical systems. In fact, the "one-shot" policy has very interesting consequences at protocol layer: in wideband regime, in order to optimize the average number of received bits per joule, sequential polling of the active users by the master is not needed. This analysis

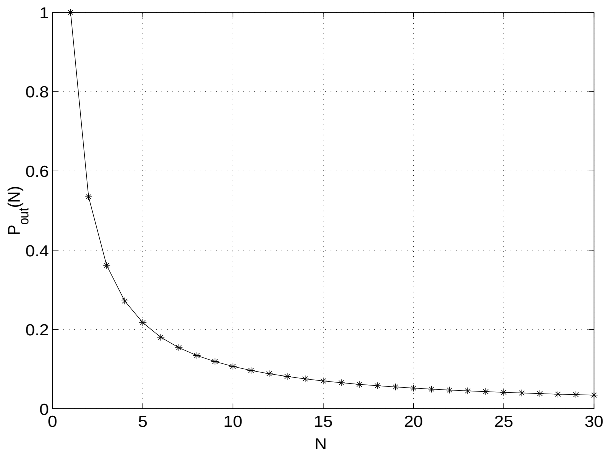

Figure 5: $\mathrm{P}_{\mathrm{out}}(N)$ vs. N for the Rayleigh channel.

suggests that the master station should send periodically a “probe” signal; if a user has a message to send, then it starts a timeout and measures the attenuation of its own channel on every slot of the frame: on the first slot where the channel gain is higher than the time varying threshold $s_\cdot$ it sends its packet with all the available energy, then it resets the timeout and waits for the next packet to send. Note that an optimal system actually does not require the time windows of the active users to be synchronous and the users can have different delay requirements this allowing for extra flexibility.

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