On the Effect of Random Snapshot Timing Jitter on the Covariance Matrix for JADE Estimation

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Outline

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- Indoor Localisation:
  1. **Radio-Based: Online estimation of AoA (Angle of Arrival) and/or ToA (Time of Arrival) , etc.**

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- *AoA* and *ToA* information found in $\mathbf{A}$.

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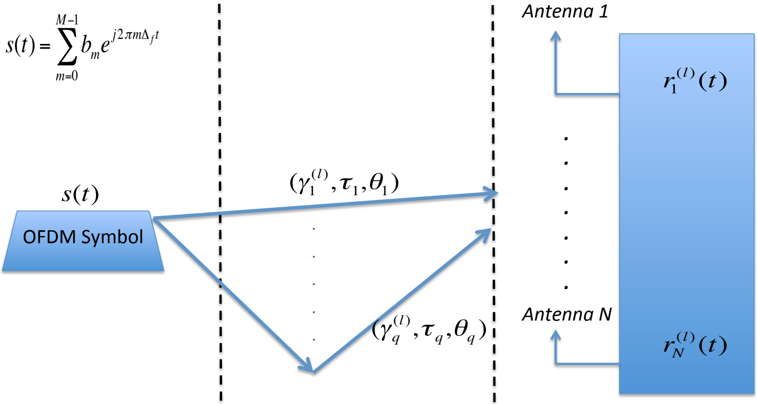
- It turns out that all existing algorithms are *Model-Sensitive*.
- In other words, such algorithms fail when model is $\mathbf{x}^{(l)} = \mathbf{C}^{(l)}\mathbf{A}\boldsymbol{\gamma}^{(l)} + \mathbf{n}^{(l)}$ with unknown $\mathbf{C}^{(l)}$.

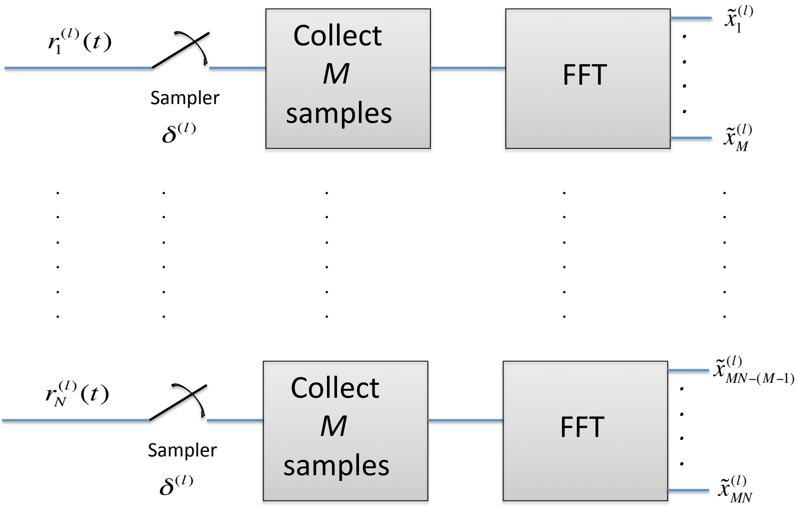
Outline

Transmitter (TX)	Multipath Channel	Receiver (RX)
$s(t)$ is the transmit OFDM symbol composed of M subcarriers.	**Channel composed of q paths.** **The i[th] path is characterized by:** • **Complex amplitude $\gamma_i^{(l)}$** • **Time of Arrival (ToA) τ_i** • **Angle of Arrival (AoA) θ_i**	The n^{th} antenna receives a sum of the q paths, $r_n^{(l)}(t)$.

$$s(t) = \sum_{m=0}^{M-1} b_m e^{j2\pi m\Delta_f t}$$



Snapshot Illustration (cont'd)



Outline

Analytical Formulation

System Model

$$\tilde{\mathbf{x}}^{(l)} = \mathbf{C}^{(l)} \mathbf{A} \boldsymbol{\gamma}^{(l)} + \mathbf{n}^{(l)}, \quad l = 1 \dots L$$

where

- $\mathbf{A} = [\mathbf{a}(\theta_1) \otimes \mathbf{c}(\tau_1), \dots, \mathbf{a}(\theta_q) \otimes \mathbf{c}(\tau_q)]$

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  - $\mathbf{a}(\theta) = [\mathbf{a}_1(\theta), \dots, \mathbf{a}_N(\theta)]^T$ depends on array geometry (ex: ULA).

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- $\mathbf{C}^{(l)} = \mathbf{I}_N \otimes \text{diag}\{\mathbf{c}(\delta^{(l)})\}$
- $\boldsymbol{\gamma}^{(l)} = [\gamma_1^{(l)} \dots \gamma_q^{(l)}]^T$
- $\mathbf{n}^{(l)}$ is an $MN \times 1$ background noise vector.

Outline

Assumptions

1. $q < MN$
2. The matrix $\mathbf{A}$ has full column rank.
3. The entries of the multipath vector $\boldsymbol{\gamma}^{(l)}$ are not fully correlated, i.e. $\mathbf{R}_{\gamma\gamma}$ is full rank.
4. $\mathbf{n}^{(l)} \sim \mathcal{N}(0, \sigma_n^2 \mathbf{I}_{MN})$ indep from signal part $\mathbf{C}^{(l)}\mathbf{A}\boldsymbol{\gamma}^{(l)}$
5. $\delta^{(l)} \sim \mathcal{N}(0, \sigma_\delta^2)$ indep from $\boldsymbol{\gamma}^{(l)}$ and $\mathbf{n}^{(l)}$.

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Problem Statement

Given the data $\left\{\mathbf{x}^{(l)}\right\}_{l=1}^{L}$, *estimate:*

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- *the times and angles of arrival* $\left\{\theta_i, \tau_i\right\}_{i=1}^{q}$.

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Jitter Effect on Covariance Matrix

$$\mathbf{R}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} \triangleq E_l\{\tilde{\mathbf{x}}^{(l)}\tilde{\mathbf{x}}^{(l)^H}\} = \boldsymbol{\Upsilon} \odot \mathbf{R}_{\mathbf{x}\mathbf{x}}$$

where

- $\mathbf{R}_{\mathbf{x}\mathbf{x}}$ is the covariance matrix of data **without** jitter, i.e.

$$\mathbf{x}^{(l)} = \mathbf{A}\boldsymbol{\gamma}^{(l)} + \mathbf{n}^{(l)}$$

Jitter Effect on Covariance Matrix

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- $\mathbf{R}_{\mathbf{xx}}$ is the covariance matrix of data **without** jitter, i.e.

$$\mathbf{x}^{(l)} = \mathbf{A}\boldsymbol{\gamma}^{(l)} + \mathbf{n}^{(l)}$$

- $\boldsymbol{\Upsilon} = \mathbf{J}_N \otimes \mathbf{T}$
  with
  - $\mathbf{J}_N$ is the all-ones square matrix of size N.
  - $\mathbf{T}^{\langle m,n\rangle} = e^{-\frac{\left(2\pi\triangle_f(m-n)\sigma_\delta\right)^2}{2}}$

Outline

Consequence on Estimating q

No Jitter

Assume no jitter, i.e. the model is:
$\mathbf{x}^{(l)} = \mathbf{A}\boldsymbol{\gamma}^{(l)} + \mathbf{n}^{(l)} \Rightarrow \mathbf{R}_{\mathbf{xx}} = \mathbf{A}\mathbf{R}_{\gamma\gamma}\mathbf{A}^H + \sigma_n^2 \mathbf{I}_{MN}$ (Using Assumption 4).

Eigenvalues of $\mathbf{R}_{\mathbf{xx}}$

1. The matrix $\mathbf{A}\mathbf{R}_{\gamma\gamma}\mathbf{A}^H$ is of rank q (Under Assumptions 1-3).
2. Sorting the non-zero eigenvalues of $\mathbf{A}\mathbf{R}_{\gamma\gamma}\mathbf{A}^H$ in decreasing order $(\lambda_1 \geq \ldots \geq \lambda_q)$, the q large eigenvalues of $\mathbf{R}_{\mathbf{xx}}$ are $\{\lambda_i + \sigma_n^2\}_{i=1}^q$ and its complementary $MN - q$ eigenvalues are equal to σ_n^2.

Conclusion

This means that, under high SNR, one could estimate the number of sources using the q largest eigenvalues of $\mathbf{R}_{\mathbf{xx}}$.

Consequence on Estimating q (cont'd)

Theorem (C. S. Ballantine[1])

Let $\mathbf{E}$ *and* $\mathbf{F}$ *be* $m \times n$ *matrices. If* $\mathbf{A}$ *is positive definite and* $\mathbf{B}$ *is positive semi-definite with* r *positive diagonal elements, then rank*$(\mathbf{A} \odot \mathbf{B}) = r$*.*

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[1] **C. S. Ballantine**; **On the Hadamard product**; **Mathematische Zeitschrift 105**; **1968**; **365-366**.

Consequence on Estimating q (cont'd)

Presence of Jitter

$$\tilde{\mathbf{x}}^{(l)} = \mathbf{C}^{(l)}\mathbf{A}\boldsymbol{\gamma}^{(l)} + \mathbf{n}^{(l)}$$
$$\Rightarrow \mathbf{R}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} = \boldsymbol{\Upsilon} \odot \mathbf{R}_{\mathbf{xx}}$$
$$\Rightarrow \mathbf{R}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} = \boldsymbol{\Upsilon} \odot \mathbf{A}\mathbf{R}_{\gamma\gamma}\mathbf{A}^{H} + \sigma_n^2\mathbf{I}_{MN}$$

Conclusion

Assuming one antenna ($N = 1$) and using the previous theorem, the rank of $\boldsymbol{\Upsilon} \odot \mathbf{A}\mathbf{R}_{\gamma\gamma}\mathbf{A}^{H}$ is M for any q.

Outline

LS approach

Definition

Let $\mathbf{B}$ be an $MN \times MN$ matrix, i.e. $\mathbf{B} = \begin{bmatrix} \mathbf{B}_{1,1} & \cdots & \mathbf{B}_{1,N} \\ \vdots & \ddots & \vdots \\ \mathbf{B}_{N,1} & \cdots & \mathbf{B}_{N,N} \end{bmatrix}$, where $\mathbf{B}_{i,j}$ is the $(i,j)^{th}$ block matrix of size $M \times M$.

We define the following function:

$$F^{\mathbf{B}}(k) = \frac{1}{2(M-k)N^2} \sum_{\substack{|m-n|=k \\ i=1\dots N \\ j=1\dots N}} |\mathbf{B}_{i,j}^{\langle m,n \rangle}|, \qquad k = 1 \dots M-1$$

LS approach (cont'd)

Recall that

$\mathbf{R}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}} = \boldsymbol{\Upsilon} \odot \mathbf{R}_{\mathbf{xx}} = (\mathbf{J}_N \otimes \mathbf{T}) \odot \mathbf{R}_{\mathbf{xx}}$

where

- $\mathbf{J}_N$ is the all-ones square matrix of size N
- $\mathbf{T}^{\langle m,n \rangle} = e^{-\frac{\left(2\pi \triangle_f (m-n)\sigma_\delta\right)^2}{2}}$

It is easy to verify that

$$F^{\mathbf{R}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}}}(k) = F^{\mathbf{R}_{\mathbf{xx}}}(k).e^{-\frac{(2\pi \triangle_f k \sigma_\delta)^2}{2}}, \qquad k = 1 \dots M-1$$

LS approach (cont'd)

LS Estimate

For large M, we can say that $F^{\mathbf{R_{xx}}}(k)$ is almost invariant compared to $e^{-\frac{(2\pi\triangle_f k\sigma_\delta)^2}{2}}$, i.e. $F^{\mathbf{R_{xx}}}(k) = F^{\mathbf{R_{xx}}}$. We have:

$$\mathbf{f} = \mathbf{P}\mathbf{g}$$

where:

- $\mathbf{f} = \left[\log\left(F^{\mathbf{R}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}}}(1)\right) \cdots \log\left(F^{\mathbf{R}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}}}(M-1)\right)\right]^T$
- $\mathbf{P} = \begin{bmatrix} -\frac{(2\pi\triangle_f)^2}{2} & 1 \\ \vdots & \vdots \\ -\frac{(2\pi(M-1)\triangle_f)^2}{2} & 1 \end{bmatrix}$
- $\mathbf{g} = \begin{bmatrix} \sigma_\delta^2 \\ \log\left(F^{\mathbf{R_{xx}}}\right) \end{bmatrix}$

LS approach (cont'd)

LS Estimate

The LS estimate is obtained by solving:

$$\hat{\mathbf{g}}^{LS} = \arg\min_{\mathbf{g}} \|\mathbf{f} - \mathbf{P}\mathbf{g}\|_2^2 = (\mathbf{P}^H\mathbf{P})^{-1}\mathbf{P}^H\mathbf{f} = \begin{bmatrix} \hat{\sigma}_\delta^2 \\ \log\left(\hat{F}^{\mathbf{R_{xx}}}\right) \end{bmatrix}$$

Compensate

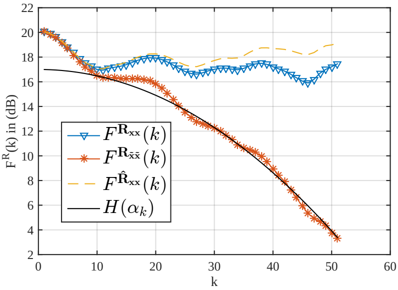
1. Form the matrix $\hat{\boldsymbol{\Upsilon}} = (\mathbf{J}_N \otimes \hat{\mathbf{T}})$, where

$$\hat{\mathbf{T}}^{\langle m,n \rangle} = e^{-\frac{\left(2\pi\triangle_f(m-n)\hat{\sigma}_\delta\right)^2}{2}}$$

2. Compensate by:

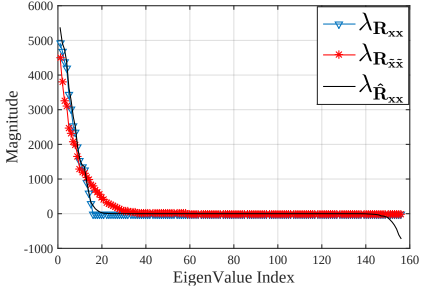
$$\hat{\mathbf{R}}_{\mathbf{xx}} = \hat{\boldsymbol{\Upsilon}}^H \odot \mathbf{R}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}}$$

LS approach (cont'd)



Simulation parameters

- $q = 15$ paths.
- $M = 52$ subcarriers and $N = 3$ antennas.
- SNR $= 20$ dB and $L = 100$ Snapshots.
- $\triangle_f = 0.3125\text{MHz}$
- $\sigma_\delta = 25$ nsec, we get $\boldsymbol{\hat{\sigma}_\delta} = \mathbf{27.2}$ **nsec**



The Over-Compensated Case

- $\sigma_\delta = 25$ nsec $< \boldsymbol{\hat{\sigma}_\delta = 27.2}$ **nsec**.
- The "over-compensated" covariance matrix $\hat{\mathbf{R}}_{\mathbf{xx}}$ admits negative eigenvalues.

Outline

Negative Eigenvalues approach

Idea

Search for the smallest value of $\breve{\sigma}_\delta \in [0, S]$ that yields

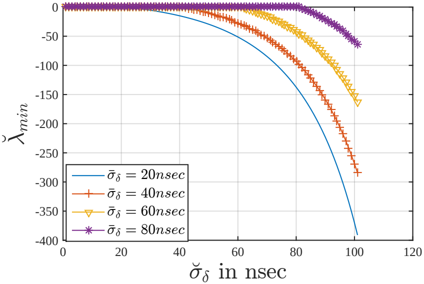
$$\boxed{\breve{\lambda}_{min} = \lambda_{min}\left(\breve{\mathbf{R}}_{\mathbf{xx}}\right) < 0}$$

where

$$\breve{\mathbf{R}}_{\mathbf{xx}} = \breve{\boldsymbol{\Upsilon}}^H \odot \mathbf{R}_{\tilde{\mathbf{x}}\tilde{\mathbf{x}}}$$

$$\breve{\boldsymbol{\Upsilon}} = \mathbf{J}_N \otimes \breve{\mathbf{T}}, \qquad \breve{\mathbf{T}}^{\langle m,n \rangle} = e^{-\frac{\left(2\pi \triangle_f (m-n) \breve{\sigma}_\delta\right)^2}{2}}$$

Negative Eigenvalues approach (cont'd)

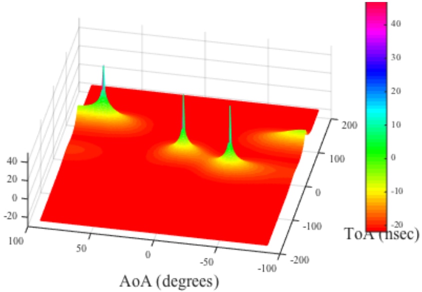


Simulation Parameters

- Same as before, but different σ_δ for each curve.

Outline

JADE with No Jitter

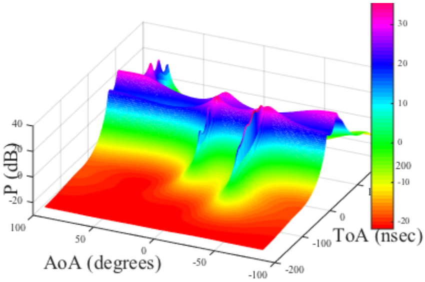


Simulation Parameters

- $M = 52$, $N = 3$, $\text{SNR} = 20\text{dB}$, $L = 100$
- $q = 3$ paths:
  1. $(\theta_1, \tau_1) = (-40°, 0\text{nsec})$
  2. $(\theta_2, \tau_2) = (0°, 30\text{nsec})$
  3. $(\theta_3, \tau_3) = (70°, 95\text{nsec})$

Outline

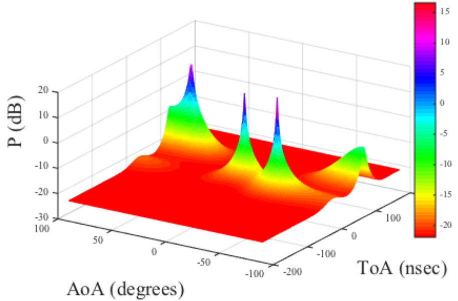
JADE in the Presence of Jitter



- Same parameters as previous slide but $\sigma_\delta = 95$nsec

Outline

JADE after Compensating for Jitter



Compensate then JADE

- True jitter variance: $\sigma_\delta = 95$nsec
- Estimated jitter variance: $\hat{\sigma}_\delta = 99$nsec

QUESTIONS ?