JOINT ANGLE AND DELAY ESTIMATION (JADE) BY PARTIAL RELAXATION

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ABSTRACT

This paper addresses the Joint Angle and Delay estimation problem of a received multi-carrier frame through multiple antennas. In particular, each source is parameterized through its complex gain, Time-of-arrival (ToA) and Angle-of-arrival (AoA). We develop three novel JADE methods in a partially relaxed model, where one relaxes the interference zone, while focusing on a certain direction at a given time. Computer simulations are given with comparison to the Cramér-Rao Bound.

Index terms— JADE, Partial Relaxation, Maximum Likelihood, Weighted Subspace Fitting, Covariance Fitting

1. INTRODUCTION

Node positioning in a wireless system requires the gathering of location information from radio signals traveling between the target node and one, or multiple, reference anchors. The JADE (Joint Angle and Delay Estimation) approach measures delays/angles between an intended node and anchors to estimate the location of the former, with the help of the position of the latter. JADE-based location estimation offers a wealth of advantages for both communications[1] and radar. Many techniques were proposed for this purpose, such as MUSIC [2] and ESPRIT [3]. Asymptotic studies were conducted on the variances of these method, in which they attain the CRB with uncorrelated sources and high SNR (or high number of antennas) [4]. The Joint Angle and Delay Estimation (JADE) [11] parameterizes each source through its AoA and its Time-of-Arrival (ToA). Even though more parameters are to be estimated, this allows to resolve more sources [6]. As a result, many methods were developed to solve the JADE problem, such as shift-invariant ones in [7], single-snapshot methods [8], mutual coupling agnostic methods [9], etc.

Recently, the partial relaxation (PR) framework has been introduced as a novel framework for the Angle-of-Arrival (AoA) problem in [5]. To be more precise, the maximum likelihood cost function is partially relaxed to include one parameterized source, through its AoA and other nonparametric sources. This relaxation results in new cost functions that reduce multi-source searches to single-source searches and are able to resolve AoAs in a reliable and computationally efficient manner.

This paper extends the cost functions presented in [5] to the JADE problem. Specifically, we derive three JADE estimators, in the context of partial relaxation, namely: (i) Deterministic Maximum Likelihood (ii) Weighted Subspace Fitting (iii) Covariance Fitting. Simulations show the closeness of the Root Means Square Error (RMSE) of the proposed JADE estimators to the Cramér-Rao Bound (CRB) [12] of the partially relaxed model.

This paper is organized as follows: Section 2 presents the system model. Section 3 introduces the partial relaxation to the JADE problem and presents the three partially relaxed JADE estimators, i.e. Deterministic Maximum Likelihood, Weighted Subspace Fitting and the Covariance Fitting. Computer simulations are given in Section 4 to show the potential of the proposed estimators, when compared to the partially relaxed Cramér-Rao Bound (CRB). The paper is concluded in Section 5. **Notations:** Upper-case and lower-case boldface letters denote matrices and vectors, respectively. $\otimes$ is the Kronecker product. $(.)^{\mathsf{T}}$ and $(.)^{\mathsf{H}}$ denote the transpose and Hermitian transpose operators. $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ denote the real and imaginary parts of z, respectively. Furthermore, $\lambda_k(\mathbf{A})$ denotes the k^{th} largest eigenvalue of matrix $\mathbf{A}$. The operator $\|\|$ is the Frobenius norm and tr represents the trace.

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2. SYSTEM MODEL

Consider an OFDM symbol consisting of M subcarriers, and centered at a carrier frequency f_c (e.g. $2.4/5$ GHz) that has been transmitted through a rich multipath channel of q taps, and received via an array of N antennas. If we parametrize the i^{th} multipath component by a Direction-of-Arrival (DoA) θ_i and Time-of-Arrival (ToA) τ_i, then the ℓ^{th} received OFDM symbol could be expressed as

$$\mathbf{x}(\ell) = \mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau})\boldsymbol{\gamma}(\ell) + \mathbf{n}(\ell) \tag{1}$$

where $\mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau}) \in \mathbb{C}^{MN\times q}$ and $\boldsymbol{\gamma}(\ell) \in \mathbb{C}^{q\times 1}$ are defined as

$$\mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau}) = \begin{bmatrix} h(\theta_1,\tau_1) & \dots & h(\theta_q,\tau_q) \end{bmatrix} \tag{2}$$

$$\boldsymbol{\gamma}(\ell) = \begin{bmatrix} \gamma_1(\ell) & \dots & \gamma_q(\ell) \end{bmatrix} \tag{3}$$

and $\mathbf{n}(\ell) \in \mathbb{C}^{MN\times 1}$ is additive circularly complex noise, i.e. $\mathbf{n}(\ell) \sim \mathcal{N}(\mathbf{0},\sigma^2\mathbf{I}_{MN})$. The vector $\mathbf{x}(\ell)$ is indexed as follows

$$\mathbf{x}(\ell) = \begin{bmatrix} \mathbf{x}_1^\top(\ell) & \dots & \mathbf{x}_N^\top(\ell) \end{bmatrix}^\top \tag{4}$$

and

$$\mathbf{x}_n(\ell) = \begin{bmatrix} x_{n,1}(\ell) & \dots & x_{n,M}(\ell) \end{bmatrix}^\top \tag{5}$$

where $x_{n,m}(\ell)$ represents the data on the m^{th} subcarrier received by the n^{th} antenna in the ℓ^{th} frame. The problem is to estimate $\boldsymbol{\tau},\boldsymbol{\theta}$ given all observations $\mathbf{x}(1)\dots\mathbf{x}(L)$.

3. JADE BY PARTIAL RELAXATION

The optimal criterion in the presence of white Gaussian noise $\{\mathbf{n}(\ell)\}_{\ell=1\dots L}$ is to solve the Deterministic Maximum Likelihood (DML) cost, i.e.

$$(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\tau}},\hat{\mathbf{G}}) = \underset{\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{G}}{\arg\min} \left\|\mathbf{X} - \mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau})\mathbf{G}\right\|^2 \tag{6}$$

where

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}(1) & \dots & \mathbf{x}(L) \end{bmatrix} \tag{7}$$

$$\mathbf{G} = \begin{bmatrix} \boldsymbol{\gamma}(1) & \dots & \boldsymbol{\gamma}(L) \end{bmatrix} \tag{8}$$

Traditional beamformers aim at minimizing the above assuming one source at a time in $\mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau})$. This leads to a simple and fast implementation of the final criterion that aims at finding $\boldsymbol{\tau},\boldsymbol{\theta}$, through peak finding, such as MUSIC. However, this is suboptimal due to existence of multiple sources, when focusing on one. Said differently, equation (6) leads to the following cost

$$(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\tau}}) = \underset{\boldsymbol{\theta},\boldsymbol{\tau}}{\arg\min} \left\|\boldsymbol{\mathcal{P}}^\perp_{\mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau})}\hat{\mathbf{R}}\right\|^2 \tag{9}$$

where $\boldsymbol{\mathcal{P}}^\perp_{\mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau})}$ is the orthogonal projector matrix onto the space spanned by the columns of $\mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau})$. In addition, $\hat{\mathbf{R}} = \mathbf{X}\mathbf{X}^{\mathrm{H}}$ is the empirical covariance matrix. Now, it is clear that one should jointly focus on all AoAs and ToAs when solving the JADE problem. Unfortunately, the cost in (9) is highly complex and may not be implementable in most applications. One might resort to suboptimal techniques such as MUSIC/ESPRIT. Another alternative is to "partially relax" the parametric structure of the interfering sources when looking in direction θ at time τ, i.e.

$$\underset{\theta,\tau,\mathbf{B}}{\arg\min} \left\|\boldsymbol{\mathcal{P}}^\perp_{[\mathbf{h}(\theta,\tau)\ \mathbf{B}]}\hat{\mathbf{R}}\right\|^2 \tag{10}$$

In the above cost, we parameterize only one column in terms of the times and angles of arrivals, whereas the other $q-1$ columns, captured by an term $\mathbf{B}$, are relaxed to have an arbitrary structure. The matrix $\mathbf{B}$ could be seen as an interference term in which $q-1$ sources contribute to, when beamforming at the remaining one source. For example, in the neighbourhood of (θ_1,τ_1), the matrix $\mathbf{B}$ will play the role of an unstructured approximation of the last $q-1$ columns of $\mathbf{H}(\boldsymbol{\theta},\boldsymbol{\tau})$. To this extent, using matrix-projector decomposition, the relaxed DML above is now

$$\underset{\theta,\tau,\mathbf{B}}{\arg\min} \left(\mathrm{tr}\left\{\boldsymbol{\mathcal{P}}^\perp_{\mathbf{h}(\theta,\tau)}\hat{\mathbf{R}}\right\} - \mathrm{tr}\left\{\boldsymbol{\mathcal{P}}_{\tilde{\mathbf{B}}(\theta,\tau)}\hat{\mathbf{R}}\right\}\right) \tag{11}$$

where $\tilde{\mathbf{B}}(\theta,\tau) = \boldsymbol{\mathcal{P}}^\perp_{\mathbf{h}(\theta,\tau)}\mathbf{B}$. First, we optimize with respect to the interference term $\mathbf{B}$ by representing the projector matrix in terms of its eigenvalue decomposition, $\boldsymbol{\mathcal{P}}_{\tilde{\mathbf{B}}(\theta,\tau)} = \mathbf{U}\mathbf{U}^H$, where $\mathbf{U}^H\mathbf{U} = \mathbf{I}$. So,

$$\max_{\mathbf{B}} \mathrm{tr}\left\{\boldsymbol{\mathcal{P}}_{\tilde{\mathbf{B}}(\theta,\tau)}\hat{\mathbf{R}}\right\} = \max_{\mathbf{U}|\mathbf{U^H U = I}} \mathrm{tr}\left\{\mathbf{U}^H\boldsymbol{\mathcal{P}}^\perp_{\mathbf{h}(\theta,\tau)}\hat{\mathbf{R}}\mathbf{U}\right\} \tag{12}$$

where is optimal value is obtained by choosing the columns of $\mathbf{U}$ to be the eigenvectors corresponding to the maximal eigenvalues, viz.

$$\max_{\mathbf{B}} \mathrm{tr}\left\{\boldsymbol{\mathcal{P}}_{\tilde{\mathbf{B}}(\theta,\tau)}\hat{\mathbf{R}}\right\} = \sum_{k=1}^{q-1}\lambda_k\left(\boldsymbol{\mathcal{P}}^\perp_{\mathbf{h}(\theta,\tau)}\hat{\mathbf{R}}\right) \tag{13}$$

which when replaced back in (11), gives us the following criterion which when replaced back in (11), gives us the following criterion

$$\underset{\theta,\tau}{\arg\max}\, f_{\mathsf{DML}}(\theta,\tau) \tag{14}$$

where

$$f_{\mathsf{DML}}(\theta,\tau) = \frac{1}{\sum_{k=q}^{NM}\lambda_k\left(\boldsymbol{\mathcal{P}}^\perp_{\mathbf{h}(\theta,\tau)}\hat{\mathbf{R}}\right)} \tag{15}$$

where $\lambda_k(\mathbf{X})$ denotes the k^{th} largest eigenvalue of $\mathbf{X}$. Using a similar procedure, one could propose a weighted subspace fitting method as

$$f_{\mathsf{WSF}}(\theta,\tau) = \frac{1}{\sum_{k=q}^{NM}\lambda_k\left(\boldsymbol{\mathcal{P}}^\perp_{\mathbf{h}(\theta,\tau)}\hat{\mathbf{U}}_s\mathbf{W}\hat{\mathbf{U}}_s^{\mathsf{H}}\right)} \tag{16}$$

In addition to the previous estimators, a covariance fitting one could be derived for the JADE problem taking into account the partial relaxation modelling. Focusing on the k^{th} path,

$$\mathbf{X} = \mathbf{h}(\theta_k, \tau_k)\boldsymbol{\gamma}_k^{\mathsf{T}} + \mathbf{J} + \mathbf{N} \tag{17}$$

where $\boldsymbol{\gamma}_k^{\mathsf{T}}$ is the k^{th} row of $\mathbf{G}$. In addition, the matrix $\mathbf{J}$ is obtained by denoising the rank-1 contribution of the k^{th} path from the signal part, i.e.

$$\mathbf{J} = \mathbf{H}(\boldsymbol{\theta}, \boldsymbol{\tau})\mathbf{G} - \mathbf{h}(\theta_k, \tau_k)\boldsymbol{\gamma}_k^{\mathsf{T}} \tag{18}$$

Assuming $\boldsymbol{\gamma}_k^{\mathsf{T}}$ is uncorrelated from the rows of $\mathbf{J}$, one could say that the "noiseless" covariance matrix of $\mathbf{X}$ is

$$\mathbf{R} = \sigma_k^2 \mathbf{h}(\theta_k, \tau_k)\mathbf{h}^{\mathsf{H}}(\theta_k, \tau_k) + \mathbf{J}\mathbf{J}^{\mathsf{H}} \tag{19}$$

Therefore, covariance fitting aims at minimizing noisy observations on the above as

$$\begin{aligned} \underset{\theta_k, \tau_k, \sigma_k^2, \mathbf{J}}{\text{minimize}} \quad & \left\| \hat{\mathbf{R}} - \sigma_k^2 \mathbf{h}(\theta_k, \tau_k)\mathbf{h}^{\mathsf{H}}(\theta_k, \tau_k) - \mathbf{J}\mathbf{J}^{\mathsf{H}} \right\|^2 \\ \text{subject to} \quad & \hat{\mathbf{R}} - \sigma_k^2 \mathbf{h}(\theta, \tau)\mathbf{h}^{\mathsf{H}}(\theta, \tau) \succcurlyeq \mathbf{0} \end{aligned} \tag{20}$$

Fixing $\theta_k, \tau_k, \sigma_k^2$ and minimizing w.r.t interferer $\mathbf{J}$, we get

$$\begin{aligned} \underset{\sigma_k^2}{\text{minimize}} \quad & \sum_{k=q}^{NM} \lambda_k^2 \left(\hat{\mathbf{R}} - \sigma_k^2 \mathbf{h}(\theta, \tau)\mathbf{h}^{\mathsf{H}}(\theta, \tau) \right) \\ \text{subject to} \quad & \hat{\mathbf{R}} - \sigma_k^2 \mathbf{h}(\theta, \tau)\mathbf{h}^{\mathsf{H}}(\theta, \tau) \succcurlyeq \mathbf{0} \end{aligned} \tag{21}$$

The optimal value of σ_k^2 is attained when the denoised matrix $\hat{\mathbf{R}} - \sigma_k^2 \mathbf{h}(\theta, \tau)\mathbf{h}^{\mathsf{H}}(\theta, \tau)$ obtains at least a zero eigenvalue, hence the covariance fitting cost function is

$$f_{\mathsf{CF}}(\theta, \tau) = \frac{1}{\sum_{k=q}^{NM} \lambda_k^2 \left(\hat{\mathbf{R}} - \frac{\mathbf{h}(\theta,\tau)\mathbf{h}^{\mathsf{H}}(\theta,\tau)}{\mathbf{h}^{\mathsf{H}}(\theta,\tau)\hat{\mathbf{R}}^{-1}\mathbf{h}(\theta,\tau)} \right)} \tag{22}$$

4. COMPUTER SIMULATIONS

In this section, computer simulations are presented to visualize the behaviour of the partially relaxed CRB in different scenarios. The performance of the proposed JADE estimators are compared with the JADE-MUSIC algorithm [7]. To this extent, the RMSE of the AoAs and ToAs are computed as follows

$$\mathsf{RMSE}(\theta_k) = \sqrt{\frac{\sum_{k=1}^{q} \sum_{p=1}^{P} (\theta_k - \hat{\theta}_k^{(p)})^2}{M}} \tag{23}$$

$$\mathsf{RMSE}(\tau_k) = \sqrt{\frac{\sum_{k=1}^{q} \sum_{p=1}^{P} (\tau_k - \hat{\tau}_k^{(p)})^2}{M}} \tag{24}$$

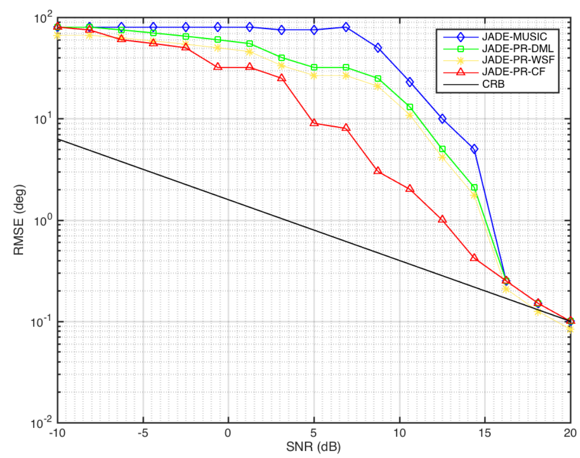

Figure 1: The RMSE of AoA estimates $\hat{\theta}_k$ per method vs SNR and the CRB.

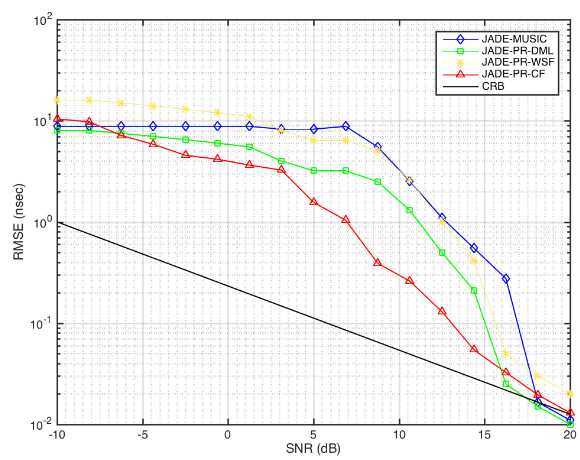

Figure 2: The RMSE of ToA estimates $\hat{\tau}_k$ per method vs SNR and the CRB.

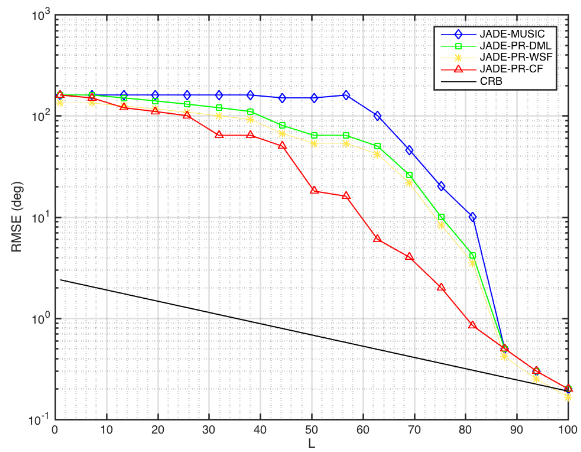

Figure 3: The RMSE of AoA estimates $\hat{\theta}_k$ per method vs Number of Snapshots and the CRB.

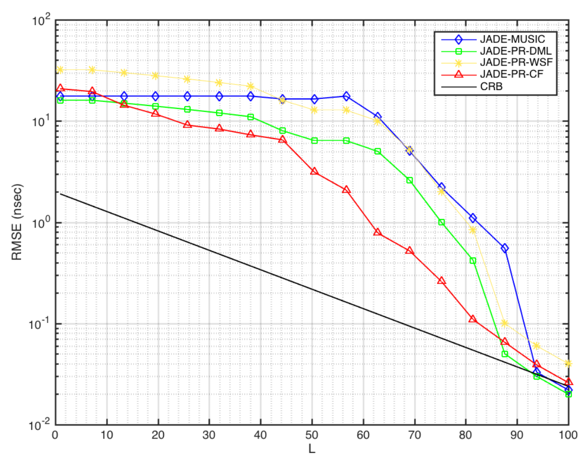

Figure 4: The RMSE of ToA estimates $\hat{\tau}_k$ per method vs Number of Snapshots and the CRB.

where P is the number of Monte-Carlo trials and $\hat{\theta}_k^{(p)}, \hat{\tau}_k^{(p)}$ are the AoA and ToA estimates of the k^{th} source obtained at the p^{th} Monte-Carlo trial.

In all simulations and otherwise stated, we fix the following parameters: $q = 2$ sources impinging at $\theta_1 = 6°$ and $\theta_2 = 66°$ and with delays $\tau_1 = 5$ nsec and $\tau_2 = 10$ nsec. The CRB used is derived in [12], i.e.

$$C_{\theta\theta} = \frac{\sigma^2}{2\alpha L} \frac{\mathbf{d}_\tau^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau}{(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\theta)(\mathbf{d}_\tau^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau) - \mathrm{Re}^2(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau)} \tag{25}$$

$$C_{\tau\tau} = \frac{\sigma^2}{2\alpha L} \frac{\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\theta}{(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\theta)(\mathbf{d}_\tau^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau) - \mathrm{Re}^2(\mathbf{d}_\theta^{\mathsf{H}} \mathcal{P}_{\mathbf{H}}^{\perp} \mathbf{d}_\tau)} \tag{26}$$

where $\alpha = \Pi_{11} - \mathbf{\Pi}_{21}^{\mathsf{H}} \mathbf{\Pi}_{22}^{-1} \mathbf{\Pi}_{21}$ is the Schur complement of the matrix $\mathbf{\Pi}$ with respect to matrix $\mathbf{\Pi}_{22}$ with

$$\mathbf{\Pi} = \left[\begin{array}{c|c} \Pi_{11} & \mathbf{\Pi}_{21}^{\mathsf{H}} \\ \hline \mathbf{\Pi}_{21} & \mathbf{\Pi}_{22} \end{array}\right] = \mathbf{P}\mathbf{H}^{\mathsf{H}}(\mathbf{H}\mathbf{P}\mathbf{H}^{\mathsf{H}} + \sigma^2 \mathbf{I})^{-1}\mathbf{H}\mathbf{P} \tag{27}$$

and $\mathbf{P}$ is the source covariance matrix, which is set throughout the simulation as

$$\mathbf{P} = \begin{bmatrix} 1 & 0.2e^{-j\frac{\pi}{6}} \\ 0.2e^{j\frac{\pi}{6}} & 1 \end{bmatrix} \tag{28}$$

Furthermore, the matrix $\mathbf{H}$ is obtained by the joint responses of the array/subcarriers with respect to the θ_1, τ_1 and θ_2, τ_2 as follows

$$\mathbf{H} = \begin{bmatrix} \mathbf{a}(\theta_1) \otimes \mathbf{c}(\tau_1) & \mathbf{a}(\theta_2) \otimes \mathbf{c}(\tau_2) \end{bmatrix} \tag{29}$$

We carry out simulations using a uniform linear array of $N = 2$ antennas and $M = 32$ subcarriers. Also we have that $\mathcal{P}_{\mathbf{H}}^{\perp}$ is the projector spanning the null-space of the columns of $\mathbf{H}$. The vectors $\mathbf{d}_\theta$ and $\mathbf{d}_\tau$ are the derivatives defined as

$$\mathbf{d}_\theta = \frac{d\mathbf{a}(\theta)}{d\theta} \otimes \mathbf{c}(\tau) \tag{30}$$

$$\mathbf{d}_\tau = \mathbf{a}(\theta) \otimes \frac{d\mathbf{c}(\tau)}{d\tau} \tag{31}$$

As we can see in Fig. 1 and Fig. 2, we vary the SNR from -10 to 20dB to study the performance of the proposed methods as a function of SNR. In Fig. 3 and Fig.4, we fix t the SNR to 30dB and vary the number of snapshots from 10 to 100. All methods seem to align with the CRB at high SNR or with large number of snapshots. In the low SNR and number-of-snapshots regime, we observe that the covariance fitting method achieves the highest performance. For example, according to Fig. 1, and at 5dB SNR, we see around 50° RMSE improvement with respect to WSF and the DML and ~ 80° RMSE improvement with respect to

JADE-MUSIC. In terms of time estimates, we observe that at the same SNR, a 1.5 nsec RMSE improvement with respect to DML and ~ 6 nsec with respect to WSF and JADE-MUSIC. Similar improvements appear in Fig.3 and Fig.4.

5. CONCLUSIONS

In this paper, we have introduced three JADE estimators to the partially relaxed model, namely a deterministic Maximum Likelihood, a Weighted Subspace Fitting, and a Covariance Fitting estimator. Simulation results demonstrate the Mean-Squared-Error convergence towards the Cramér-Rao Bound of each of these methods, either in SNR or in number of snapshots. Future work will address better

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