Cramér-Rao Bound for Joint Angle and Delay Estimators by Partial Relaxation

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Abstract

Novel Fisher-Information Matrix (FIM) and Cramér-Rao Bound (CRB) expressions for the problem of the "partially relaxed" Joint Angle and Delay Estimation (JADE) are derived and analyzed in this paper. In particular, exact closed form expressions of the CRB on the Angles and Times of Arrival of multiple sources are presented. Furthermore, interesting asymptotic and desirable properties are demonstrated, such as high SNR behaviour and lower bound expressions on the CRBs of Angles and Times of Arrival of multiple sources. Computer simulations are also given to visualize CRB behaviour in regimes of interest.

Introduction

- Localization has been a challenging topic over the past 70 years. Applications include seismology, radar, sonar, communications, etc.
- Recently, the partial relaxation (PR) framework has been introduced as a novel framework for the Angle-of-Arrival (AoA) problem.
- This paper derives and analyzes the Cramér-Rao Bound (CRB) of the partially-relaxed JADE problem.

Contributions

- The Fisher-Information-Matrix (FIM) and CRB of the partially relaxed JADE problem are derived. Exact closed form expressions are given.
- Some interesting asymptotic properties are revealed, i.e. lower bounds on the CRBs of the AoAs/ToAs are given.
- The cross-correlation CRB between ToA and AoA vanishes exponentially with linear increase of number of subcarriers/antennas.

System Model

Consider an OFDM symbol $s(t)$ composed of M subcarriers and centered at a carrier frequency f_c, impinging an antenna array of N antennas via q multipath components, each arriving at different AoAs $\{\theta_i\}_{i=1}^q$ and ToAs $\{\tau_i\}_{i=1}^q$. In baseband, we could write the l^{th} received OFDM symbol at the n^{th} antenna as:

$$\mathbf{x}(\ell) = \mathbf{H}(\boldsymbol{\theta}, \boldsymbol{\tau})\boldsymbol{\gamma}(\ell) + \mathbf{n}(\ell) \tag{1}$$

where $\mathbf{H}(\boldsymbol{\theta}, \boldsymbol{\tau}) = \begin{bmatrix}\mathbf{h}(\theta_1, \tau_1) & \dots & \mathbf{h}(\theta_q, \tau_q)\end{bmatrix}$ is the response of the channel to the ToA/AoAs and $\boldsymbol{\gamma}(\ell) = \begin{bmatrix}\gamma_1(\ell) & \dots & \gamma_q(\ell)\end{bmatrix}$ are the multipath complex gains. The problem is to estimate $\boldsymbol{\tau}, \boldsymbol{\theta}$ given all observations $\mathbf{x}(1) \dots \mathbf{x}(L)$.

JADE by Partial Relaxation

We generalize the notion of partial relaxation to the JADE case by optimizing the following cost

$$\arg\min_{\theta,\tau,\mathbf{B}} \left\| \boldsymbol{\mathcal{P}}^{\perp}_{[\mathbf{h}(\theta,\tau)\ \mathbf{B}]} \hat{\mathbf{R}} \right\|^2 \tag{2}$$

under suitable constraints. In the above cost, we parameterize only one column in terms of the times and angles of arrivals, whereas the other $q-1$ columns, captured by an term $\mathbf{B}$, are relaxed to have an arbitrary structure. The matrix $\mathbf{B}$ could be seen as an interference term in which $q-1$ sources contribute to, when beamforming at the remaining one source. For example, in the neighbourhood of (θ_1, τ_1), the matrix $\mathbf{B}$ will play the role of an unstructured approximation of the last $q-1$ columns of $\mathbf{H}(\boldsymbol{\theta}, \boldsymbol{\tau})$.

Cramér-Rao Bound for Times and Angles of Arrival

The Fisher-Information Matrix (FIM) measures the quantity of information embedded in random parameters. We find it useful to partition the FIM into smaller block FIMs to separate the nuisance from parameters of interest as follows

$$I_{\beta,\beta} = \begin{bmatrix} I_{\theta,\theta} & I_{\theta,\tau} & I_{\theta,\epsilon} & I_{\theta,\eta} \\ I_{\tau,\theta} & I_{\tau,\tau} & I_{\tau,\epsilon} & I_{\tau,\eta} \\ I_{\epsilon,\theta} & I_{\epsilon,\tau} & I_{\epsilon,\epsilon} & I_{\epsilon,\eta} \\ I_{\eta,\theta} & I_{\eta,\tau} & I_{\eta,\epsilon} & I_{\eta,\eta} \end{bmatrix} \tag{3}$$

$$I_{\beta_i,\beta_j} = \frac{2L}{\sigma^2} \operatorname{Re}\left(\operatorname{tr}\left\{ \boldsymbol{\Pi} \frac{\partial H^{\mathsf{H}}}{\partial \beta_i} \mathcal{P}^{\perp}_{H} \frac{\partial H}{\partial \beta_j} \right\} \right) \tag{4}$$

where $\boldsymbol{\Pi} = PH^{\mathsf{H}}R^{-1}HP$ and $\mathbf{P} = \mathbb{E}\left[\boldsymbol{\gamma}(\ell)\boldsymbol{\gamma}^{\mathsf{H}}(\ell)\right]$ represents the source covariance matrix. Also ϵ, η are nuissance vectors that contain the real and imaginary parts of $\boldsymbol{\gamma}(\ell)$. Using straightforward manipulations, we can say that

$$I_{\theta,\theta} = \frac{2L}{\sigma^2}\boldsymbol{\Pi}_{11}\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\theta} \qquad I_{\tau,\tau} = \frac{2L}{\sigma^2}\boldsymbol{\Pi}_{11}\mathbf{d}_{\tau}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau} \qquad I_{\theta,\tau} = \frac{2L}{\sigma^2}\operatorname{Re}\left(\boldsymbol{\Pi}_{11}\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau}\right) \tag{5}$$

where $\mathbf{d}_{\theta} = \frac{d\mathbf{a}(\theta)}{d\theta} \otimes \mathbf{c}(\tau)$ and $\mathbf{d}_{\tau} = \mathbf{a}(\theta) \otimes \frac{d\mathbf{c}(\tau)}{d\tau}$. Now, taking a look at the $(i,j)^{th}$ entry at the following block matrices, we have

$$[I_{\theta,\epsilon}]_{i,j} = \frac{2L}{\sigma^2}\operatorname{Re}\left(\operatorname{tr}\left\{\boldsymbol{\Pi}\mathbf{e}_1\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{e}_{i+1}\mathbf{e}_{j+1}^{\mathsf{H}}\right\}\right) \qquad [I_{\theta,\eta}]_{i,j} = \frac{2L}{\sigma^2}\operatorname{Re}\left(\operatorname{tr}\left\{j\boldsymbol{\Pi}\mathbf{e}_1\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{e}_{i+1}\mathbf{e}_{j+1}^{\mathsf{H}}\right\}\right) \tag{6}$$

$$[I_{\tau,\epsilon}]_{i,j} = \frac{2L}{\sigma^2}\operatorname{Re}\left(\operatorname{tr}\left\{\boldsymbol{\Pi}\mathbf{e}_1\mathbf{d}_{\tau}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{e}_{i+1}\mathbf{e}_{j+1}^{\mathsf{H}}\right\}\right) \qquad [I_{\tau,\eta}]_{i,j} = \frac{2L}{\sigma^2}\operatorname{Re}\left(\operatorname{tr}\left\{j\boldsymbol{\Pi}\mathbf{e}_1\mathbf{d}_{\tau}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{e}_{i+1}\mathbf{e}_{j+1}^{\mathsf{H}}\right\}\right) \tag{7}$$

In compact matrix form, the above could be expressed as

$$I_{\theta,\epsilon} = \frac{2L}{\sigma^2}\operatorname{Re}\left(\Pi_{21}^{\mathsf{H}} \otimes (\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{E})\right) \qquad I_{\theta,\eta} = \frac{2L}{\sigma^2}\operatorname{Re}\left(j\Pi_{21}^{\mathsf{H}} \otimes (\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{E})\right) \qquad I_{\tau,\epsilon} = \frac{2L}{\sigma^2}\operatorname{Re}\left(\Pi_{21}^{\mathsf{H}} \otimes (\mathbf{d}_{\tau}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{E})\right) \tag{8}$$

$$I_{\tau,\eta} = \frac{2L}{\sigma^2}\operatorname{Re}\left(j\Pi_{21}^{\mathsf{H}} \otimes (\mathbf{d}_{\tau}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{E})\right) \qquad I_{\epsilon,\epsilon} = I_{\eta,\eta} = \frac{2L}{\sigma^2}\operatorname{Re}\left(\Pi_{22} \otimes (\mathbf{E}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{E})\right) \qquad I_{\epsilon,\eta} = \frac{2L}{\sigma^2}\operatorname{Re}\left(j\Pi_{22} \otimes (\mathbf{E}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{E})\right) \tag{9}$$

The Cramer-Rao bound is the inverse of the FIM. In our setting, we can say that

$$C_{\beta\beta} = I_{\beta,\beta}^{-1} = \begin{bmatrix} C_{\theta\theta} & C_{\theta\tau} & C_{\theta\epsilon} & C_{\theta\eta} \\ C_{\tau\theta} & C_{\tau\tau} & C_{\tau\epsilon} & C_{\tau\eta} \\ C_{\epsilon\theta} & C_{\epsilon\tau} & C_{\epsilon\epsilon} & C_{\epsilon\eta} \\ C_{\eta\theta} & C_{\eta\tau} & C_{\eta\epsilon} & C_{\eta\eta} \end{bmatrix} \tag{10}$$

Finally, the CRBs of the parameters of interest are given as

$$C_{\theta\theta} = \frac{\sigma^2}{2\alpha L}\frac{\mathbf{d}_{\tau}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau}}{(\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\theta})(\mathbf{d}_{\tau}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau}) - \operatorname{Re}^2(\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau})} \quad C_{\tau\tau} = \frac{\sigma^2}{2\alpha L}\frac{\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\theta}}{(\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\theta})(\mathbf{d}_{\tau}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau}) - \operatorname{Re}^2(\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau})} \quad C_{\theta\tau} = -\frac{\sigma^2}{2\alpha L}\frac{\operatorname{Re}(\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau})}{(\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\theta})(\mathbf{d}_{\tau}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau}) - \operatorname{Re}^2(\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau})} \tag{11}$$

where $\alpha = \boldsymbol{\Pi}_{11} - \boldsymbol{\Pi}_{21}^{\mathsf{H}}\boldsymbol{\Pi}_{22}^{-1}\boldsymbol{\Pi}_{21}$ is the Schur's complement of $\boldsymbol{\Pi}$ w.r.t its block matrix $\boldsymbol{\Pi}_{22}$. Notice that when the cross-term $\operatorname{Re}(\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau}) = 0$, the CRB on θ.

Properties and Results

In this section, we discuss some useful insights related to the derived CRBs. First and foremost, we note that the cross-correlation CRB term, $C_{\theta\tau}$ vanishes in the large regime (either in space or frequency). This is easily seen as the term $\operatorname{Re}(\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau}) \longrightarrow 0$ for large M given a fixed N, or vice versa. Even more, this regime allows us to lower bound the CRBs on θ and τ, i.e. $C_{\theta\theta} > C_{\theta\theta}^*$ and $C_{\tau\tau} > C_{\tau\tau}^*$ where $C_{\theta\theta}^* = \frac{\sigma^2}{2\alpha L}\left(\mathbf{d}_{\theta}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\theta}\right)^{-1}$ and $C_{\tau\tau}^* = \frac{\sigma^2}{2\alpha L}\left(\mathbf{d}_{\tau}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{d}_{\tau}\right)^{-1}$ Secondly, it is worth noting that the traditional CRB of the Joint Angle and Delay Estimation problem serves as a lower bound on $C_{\theta\theta}^*$ and $C_{\tau\tau}^*$, i.e. $C_{\theta\theta} > C_{\theta\theta}^* > C_{\theta\theta}^{\mathsf{trad}}$ and $C_{\tau\tau} > C_{\tau\tau}^* > C_{\tau\tau}^{\mathsf{trad}}$ where $C_{\theta\theta}^{\mathsf{trad}}, C_{\tau\tau}^{\mathsf{trad}}$ are extracted from the following quantity

$$\mathsf{CRB}(\theta, \tau) = \frac{\bar{\sigma}}{2}\sum_{\ell=1}^{L}\operatorname{Re}(\boldsymbol{B}_{\ell}^{\mathsf{H}}\mathbf{F}^{\mathsf{H}}\mathcal{P}^{\perp}_{H}\mathbf{F}\boldsymbol{B}_{\ell}) \tag{12}$$

where $\bar{\sigma}$ is the estimation noise variance and $\mathbf{F} = \begin{bmatrix}\mathbf{d}_{\theta} & \mathbf{d}_{\tau}\end{bmatrix}$ and $\boldsymbol{B}_{\ell} = \mathbf{I}_2 \otimes \operatorname{diag}\{\boldsymbol{\gamma}(\ell)\}$. Note that $C_{\theta\theta}^{\mathsf{trad}}, C_{\tau\tau}^{\mathsf{trad}}$ is attained only for large N or M and at high SNR for uncorrelated sources, i.e. when $\boldsymbol{\Gamma}$ is diagonal.

Simulations

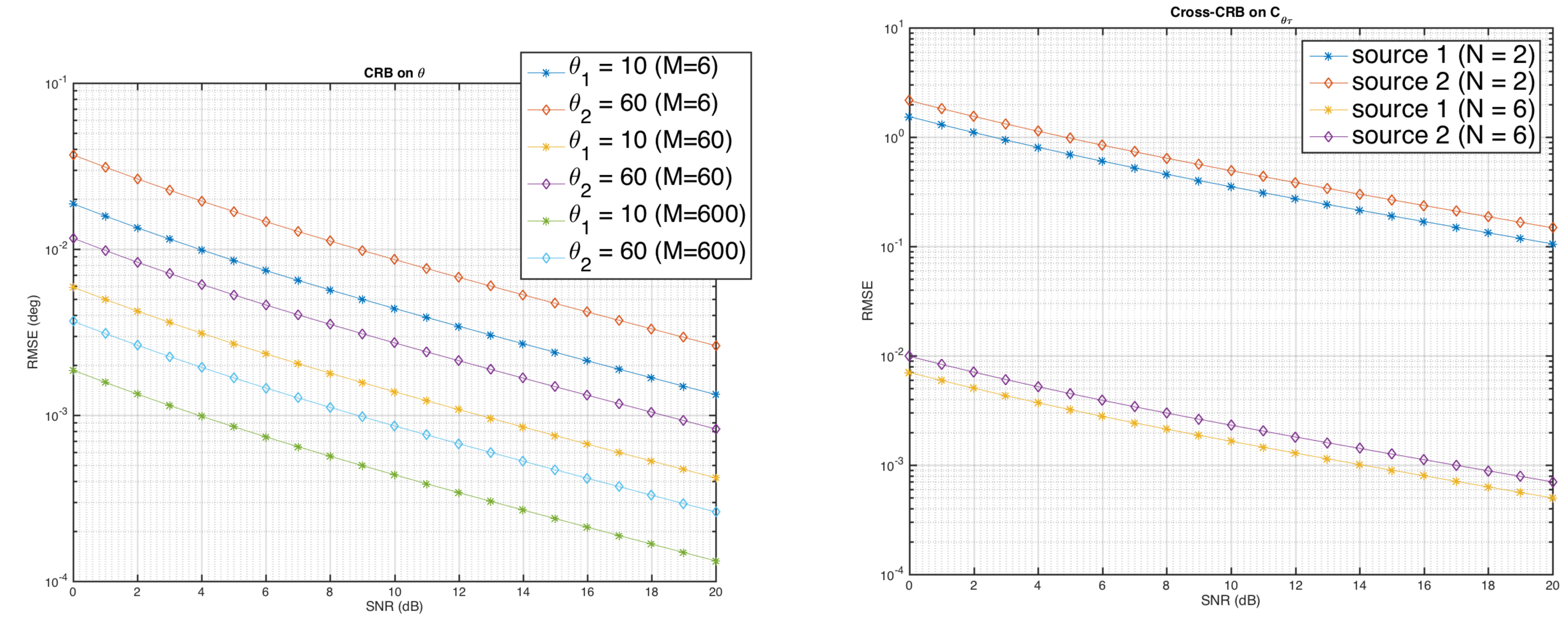

Figure 1: CRB $C_{\theta\theta}$ (left) and $C_{\theta\tau}$ (right)

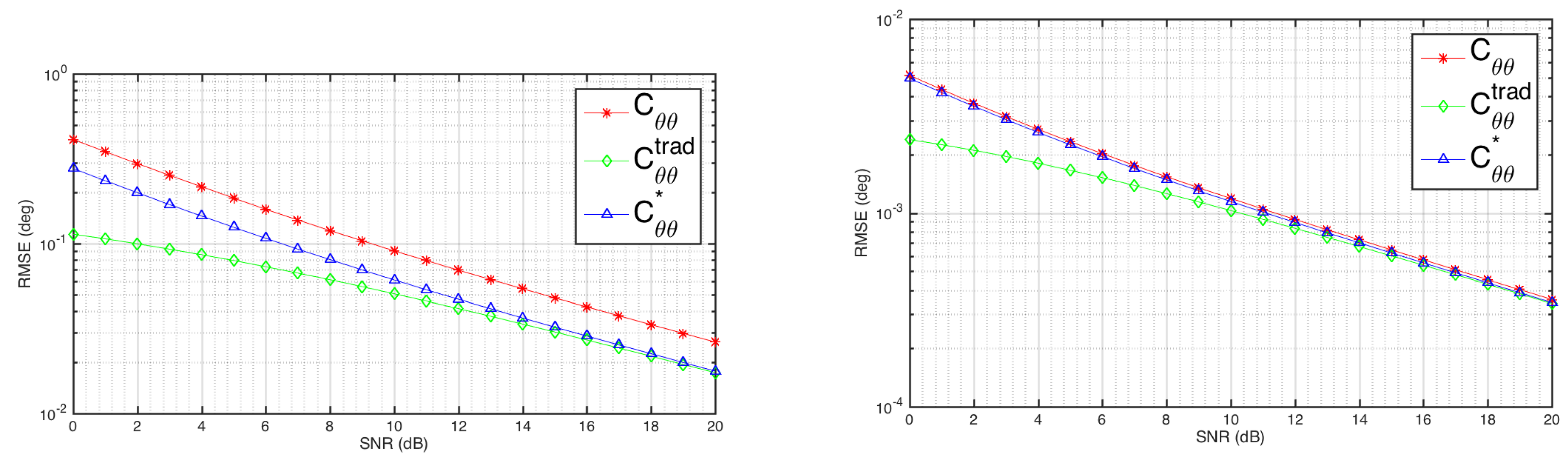

Figure 2: Traditional JADE CRB vs PR-JADE CRB for ($M = 5, N = 2$ on the left) and ($M = 100, N = 2$ on the right)

Conclusions

- We have extended the CRB of the partial relaxation framework to the case of joint angle and delay estimation (JADE).
- The exact closed form expressions of the Fisher Information Matrix (FIM), as well as the Cramér-rao Bound (CRB) are derived.
- Some interesting asymptotic results are presented, which reveals desired properties and results of the partial relaxation framework, in the context of JADE.

References

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