Detection of the Number of Superimposed Signals using Modified MDL Criterion: A Random Matrix Approach

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Abstract

The problem of estimating the number of superimposed signals using noisy observations from N antennas is addressed. In particular, we are interested in the case where a low number of snapshots $L = \mathcal{O}(N)$ is available. We focus on the Minimum Description Length (MDL) estimator, which is revised herein. Furthermore, we propose a modified MDL estimator, with the help of random matrix tools, which improves the estimation of the number of sources. Simulation results demonstrate the potential of the modified MDL estimator over the traditional one, in the case where $L = \mathcal{O}(N)$.

Introduction

- The problem of model selection from the observed data, or snapshots, is a fundamental problem in diverse areas of signal processing.
- Many detection techniques depend, merely, on the sample eigenvalues of the data covariance matrix. In practice, this data covariance matrix is not the true one, but an estimate.
- Modelling by Minimum Description Length (MDL) [1] is one approach for a detection problem. However, MDL is known to be L-consistent, but is not (N, L)-consistent.

Contributions

- In this contribution and with the help of random matrix tools, we have present a modified MDL (MMDL) estimator for detecting the number of superimposed signals.

System Model

Assume a planar arbitrary array of N antennas. Furthermore, consider $q < N$ narrowband sources attacking the array from different angles, i.e. $\boldsymbol{\Theta} = [\theta_1 \dots \theta_q]$. Collecting L time snapshots, we can write

$$\mathbf{X} = [\mathbf{x}(t_1)\dots\mathbf{x}(t_\mathrm{L})] = \mathbf{AS} + \mathbf{W} \qquad \text{where} \qquad \mathbf{S} = [\mathbf{s}(t_1)\dots\mathbf{s}(t_\mathrm{L})] \in \mathbb{C}^{\mathrm{q}\times\mathrm{L}} \qquad \text{and} \qquad \mathbf{s}(t) = [s_1(t)\dots s_\mathrm{q}(t)]^\mathrm{T} \tag{1}$$

The matrix $\mathbf{A}$ is modelled as

$$\mathbf{A} = [\mathbf{a}(\theta_1)\dots\mathbf{a}(\theta_q)] \qquad \text{where} \qquad \mathbf{a}(\theta_i) = [e^{-jk(\bar{x}_1 sin\theta_i + \bar{y}_1 cos\theta_i)}\dots e^{-jk(\bar{x}_N sin\theta_i + \bar{y}_N cos\theta_i)}]^\mathrm{T} \tag{2}$$

where $k = \frac{2\pi}{\lambda}$ is the wavenumber, and λ is the wavelength. The position of the n^{th} antenna is $(\bar{x}_n, \bar{y}_n)$ in xy-plane. The matrix $\mathbf{W} \in \mathbb{C}^{\mathrm{N}\times\mathrm{L}}$ is background noise.

Assumptions

We assume the following:

- **(A.1)** The matrix $\mathbf{A}$, is full column rank. This is valid when $q \leq N$ and all angles of arrival are distinct, i.e. $\theta_i \neq \theta_j$ for all $i \neq j$.
- **(A.2)** The sources are assumed to be non-coherent, i.e. $\mathbf{R}_{ss} = E\{\mathbf{s}(t)\mathbf{s}^\mathrm{H}(t)\}$ is full rank.
- **(A.3)** The noise is modelled as complex Gaussian vectors, i.i.d over time, with zero-mean and covariance $\sigma^2\mathbf{I}_\mathrm{N}$, and independent from the signal.

Now, we are ready to address our *detection* problem:

"Given the available snapshots $\mathbf{X}$, estimate the number of source signals, i.e. q."

Estimating number of signals using MDL [1]: A Recap

As stated earlier, the problem of estimating the number of incoming signals could be seen as a model selection one, i.e. finding the model that best fits the data $\mathbf{X}$. More specifically, the problem is to select one of the N following models:

$$\mathbf{R}_{xx}^{(k)} = \sum_{i=1}^{k}(\lambda_i - \sigma^2)\mathbf{v}_i\mathbf{v}_i^\mathrm{H} + \sigma^2\mathbf{I}_\mathrm{N}, \qquad k = 0\dots N-1 \tag{3}$$

where $\mathbf{v}_i$ is the eigenvector corresponding to the eigenvalue λ_i of $\mathbf{R}_{xx}^{(k)}$. After maximising the likelihood function w.r.t $\Theta^{(k)} = [\lambda_1, \dots, \lambda_k, \sigma^2, \mathbf{v}_1^T, \dots, \mathbf{v}_k^T]$, the best model is the one that minimises:

$$\mathrm{MDL}(k) = -\log\left(\frac{\prod_{i=k+1}^{N} \hat{l}_i^{\frac{1}{N-k}}}{\frac{1}{N-k}\sum_{i=k+1}^{N}\hat{l}_i}\right)^{L(N-k)} + \frac{k}{2}(2N-k)\log(L) \qquad \text{where } \hat{l}_i \text{ is the } i^{th} \text{ largest eigenvalue of the sample covariance matrix } \hat{\mathbf{R}} = \frac{1}{L}\mathbf{X}\mathbf{X}^\mathrm{H} \tag{4}$$

Therefore, according to the MDL criterion, the number of sources q is the argument k that minimises equation (4).

A Modified MDL Estimator

It has been shown in [2] that the sample eigenvalues $\hat{l}_1 \dots \hat{l}_N$ extracted from the sample covariance matrix $\hat{\mathbf{R}}$ are (N, L)-inconsistent estimators of the true eigenvalues of the covariance matrix $\mathbf{R}_{xx}$, that is, the sample eigenvalues do not converge towards the true ones as $(N, L) \longrightarrow \infty$ at the same rate $(0 < c = \frac{N}{L} < \infty)$. The MDL estimator in (4) depends on the sample eigenvalues of $\hat{\mathbf{R}}$, therefore, it seems natural that the performace of the MDL estimator would perform poorly in the asymptotic regime, i.e. $(N, L) \longrightarrow \infty$ at the same rate $(0 < c = \frac{N}{L} < \infty)$. Before presenting the improved estimators of the eigenvalues of the covariance matrix $\mathbf{R}_{xx}$, we proceed as in [3] and pose the following assumptions:

- **(B.1)** The covariance matrix $\mathbf{R}_{xx}$ has uniformly bounded spectral norm for all N, i.e. $\mathrm{Sup}_\mathrm{N}\|\mathbf{R}_{xx}\| < \infty$ where $\|.\|$ denotes spectral norm.
- **(B.2)** The sample covariance matrix written as $\hat{\mathbf{R}} = \sqrt{\mathbf{R}}_{xx}\mathbf{V}\mathbf{V}^H\sqrt{\mathbf{R}}_{xx}$ where $\sqrt{\mathbf{R}}_{xx}$ denotes the square root of $\mathbf{R}_{xx}$. The matrix $\mathbf{V}$ is of size $N \times L$ with complex i.i.d. absolutely continous random entries, with each entry having i.i.d. real and imaginary parts of zeros mean, variance $\frac{1}{2L}$, and finite eighth-order moments.
- **(B.3)** For all distinct $q+1$ eigenvalues of $\mathbf{R}_{xx}$, which are $l_1 > \dots > l_q > l_{q+1} = \sigma^2$, we assume $\mathrm{inf}_\mathrm{N}\{\frac{L}{N} - \kappa_N(m)\} > 0$, where $\kappa_N(m)$ is given in (5).

$$\kappa_N(m) = \begin{cases} \frac{1}{N}\sum_{i=1}^{q+1}\phi_{i,1}, & \text{if} \quad m = 1 \\ \max\{\sum_{i=1}^{q+1}\phi_{i,m-1}, \sum_{i=1}^{q+1}\phi_{i,m}\}, & \text{if} \quad 1 < m < q+1 \\ \frac{1}{N}\sum_{i=1}^{q+1}\phi_{i,q}, & \text{if} \quad m = q+1 \end{cases} \qquad \text{where} \qquad \phi_{i,k} = K_i\Big(\frac{l_i}{l_i - f_k}\Big)^2 \tag{5}$$

where K_i is the multiplicity of the i^{th} largest eigenvalue of $\mathbf{R}_{xx}$, i.e. $K_1 = \dots = K_q = 1$ and $K_{q+1} = N - q$. Also, $f_1 < f_2 < \dots < f_q$ are the real-valued roots of

$$\frac{1}{N}\sum_{i=1}^{q+1} K_i \frac{l_i^2}{(l_i - f)^3} = 0 \tag{6}$$

Improved Eigenvalue Estimates [3]

We revise a theorem [3] regarding improved eigenvalue estimates, which are not only L-consistent, but also (N, L)-consistent. The theorem is as follows:

Theorem: Under assumptions **(B.1)** to **(B.3)**, the following quantities are strongly (N, L)-consistent estimators of l_j $(j = 1, \dots, q+1)$.

$$\hat{l}_j^{\mathrm{imp}} = L\big(\hat{l}_j - \mu_j\big), \quad j = 1\dots q \qquad \text{and} \qquad \hat{l}_{q+1}^{\mathrm{imp}} = \frac{L}{N-q}\sum_{i=q+1}^{N}\big(\hat{l}_i - \mu_i\big) \tag{7}$$

where $\mu_1 \leq \mu_2 \leq \dots \leq \mu_N$ are the real-valued solutions of the following equation in μ:

$$\frac{1}{N}\sum_{i=1}^{N}\frac{\hat{l}_i}{\hat{l}_i - \mu} = \frac{1}{c} \tag{8}$$

Modified MDL estimator

With the improved eigenvalue estimates of $\mathbf{R}_{xx}$ in hand from (7), it is easily verified that

$$\hat{\lambda}_i = L\big(\hat{l}_j - \mu_j\big), \quad j = 1\dots k \qquad \text{and} \qquad \hat{\sigma}^2 = \frac{L}{N-k}\sum_{i=k+1}^{N}\big(\hat{l}_i - \mu_i\big) \tag{9}$$

Using these improved estimates in (9), one could also verify that the improved MDL estimator finally becomes

$$\hat{q} = \arg\min_k \mathrm{MDL}^{\mathrm{imp}}(k) \qquad \text{where} \qquad \mathrm{MDL}^{\mathrm{imp}}(k) = -\log\left(\frac{\prod_{i=k+1}^{N}(\hat{l}_i - \mu_i)^{\frac{1}{N-k}}}{\frac{1}{N-k}\sum_{i=k+1}^{N}(\hat{l}_i - \mu_i)}\right)^{L(N-k)} + \frac{k}{2}(2N-k)\log(L) \tag{10}$$

Remark: As $c \longrightarrow 0$, then we have $\hat{l}_i^{\mathrm{imp}} \longrightarrow \hat{l}_i$ for all $i = 1\dots q+1$. Consequently, one could show that $\mathrm{MDL}^{\mathrm{imp}}(k) \longrightarrow \mathrm{MDL}(k)$ for all k as $c \longrightarrow 0$.

Simulations

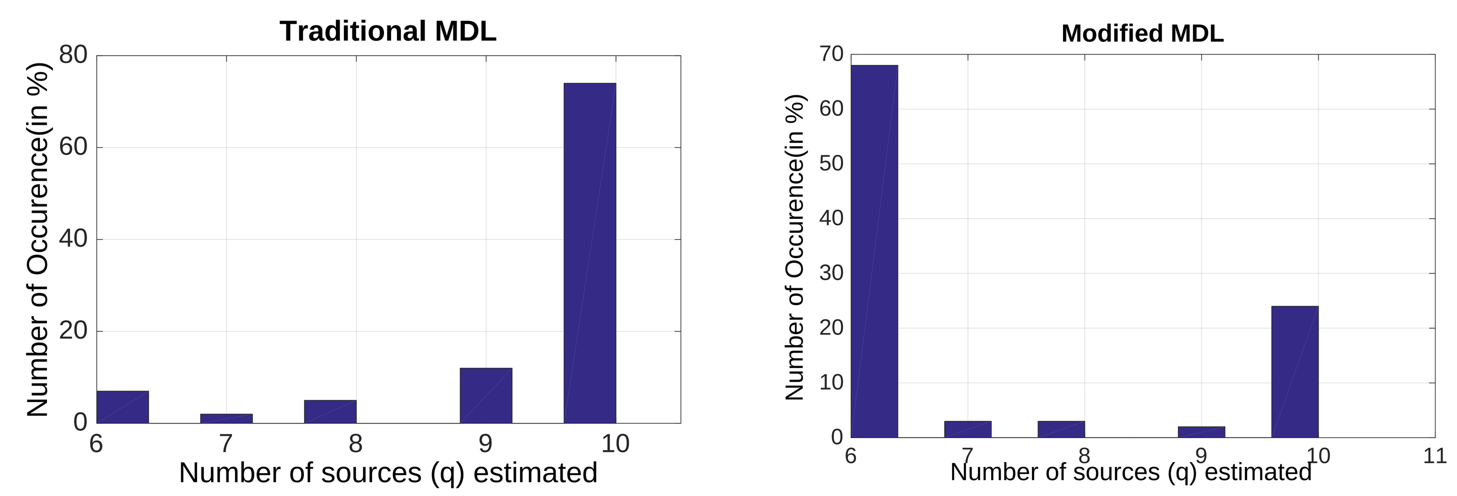

Figure 1: Histogram comparing the traditional MDL with Modified MDL. Simulations were done under at SNR = 10 dB and $q = 6$ sources with sufficiently spaced AoAs. The sources were non-coherent and the array geometry consists of $N = 10$ antennas uniformly spaced by half a wavelength. The number of snapshots collected was $L = 10$, i.e. $c = \frac{N}{L} = 1$. Note that both histograms were done using 1000 trials.

Conclusions

- With the help of random matrix tools, we have presented a modified MDL (MMDL) estimator for detecting the number of superimposed signals.
- This MMDL estimator dominates the traditional MDL especially at the low number of snapshots regime, i.e. when $L = \mathcal{O}(N)$.

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