DETECTION OF THE NUMBER OF SUPERIMPOSED SIGNALS USING MODIFIED MDL CRITERION: A RANDOM MATRIX APPROACH

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ABSTRACT

The problem of estimating the number of superimposed signals using noisy observations from N antennas is addressed. In particular, we are interested in the case where a low number of snapshots $L = \mathcal{O}(N)$ is available. We focus on the Minimum Description Length (MDL) estimator, which is revised herein. Furthermore, we propose a modified MDL estimator, with the help of random matrix tools, which improves the estimation of the number of sources. Simulation results demonstrate the potential of the modified MDL estimator over the traditional one, in the case where $L = \mathcal{O}(N)$.

Index Terms— Detection, Number of Signals, MDL, G-estimation, Random Matrix Theory

1. INTRODUCTION

The problem of model selection from the observed data, or snapshots, is a fundamental problem in diverse areas of signal processing, such as system identification [1] and array signal processing [2]. In particular, the detection of number of sources could be classified as a model selection problem. Furthermore, it is well known that the performance of parametric modelling techniques, such as those found in [3], depends on the knowledge of the number of signals.

The first and most basic techniques developed to determine the number of sources were based on hypothesis testing [4]. However, they require a pre-defined threshold to accept, or reject a certain hypothesis. Then, methods based on information theoretic criteria, such as *Akaike's Information Criteria* (AIC) introduced by Akaike in [5] and *Minimum Description Length* (MDL) introduced by Rissanen [6] and Schwartz [7], were used for the detection of number of signals in [8]. Similar criteria such as the Benjamin-Hochberg procedure [9, 10] were used in [11] to detect the number of signals. Other methods, such as bootstrap [12], were developed to tackle the same issue.

All detection techniques just mentioned depend, merely, on the sample eigenvalues of the data covariance matrix. In practice, this data covariance matrix is not the true one, but an estimate of it. It is referred to as the sample covariance matrix. Usually, the sample covariance matrix is computed through L snapshots from N measurements, or sensors. It is well known that the sample eigenvalues are L-consistent [13], but (N, L)-inconsistent [14], estimates of the true ones. In other words, when $L \longrightarrow \infty$ at finite N ($L \gg N$), the sample eigenvalues converge towards the true ones. However, as $(N, L) \longrightarrow \infty$ at the same rate ($0 < c = \frac{N}{L} < \infty$), the sample eigenvalues do not converge towards the true ones.

Random matrix theory is a branch of statistics that is devoted to the study of the asymptotic behavior of the eigenvalues and eigenvectors of some random matrix models when the dimensions of the matrices increase without bound at the same rate [16]. It should be noted that dimensions, in our case, are N and L. One might think that these dimensions tend towards a large number since both quantities are increasing without bound, but this is not the only case. The important factor here is the ratio of both quantities, i.e. $c = \frac{N}{L}$ [17].

The motivation of this paper is to observe the performance of the traditional MDL criterion when the quantity c is not neglected, and to propose a modified MDL criterion using (N, L)-consistent estimates of the eigenvalues of the data covariance matrix, such as the ones derived in [15]. Indeed, other modified estimates of the eigenvalues that are (N, L)-consistent are discussed in [18, 19]. Prior work has been done on using random matrix theory for the detection problem, such as [20–23]. However, their main focus was on the hypothesis testing problem.

This paper is divided as follows: Section 2 present the system model followed by a recap of the MDL criterion in Section 3. The modified MDL estimator is presented in Section 4 with simulation results in Section 5. We conclude the

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paper in Section 6.

Notations: Upper-case and lower-case boldface letters denote matrices and vectors, respectively. $(.)^T$ and $(.)^{\mathrm{H}}$ represent the transpose and the transpose-conjugate operators. $E\{.\}$ is the statistical expectation. The matrix $\mathbf{I}_N$ is the identity matrix of dimensions $N \times N$. The operators "det$\mathbf{X}$" and "tr$\mathbf{X}$" denote the determinant and trace of a square matrix $\mathbf{X}$, respectively.

2. SYSTEM MODEL

Assume a planar arbitrary array of N antennas. Furthermore, consider $q < N$ narrowband sources attacking the array from different angles, i.e. $\boldsymbol{\Theta} = [\theta_1 \dots \theta_q]$. Collecting L time snapshots and following [2], we can write

$$\mathbf{X} = [\mathbf{x}(t_1) \dots \mathbf{x}(t_{\mathrm{L}})] = \mathbf{AS} + \mathbf{W} \quad (1)$$

where $\mathbf{X} \in \mathbb{C}^{\mathrm{N}\times\mathrm{L}}$ is the data matrix with l^{th} time snapshot, $\mathbf{x}(t_l)$, stacked in its l^{th} column. The matrix $\mathbf{S} = [\mathbf{s}(t_1) \dots \mathbf{s}(t_{\mathrm{L}})] \in \mathbb{C}^{\mathrm{q}\times\mathrm{L}}$ is the source matrix, with $\mathbf{s}(t)$ written as

$$\mathbf{s}(t) = [s_1(t) \dots s_{\mathrm{q}}(t)]^{\mathrm{T}} \quad (2)$$

The steering matrix $\mathbf{A} \in \mathbb{C}^{\mathrm{N}\times\mathrm{q}}$ is composed of q steering vectors, i.e. $\mathbf{A} = [\mathbf{a}(\theta_1) \dots \mathbf{a}(\theta_q)]$. Each vector $\mathbf{a}(\theta_i)$ is the response of the array to a source impinging the array from direction θ_i. The form of $\mathbf{a}(\theta_i)$ is given as

$$\mathbf{a}(\theta_i) = \begin{bmatrix} e^{-jk(\bar{x}_1 sin\theta_i + \bar{y}_1 cos\theta_i)} \\ \vdots \\ e^{-jk(\bar{x}_N sin\theta_i + \bar{y}_N cos\theta_i)} \end{bmatrix} \quad (3)$$

where $k = \frac{2\pi}{\lambda}$ is the wavenumber, and λ is the wavelength. The position of the n^{th} antenna is $(\bar{x}_n, \bar{y}_n)$ in xy-plane. The matrix $\mathbf{W} \in \mathbb{C}^{\mathrm{N}\times\mathrm{L}}$ is background noise. Now, we are ready to address our *detection* problem:

"Given the available snapshots $\mathbf{X}$, estimate the number of source signals, i.e. q."

Before moving on, we assume the following:

(A.1) The matrix of spatial signatures, i.e. $\mathbf{A}$, is full column rank. This is valid when $q \leq N$ and all angles of arrival are distinct, i.e. $\theta_i \neq \theta_j$ for all $i \neq j$.

(A.2) The sources are assumed to be non-coherent, i.e. $\mathbf{R}_{ss} = E\{\mathbf{s}(t)\mathbf{s}^{\mathrm{H}}(t)\}$ is full rank.

(A.3) The noise is modelled as complex Gaussian vectors, i.i.d over time, with zero-mean and covariance $\sigma^2\mathbf{I}_{\mathrm{N}}$. Also, the noise is independent from the signal.

Under assumption **(A.3)**, the true covariance matrix of the received signal could be written as

$$\mathbf{R}_{xx} = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^{\mathrm{H}} + \sigma^2\mathbf{I}_{\mathrm{N}} \quad (4)$$

Let $l_1 \geq l_2 \geq \dots \geq l_N$ denote the eigenvalues of $\mathbf{R}_{xx}$. Then, under assumptions **(A.1)** to **(A.3)**, the smallest $N - q$ eigenvalues of $\mathbf{R}_{xx}$ are all equal, i.e.

$$l_{q+1} = \dots = l_N = \sigma^2 \quad (5)$$

We also consider that the q largest eigenvalues are distinct, i.e. $l_1 > l_2 > \dots > l_q$. The most straightforward way in determining the number of signals is based on the multiplicity of the smallest eigenvalues of $\mathbf{R}_{xx}$ as done in the MUSIC algorithm [24]. However, in practical scenarios, we only have access to the sample eigenvalues and not the true ones, which makes it more difficult to distinguish the largest q eigenvalues from the smallest $N - q$ ones, especially at low SNR or low number of snapshots.

In the next section, we revise the MDL principle applied for estimating the number of sources.

3. ESTIMATING NUMBER OF SIGNALS USING MDL: A RECAP

As stated earlier, the problem of estimating the number of incoming signals could be seen as a model selection one, i.e. finding the model that best fits the data $\mathbf{X}$. More specifically, the problem is to select one of the N following models:

$$\mathbf{R}_{xx}^{(k)} = \sum_{i=1}^{k} (\lambda_i - \sigma^2)\mathbf{v}_i\mathbf{v}_i^{\mathrm{H}} + \sigma^2\mathbf{I}_{\mathrm{N}}, \qquad k = 0 \dots N-1 \quad (6)$$

where $\mathbf{v}_i$ is the eigenvector corresponding to the eigenvalue λ_i of $\mathbf{R}_{xx}^{(k)}$. Denoting $\Theta^{(k)}$ the vector to be estimated, then

$$\Theta^{(k)} = [\lambda_1, \dots, \lambda_k, \sigma^2, \mathbf{v}_1^T, \dots, \mathbf{v}_k^T] \quad (7)$$

Due to **(A.3)**, the likelihood function is as follows

$$f(\mathbf{X}|\Theta^{(k)}) = \prod_{i=1}^{L} \frac{1}{\pi^{\mathrm{N}} \det\mathbf{R}_{xx}^{(k)}} \exp\{-\mathbf{x}(t_i)^{\mathrm{H}}[\mathbf{R}_{xx}^{(k)}]^{-1}\mathbf{x}(t_i)\} \quad (8)$$

The log-likelihood function, with omitted terms that do not depend on $\Theta^{(k)}$, becomes

$$L(\Theta^{(k)}) = -L \log \det\{\mathbf{R}_{xx}^{(k)}\} - \mathrm{tr}\{[\mathbf{R}_{xx}^{(k))}]^{-1}\hat{\mathbf{R}}\} \quad (9)$$

where $\hat{\mathbf{R}}$ is the sample covariance matrix computed by

$$\hat{\mathbf{R}} = \frac{1}{L}\mathbf{X}\mathbf{X}^{\mathrm{H}} \quad (10)$$

Maximising (9) gives the maximum likelihood estimates of $\Theta^{(k)}$. As in [13], these estimates are

$$\hat{\lambda}_i = \hat{l}_i, \qquad i = 1 \dots k \quad (11a)$$

$$\hat{\sigma}^2 = \frac{1}{N-k}\sum_{i=k+1}^{N}\hat{l}_i \tag{11b}$$

$$\hat{\mathbf{v}}_i = \hat{\mathbf{u}}_i, \qquad i = 1\ldots k \tag{11c}$$

where $\hat{l}_1 \geq \ldots \geq \hat{l}_N$ and $\hat{\mathbf{u}}_1 \ldots \hat{\mathbf{u}}_N$ are the sample eigenvalues and their corresponding eigenvectors, respectively. In other words, they are the eigenvalues and eigenvectors of the matrix $\hat{\mathbf{R}}$. Plugging (11) in (9), we get

$$L(\hat{\Theta}^{(k)}) = \log\left(\frac{\prod_{i=k+1}^{N}\hat{l}_i^{\frac{1}{N-k}}}{\frac{1}{N-k}\sum_{i=k+1}^{N}\hat{l}_i}\right)^{L(N-k)} \tag{12}$$

The model selection based on the MDL principle is the one that minimises the following

$$\mathrm{MDL}(k) = -L(\hat{\Theta}^{(k)}) + \frac{1}{2}\eta\log(L) \tag{13}$$

where η is the number of free adjusted parameters in the parameter vector Θ. Substituting (12) in (13) and plugging in the number of free adjusted parameters η (See [8]), we get

$$\mathrm{MDL}(k) = -\log\left(\frac{\prod_{i=k+1}^{N}\hat{l}_i^{\frac{1}{N-k}}}{\frac{1}{N-k}\sum_{i=k+1}^{N}\hat{l}_i}\right)^{L(N-k)} + \frac{k}{2}(2N-k)\log(L) \tag{14}$$

Therefore, according to the MDL criterion, the number of sources q is the argument k that minimises equation (14).

4. A MODIFIED MDL ESTIMATOR

It has been shown in [14] that the sample eigenvalues $\hat{l}_1 \ldots \hat{l}_N$ extracted from the sample covariance matrix $\hat{\mathbf{R}}$ are (N, L)-inconsistent estimators of the true eigenvalues of the covariance matrix $\mathbf{R}_{xx}$, that is, the sample eigenvalues do not converge towards the true ones as $(N, L) \longrightarrow \infty$ at the same rate ($0 < c = \frac{N}{L} < \infty$). The MDL estimator in (14) depends on the sample eigenvalues of $\hat{\mathbf{R}}$, therefore, it seems natural that the performace of the MDL estimator would perform poorly in the asymptotic regime, i.e. $(N, L) \longrightarrow \infty$ at the same rate ($0 < c = \frac{N}{L} < \infty$). In other words, when insufficient number of snapshots L are available with respect to the number of antennas N in such a way that the ratio $c = \frac{N}{L}$ is not negligible, then the MDL estimator would perform poorly. In this section, we present a modified MDL estimator to cope with this aforementioned issue. The modified MDL estimator is based on using improved estimators of eigenvalues of the covariance matrix $\mathbf{R}_{xx}$, which turn out to be (N, L)-consistent, as shown in [15]. Before presenting the improved estimators of the eigenvalues of the covariance matrix $\mathbf{R}_{xx}$, we proceed as in [15] and pose the following assumptions:

(B.1) The covariance matrix $\mathbf{R}_{xx}$ has uniformly bounded spectral norm for all N, i.e. $\mathrm{Sup_N}\|\mathbf{R}_{xx}\| < \infty$ where $\|.\|$ denotes spectral norm.

(B.2) The sample covariance matrix written as

$$\hat{\mathbf{R}} = \sqrt{\mathbf{R}}_{xx}\mathbf{W}\mathbf{W}^H\sqrt{\mathbf{R}}_{xx} \tag{15}$$

where $\sqrt{\mathbf{R}}_{xx}$ denotes the square root of $\mathbf{R}_{xx}$. The matrix $\mathbf{W}$ is of size $N \times L$ with complex i.i.d. absolutely continous random entries, with each entry having i.i.d. real and imaginary parts of zeros mean, variance $\frac{1}{2L}$, and finite eighth-order moments.

(B.3) For all distinct $q+1$ eigenvalues of $\mathbf{R}_{xx}$, which are $l_1 > \ldots > l_q > l_{q+1} = \sigma^2$, we assume $\mathrm{inf_N}\{\frac{L}{N} - \kappa_N(m)\} > 0$, where $\kappa_N(m)$ is given in (16). In (16), K_i is the multiplicity of the i^{th} largest eigenvalue of $\mathbf{R}_{xx}$, i.e. $K_1 = \ldots = K_q = 1$ and $K_{q+1} = N - q$. Furthermore, $f_1 < f_2 < \ldots < f_q$ are the real-valued roots of equation (17).

$$\kappa_N(m) = \begin{cases} \frac{1}{N}\sum_{i=1}^{q+1}\phi_{i,1}, & \text{if} \quad m = 1 \\ \max\{\sum_{i=1}^{q+1}\phi_{i,m-1}, \sum_{i=1}^{q+1}\phi_{i,m}\}, & \text{if} \quad 1 < m < q+1 \\ \frac{1}{N}\sum_{i=1}^{q+1}\phi_{i,q}, & \text{if} \quad m = q+1 \end{cases} \tag{16a}$$

with

$$\phi_{i,k} = K_i\big(\frac{l_i}{l_i - f_k}\big)^2 \tag{16b}$$

and

$$\frac{1}{N}\sum_{i=1}^{q+1}K_i\frac{l_i^2}{(l_i - f)^3} = 0 \tag{17}$$

4.1. Improved Eigenvalue Estimates

We present a theorem [15] regarding improved eigenvalue estimates, which are not only L-consistent, but also (N, L)-consistent. The theorem is as follows:

Theorem: Under assumptions **(B.1)** to **(B.3)**, the following quantities are strongly (N, L)-consistent estimators of l_j $(j = 1, \ldots, q+1)$.

$$\hat{l}_j^{\text{imp}} = L\big(\hat{l}_j - \mu_j\big), \quad j = 1\ldots q \tag{18a}$$

and

$$\hat{l}_{q+1}^{\text{imp}} = \frac{L}{N-q}\sum_{i=q+1}^{N}\big(\hat{l}_i - \mu_i\big) \tag{18b}$$

where $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_N$ are the real-valued solutions of the following equation in μ:

$$\frac{1}{N}\sum_{i=1}^{N}\frac{\hat{l}_i}{\hat{l}_i-\mu}=\frac{1}{c} \tag{18c}$$

4.2. Improved MDL estimator

With the improved eigenvalue estimates of $\mathbf{R}_{xx}$ in hand from (18), we can modify equations (11a) and (11b) to get

$$\hat{\lambda}_i = L\big(\hat{l}_j-\mu_j\big), \quad j=1\ldots k \tag{19a}$$

$$\hat{\sigma}^2=\frac{L}{N-k}\sum_{i=k+1}^{N}\big(\hat{l}_i-\mu_i\big) \tag{19b}$$

Using these improved estimates in (19), one could easily verify that the improved MDL estimtor finally becomes

$$\mathrm{MDL}^{\mathrm{imp}}(k) = -\log\left(\frac{\prod_{i=k+1}^{N}(\hat{l}_i-\mu_i)^{\frac{1}{N-k}}}{\frac{1}{N-k}\sum_{i=k+1}^{N}(\hat{l}_i-\mu_i)}\right)^{L(N-k)} + \frac{k}{2}(2N-k)\log(L) \tag{20}$$

and, therefore the number of sources are estimated by

$$\hat{q}=\arg\min_{k}\mathrm{MDL}^{\mathrm{imp}}(k) \tag{21}$$

Remark: As $c \longrightarrow 0$, then we have $\hat{l}_i^{\mathrm{imp}} \longrightarrow \hat{l}_i$ for all $i = 1\ldots q+1$. Consequently, one could show that $\mathrm{MDL}^{\mathrm{imp}}(k) \longrightarrow \mathrm{MDL}(k)$ for all k as $c \longrightarrow 0$.

5. SIMULATIONS

In order to show the improvement of the modified MDL estimator, we compare it with the traditional one. We have plotted two histograms that show the percentage of occurance of an estimate of the number of sources $\hat{q}$. Simulations were done under an SNR of 10 dB and in the presence of 6 sources with arbitrary (but sufficiently spaced) angles of arrival. The sources were non-coherent and the array geometry consists of $N = 10$ antennas uniformly spaced by half a wavelength. The number of snapshots collected was $L = 10$, i.e. $c = \frac{N}{L} = 1$. Note that both histograms were done using 1000 trials.

Figure 1 shows the histogram of the percentage of occurance of $\hat{q}$ using the "traditional" MDL criterion, i.e equation (14). Indeed, the performance is poor because only 8% of the estimates of number of sources correspond to the true one, i.e. $\hat{q} = 6$.

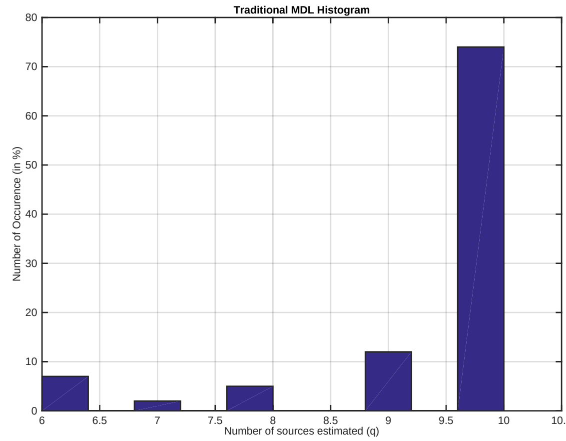

Fig. 1: Histogram of the number of signals resolved by the traditional MDL estimator.

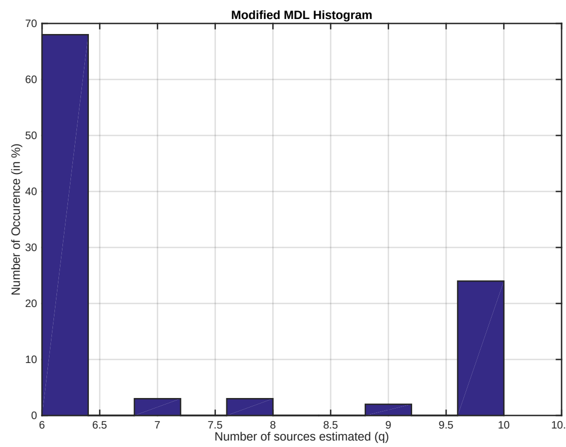

Fig. 2: Histogram of the number of signals resolved by the modified MDL estimator.

On the other hand, figure 2 depicts the histogram of the percentage of occurance of $\hat{q}$ using the "modified" MDL criterion, i.e equation (20). There is a great improvement as almost 68% of the estimates of number of sources correspond to the true one.

6. CONCLUSION

In this contribution and with the help of random matrix tools, we have presented a modified MDL (MMDL) estimator for detecting the number of superimposed signals. This MMDL estimator dominates the traditional MDL especially at the low number of snapshots regime, i.e. when $L = \mathcal{O}(N)$. Simulation results have shown the potential of MMDL over the traditional MDL.

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