Gaussian MIMO Half-Duplex Relay Networks: Approximate Optimality of *Simple* Schedules

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***Abstract*—This paper considers a Gaussian network where N half-duplex multiple-antenna relays assist the communication between a source and a destination. A novel antenna switching policy is proposed, where each relays' antenna can be configured to either receive or transmit independently of the others. The rate achieved by noisy network coding is shown to be to within a constant gap from the cut-set bound, where the gap only depends on the total number of antennas in the system. Moreover, the optimal number of different relay antenna configurations needed to attain the constant gap is proved to be at most $N+1$, that is, it only depends on the number of relays but not on the total number of antennas. Such a relay scheduling policy is referred to as *simple*. Through an example, it is shown that independently switching the antennas at the relays not only achieves in general strictly higher rates compared to using the antennas for the same purpose, but can actually provide a strictly larger pre-log factor. This implies that in broadband wireless networks with half-duplex multiple-antenna relays, the relay antennas should be *dynamically* configured to either transmit of receive depending on the channel conditions.**

***Index Terms*—Approximate capacity, half-duplex networks, multiple-antenna nodes, relay scheduling policies.**

I. Introduction

In this paper we study Gaussian networks where the communication between a source and a destination is assisted by N relays. It is assumed that each node in the network is equipped with multiple antennas and that the relays operate in Half-Duplex (HD) mode. The capacity of such a network is not known in general. For single-antenna nodes, in [1] we evaluated the cut-set outer bound by following the approach first proposed in [2] and we showed that Noisy Network Coding (NNC) [3] attains it to within $1.96(N+2)$ bits per channel use. In this work we prove that NNC is optimal to within 1.96 bits per channel use *per antenna* uniformly over all channel gains for the case of multiple-antenna nodes.

Finding the capacity of single-antenna Gaussian networks with N HD relays is computationally expensive since the cut-set upper bound has to be optimized over 2^N bounds (one for each possible cut in the network), each of which is a linear combination of 2^N relay states (since each relay can either transmit or receive). In [4], for single-antenna Gaussian HD diamond networks (where the source-destination link is absent and where the N relays cannot communicate among themselves) with $N=2$ relays, the authors showed that at most $N+1=3$ states, out of the $2^N=4$ possible ones, suffice to characterize the capacity to within a constant gap; the states with a strictly positive probability are referred to as *active*. In [5], the authors numerically verified that for single-antenna Gaussian HD diamond networks with $N \leq 7$ relays at most $N+1$ states are active and nicknamed these policies *simple schedules*; they conjectured that approximately optimal schedules are simple for any N. By using properties of submodular functions and linear programming duality, in [6] the authors proved the conjecture for $N \leq 6$ relays. In [7], by leveraging properties of the Lovász extension of a submodular function and the optimality of basic feasible solutions in linear programs, we proved the conjecture of [5] for any memoryless HD N-relay network (not necessary Gaussian or with a diamond topology) with independent noises and for which independent inputs are approximately optimal in the cut-set upper bound; Gaussian relay networks with single-antenna nodes satisfy these assumptions and thus admit an approximately optimal simple relay scheduling policy.

A. Contribution

In this work we show that the result of [7] applies to Gaussian relay networks with multiple-antenna nodes as well. In particular, we propose a novel antenna switching policy, where each antenna can be configured to either receive or transmit independently of the others as a function of the channel gains. If the total number of antennas at the relays is $m_{\rm tot}$, then $2^{m_{\rm tot}}$ possible receive/transmit configurations are possible. The main contribution of the paper is the proof that simple relay scheduling policies exist with at most $N+1$ active states, out of the $2^{m_{\rm tot}}$ possible ones. This result, beyond being quite surprising, can have important practical consequences for the design of reduced complexity relay scheduling policies, as the complexity of schedules is independent of the number of antennas.

B. Paper Organization

Section II describes the channel model. Section III first shows that the rate achieved by NNC is a constant number of bits apart from the cut-set outer bound, where the gap only depends on the total number of antennas; it then proves that an approximately optimal schedule is simple and that the number of active states only depends on the number of relays and not on the total number of antennas. Section IV presents an example to show that independently switching the antennas at the relays not only achieves higher rates than using all the antennas for the same purpose (either to receive or to transmit) but it can also provide a strictly larger pre-log factor. Section V concludes the paper.

C. Notation

With $[n_1 : n_2]$ we indicate the set of integers from n_1 to $n_2 \geq n_1$. Lower and upper case letters indicate scalars, boldface lower case letters denote vectors and boldface upper case letters indicate matrices. $\mathbf{0}_{i\times j}$ is the all-zero matrix of dimension $i \times j$, $\mathbf{1}_j$ is a column vector of length j of all ones and $\mathbf{I}_j$ is the identity matrix of dimension j. With $\mathcal{A}$ we indicate an index set and with $\emptyset$ we denote the empty set. For a square matrix $\mathbf{A}$, $\mathrm{Tr}[\mathbf{A}]$ is the trace. $|a|$ is the absolute value of a, $\|\mathbf{a}\|$ is the norm of the vector $\mathbf{a}$ and $\mathbf{A}^H$ is the Hermitian transpose of the matrix $\mathbf{A}$. The partitioned matrix $\mathbf{A} = \begin{bmatrix}\mathbf{A}_{1,1} & \mathbf{A}_{1,2}\\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2}\end{bmatrix}$ is indicated as $\mathbf{A} = [\mathbf{A}_{1,1}, \mathbf{A}_{1,2}; \mathbf{A}_{2,1}, \mathbf{A}_{2,2}]$. For matrices $\mathbf{A}_1, \ldots, \mathbf{A}_N$ we let $\mathrm{diag}[\mathbf{A}_1, \ldots, \mathbf{A}_N]$ be a block matrix with the matrix $\mathbf{A}_i$ in position (i, i) for $i \in [1 : N]$. $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ indicates that X is a proper-complex Gaussian random variable with mean μ and variance σ^2. $\mathbb{E}[\cdot]$ indicates the expected value.

II. System Model

A multi-relay network consists of N HD relay nodes (numbered 1 through N) assisting the communication between a source (node 0) and a destination (node $N+1$), through a shared memoryless channel. We use standard definitions for codes, achievable rates and capacity (see for example [1]).

A multiple-antenna complex-valued power-constrained Gaussian HD relay network has input-output relationship

$$\mathbf{y} = (\mathbf{I}_{M_{\mathrm{tot}}} - \mathbf{S})\mathbf{H}\mathbf{S}\mathbf{x} + \mathbf{z}, \tag{1}$$

where: (i) m_i is the number of antennas at node $i \in [0 : N+1]$, with $m_{\mathrm{tot}} := \sum_{k=1}^{N} m_k$ being the total number of antennas at the relays and $M_{\mathrm{tot}} := m_0 + m_{\mathrm{tot}} + m_{N+1}$ being the total number of antennas in the system; (ii) $\mathbf{y} := [\mathbf{y}_0; \ldots; \mathbf{y}_{N+1}] \in \mathbb{C}^{M_{\mathrm{tot}}\times 1}$ is the vector of the received signals with $\mathbf{y}_i \in \mathbb{C}^{m_i\times 1}$ being the received signal at node $i \in [0 : N+1]$; (iii) $\mathbf{x} := [\mathbf{x}_0; \ldots; \mathbf{x}_{N+1}] \in \mathbb{C}^{M_{\mathrm{tot}}\times 1}$ is the vector of the transmitted signals with $\mathbf{x}_i \in \mathbb{C}^{m_i\times 1}$ being the transmitted signal by node $i \in [0 : N+1]$; (iv) $\mathbf{z} := [\mathbf{z}_0; \ldots; \mathbf{z}_{N+1}] \in \mathbb{C}^{M_{\mathrm{tot}}\times 1}$ is the Gaussian noise vector assumed to have i.i.d. $\mathcal{N}(0, 1)$ components; (v) $\mathbf{S} := \mathrm{diag}[\mathbf{S}_0, \ldots, \mathbf{S}_{N+1}] \in \mathbb{C}^{M_{\mathrm{tot}}\times M_{\mathrm{tot}}}$ is the block diagonal matrix to account for the state (either transmit or receive) of the different antennas, with $\mathbf{S}_i := \mathrm{diag}[S_{i,1}, \ldots, S_{i,m_i}] \in \mathbb{C}^{m_i\times m_i}, i \in [0 : N+1]$; in particular, $\mathbf{S}_0 := \mathbf{I}_{m_0}$ since all the m_0 antennas at the source are always used to transmit, $\mathbf{S}_{N+1} := \mathbf{0}_{m_{N+1}\times m_{N+1}}$ since all the m_{N+1} antennas at the destination are always used to receive, and $S_{i,j} = 1$ if the j-th antenna of the i-th relay is transmitting and $S_{i,j} = 0$ if it is receiving; in this model the antennas of each relay can be switched independently of one another to either transmit or receive, for a total of $2^{m_{\mathrm{tot}}}$ possible states; (vi) $\mathbf{H} \in \mathbb{C}^{M_{\mathrm{tot}}\times M_{\mathrm{tot}}}$ is the constant (i.e., known to all nodes) block channel matrix, where $\mathbf{H}_{i,j} \in \mathbb{C}^{m_i\times m_j}$ with $(i, j) \in [0 : N+1]^2$ represents the channel matrix from node j to node i. Without loss of generality, we assume that the channel inputs are subject to the average power constraint $\mathbb{E}\left[\|\mathbf{x}_k\|^2\right] \leq 1$, $k \in [0 : N+1]$.

The exact capacity $\mathcal{C}$ of the channel in (1) is not known. $\mathcal{C}$ is said to be known to within GAP bits if one can show an achievable rate $R^{(\mathrm{in})}$ and an outer bound $R^{(\mathrm{out})}$ such that $R^{(\mathrm{in})} \leq \mathcal{C} \leq R^{(\mathrm{out})} \leq R^{(\mathrm{in})} + \mathsf{GAP}$, where GAP is a non-negative constant that may depend on N and M_{tot}, but not on the channel matrix $\mathbf{H}$ in (1).

III. Main Result

The main contributions of this paper are to first extend the result in [1, Theorem 1] from single-antenna to multiple-antenna nodes and then to prove that an approximately optimal simple schedule exists. The main result of the paper is:

Theorem 1. *For a multiple-antenna Gaussian HD N-relay network the following holds: (i) under the assumption of independent noises, NNC is optimal to within* $\mathsf{GAP} \leq 1.96$ *bits per channel use per antenna universally over all channel gains; (ii) the approximately optimal schedule is simple, i.e., it has at most $N+1$ non-zero entries independently of the total number of antennas in the network.*

The proof of the theorem is presented in the rest of this Section and an example to show the advantages of independently switching the relay antennas is reported in Section IV.

A. Proof of Theorem 1

1) Constant Gap: We argue here that [1, Theorem 1], valid for Gaussian HD relay networks with single-antenna nodes, gives a constant gap result also for the case of multiple-antenna nodes. Actually [1, Theorem 1] holds for the more general *Multicast Gaussian Network* (MGN) in which one has $K = N+2$ HD nodes (N relays, 1 source and 1 destination); thus, we shall argue that the gap result for the general single-antenna MGN extends to the multiple-antenna case. The key observation is to consider a MGN with multiple-antenna nodes as a new MGN with single-antenna nodes, where: (i) each node in the new MGN corresponds to a different antenna in the original MGN model and (ii) in the new MGN, the links connecting the nodes corresponding to different antennas at the same node in the original MGN are of infinite capacity. Now, since our original gap result applies to the new MGN (as the gap result in [1, Theorem 1] holds uniformly over all channel gains), then for the original MGN we have that $\mathsf{GAP} \leq 1.96 M_{\mathrm{tot}}$ bits per channel use, with M_{tot} being the total number of nodes in the new MGN, i.e., the total number of antennas in the original MGN.

2) Optimality of Simple Schedules: In [7, Theorem 1] we proved that an approximately optimal simple relay scheduling policy exists for any memoryless HD N-relay network under certain assumptions. Gaussian relay networks with multiple-antenna nodes satisfy all the conditions in [7, Theorem 1], namely: (i) the NNC strategy uses independent inputs and achieves the cut-set upper bound to within a constant gap; (ii) in (1) the noises are independent; (iii) a constant power

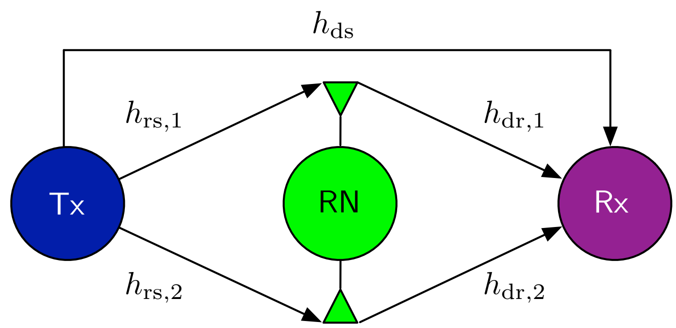

Fig. 1: Network with $N=1$ relay with $m_{\mathrm{r}}=2$ antennas, and single-antenna source and destination.

allocation across the relay states is optimal to within a constant gap (please refer to the example in Section IV). As remarked in [7], the optimization of the cut-set bound involves a minimization over all possible cuts in the network $\mathcal{A} \subseteq [1:N]$ and then a maximization over all the possible $2^{m_{\mathrm{tot}}}$ antenna receive/transmit configurations. Under the conditions in [7, Theorem 1], this problem is equivalent to solving $N!$ linear programs, each of which has $N+2$ constraints and $2^{m_{\mathrm{tot}}}+1$ unknowns. An optimal basic feasible solution for each of these problems has at most $N+2$ non-zero elements, of which one is the objective function (rate) and the others $N+1$ non-zero entries belong to the schedule. In other words, what dictates the number of active states of the relay scheduling policy is related to the minimization over $\mathcal{A} \subseteq [1:N]$ and not to the maximization over the $2^{m_{\mathrm{tot}}}$ possible relay configurations. This proves that an approximately optimal schedule has at most $N+1$ active states regardless of the total number of relay antennas. This result is rather surprising and can have important implications in the design of practical low-complexity and still approximately optimal relay scheduling policies.

IV. Example

We consider the network in Fig. 1, which consists of a single-antenna source (Tx), a single-antenna destination (Rx) and $N=1$ relay (RN) equipped with two antennas. For readability, we use here a different convention for the subscripts compared to the rest of the paper and indicate the input-output relationship as

$$\mathbf{y}_{\mathrm{r}} = \begin{bmatrix}(1-S_1)h_{\mathrm{rs},1}\\(1-S_2)h_{\mathrm{rs},2}\end{bmatrix} x_0 + \mathbf{z}_{\mathrm{r}}, \tag{2a}$$

$$y_{\mathrm{d}} = \begin{bmatrix}h_{\mathrm{ds}} & h_{\mathrm{dr},1} & h_{\mathrm{dr},2}\end{bmatrix} \begin{bmatrix}x_0\\S_1x_1\\S_2x_2\end{bmatrix} + z_{\mathrm{d}}, \tag{2b}$$

where: (i) x_0 and $\mathbf{x}_{\mathrm{r}} = [x_1;\ x_2]$ are the signals transmitted by the source and the relay, respectively; (ii) $\mathbf{y}_{\mathrm{r}} = [y_1;\ y_2]$ and y_{d} are the signals received at the relay and destination, respectively; (iii) $\mathbf{z}_{\mathrm{r}} = [z_1;\ z_2]$ and z_{d} are the noises at the relay and destination, respectively; (iv) $\mathbf{s}_{\mathrm{r}} = [S_1;\ S_2]$ is the state of the relay antennas; (v) the inputs are subject to the power constraints

$$\mathbb{E}[|x_0|^2] = \sum_{s\in[0:1]^2} \lambda_s \mathbb{E}[|x_0|^2|\mathbf{s}_{\mathrm{r}}=s] = \sum_{s\in[0:1]^2} \lambda_s P_{0|s} \le 1, \tag{3a}$$

$$\mathbb{E}\left[\|\mathbf{x}_{\mathrm{r}}\|^2\right] = \mathrm{Tr}\left[\sum_{s\in[0:1]^2} \lambda_s \mathbb{E}\left[\mathbf{x}_{\mathrm{r}}\mathbf{x}_{\mathrm{r}}^H|\mathbf{s}_{\mathrm{r}}=s\right]\right]$$
$$= \mathrm{Tr}\left[\sum_{s\in[0:1]^2} \lambda_s \begin{bmatrix} P_{1|s} & \rho_s\sqrt{P_{1|s}P_{2|s}} \\ \rho_s^*\sqrt{P_{1|s}P_{2|s}} & P_{2|s}\end{bmatrix}\right] \le 1, \tag{3b}$$

where $\rho_s : |\rho_s| \in [0,1]$ is the correlation coefficient among the relay antennas in state $s \in [0:1]^2$ and $P_{k|s}$ is the power allocated on x_k, $k \in [0:2]$, in state $s \in [0:1]^2$.

In what follows we consider two different possible switching strategies at the relay: (i) $\mathbf{s}_{\mathrm{r}} \in [0:1]^2$: the $m_{\mathrm{r}}=2$ antennas at the relay are switched independently of one another, and (ii) $\mathbf{s}_{\mathrm{r}} = S\mathbf{1}_2 : S \in [0:1]$: the $m_{\mathrm{r}}=2$ antennas at the relay are used for the same purpose, either transmit or receive.

A. Case (i): independent use of the relay antennas

For the cut-set upper bound, two cuts must be considered, namely, $\mathcal{A}=\emptyset$ (the relay is in the cut of the source) and $\mathcal{A}=\{1\}$ (the relay is in the cut of the destination). In this case the capacity $\mathcal{C}_{\text{case (i)}}$ is upper bounded as

$$\mathcal{C}_{\text{case (i)}} \le \max_{\mathbb{P}_{x_0,\mathbf{x}_{\mathrm{r}},\mathbf{s}_{\mathrm{r}}}} \min\left\{I\left(x_0,\mathbf{x}_{\mathrm{r}},\mathbf{s}_{\mathrm{r}};y_{\mathrm{d}}\right), I\left(x_0;y_{\mathrm{d}},\mathbf{y}_{\mathrm{r}}|\mathbf{x}_{\mathrm{r}},\mathbf{s}_{\mathrm{r}}\right)\right\}$$
$$\le H(\mathbf{s}_{\mathrm{r}}) + \max_{\mathbb{P}_{x_0,\mathbf{x}_{\mathrm{r}}}} \min\left\{I\left(x_0,\mathbf{x}_{\mathrm{r}};y_{\mathrm{d}}|\mathbf{s}_{\mathrm{r}}\right), I\left(x_0;y_{\mathrm{d}},\mathbf{y}_{\mathrm{r}}|\mathbf{x}_{\mathrm{r}},\mathbf{s}_{\mathrm{r}}\right)\right\},$$

where the last inequality follows since $I\left(\mathbf{s}_{\mathrm{r}};y_{\mathrm{d}}\right) \le H\left(\mathbf{s}_{\mathrm{r}}\right) \le 2$ bits. Note that, in general, Gaussian inputs are not optimal for Gaussian networks with HD relays since useful information can be conveyed to the destination through random switch [2]. However, as seen above, to within a constant gap a fixed switching policy between receive and transmit states is optimal, in which case a Gaussian input for each state is optimal. Moreover, the optimal choice of the correlation coefficients is $\rho_{00}=\rho_{01}=\rho_{10}=0$ and $\rho_{11}=\mathrm{e}^{\mathrm{j}\angle(h_{\mathrm{dr},1}^H h_{\mathrm{dr},2})}$. With this we have

$$\begin{aligned}
&I\left(x_0,\mathbf{x}_{\mathrm{r}};y_{\mathrm{d}}|\mathbf{s}_{\mathrm{r}}\right) \le I_{\emptyset} \\
&:= m_{\mathrm{d}}\log(2) + \lambda_{00}\log\left(1+|h_{\mathrm{ds}}|^2P_{0|00}\right) \\
&+ \lambda_{01}\log\left(1+|h_{\mathrm{ds}}|^2P_{0|01}+|h_{\mathrm{dr},2}|^2P_{2|01}\right) \\
&+ \lambda_{10}\log\left(1+|h_{\mathrm{ds}}|^2P_{0|10}+|h_{\mathrm{dr},1}|^2P_{1|10}\right) + \\
&\lambda_{11}\log\left(1+|h_{\mathrm{ds}}|^2P_{0|11}+\left(\sqrt{|h_{\mathrm{dr},1}|^2P_{1|11}}+\sqrt{|h_{\mathrm{dr},2}|^2P_{2|11}}\right)^2\right),
\end{aligned} \tag{6}$$

where the term $m_{\mathrm{d}}\log(2)$ (with m_{d} being the number of antennas at the destination) accounts for the loss of considering independent inputs at Tx and at RN. Similarly, we have

$$\begin{aligned}
&I\left(x_0;y_{\mathrm{d}},\mathbf{y}_{\mathrm{r}}|\mathbf{x}_{\mathrm{r}},\mathbf{s}_{\mathrm{r}}\right) \le I_{\{1\}} \\
&:= \lambda_{00}\log\left(1+(|h_{\mathrm{ds}}|^2+|h_{\mathrm{rs},1}|^2+|h_{\mathrm{rs},2}|^2)P_{0|00}\right) \\
&+ \lambda_{01}\log\left(1+(|h_{\mathrm{ds}}|^2+|h_{\mathrm{rs},1}|^2)P_{0|01}\right)
\end{aligned} \tag{7}$$

$$R_{\text{case (i)}} = \max_{\lambda_s} \min \Big\{ \lambda_{00} \log\left(1+|h_{\rm ds}|^2\right) + \lambda_{01}\log\left(1+|h_{\rm ds}|^2+|h_{\rm dr,2}|^2\right) + \lambda_{10}\log\left(1+|h_{\rm ds}|^2+|h_{\rm dr,1}|^2\right) + \lambda_{11}\log\left(1+|h_{\rm ds}|^2+\left(\sqrt{|h_{\rm dr,1}|^2}+\sqrt{|h_{\rm dr,2}|^2}\right)^2\right), \lambda_{00}\log\left(1+|h_{\rm ds}|^2+|h_{\rm rs,1}|^2+|h_{\rm rs,2}|^2\right)+\lambda_{01}\log\left(1+|h_{\rm ds}|^2+|h_{\rm rs,1}|^2\right) + \lambda_{10}\log\left(1+|h_{\rm ds}|^2+|h_{\rm rs,2}|^2\right)+\lambda_{11}\log\left(1+|h_{\rm ds}|^2\right)\Big\} \quad (4)$$

$$R_{\text{case (ii)}} = \log\left(1+|h_{\rm ds}|^2\right) + \frac{\log\left(1+\frac{\left(\sqrt{|h_{\rm dr,1}|^2}+\sqrt{|h_{\rm dr,2}|^2}\right)^2}{1+|h_{\rm ds}|^2}\right)\log\left(1+\frac{|h_{\rm rs,1}|^2+|h_{\rm rs,2}|^2}{1+|h_{\rm ds}|^2}\right)}{\log\left(1+\frac{\left(\sqrt{|h_{\rm dr,1}|^2}+\sqrt{|h_{\rm dr,2}|^2}\right)^2}{1+|h_{\rm ds}|^2}\right)+\log\left(1+\frac{|h_{\rm rs,1}|^2+|h_{\rm rs,2}|^2}{1+|h_{\rm ds}|^2}\right)} \quad (5)$$

$$+ \lambda_{10}\log\left(1+(|h_{\rm ds}|^2+|h_{\rm rs,2}|^2)P_{0|10}\right) + \lambda_{11}\log\left(1+|h_{\rm ds}|^2P_{0|11}\right).$$

Note that to determine the NNC achievable rate it suffices to remove the term $I\left(\mathbf{y}_{\rm r};\hat{\mathbf{y}}_{\rm r}|x_0,\mathbf{x}_{\rm r},\mathbf{s}_{\rm r},y_{\rm d}\right) = m_{\rm r}\log(1+1/\sigma^2)$ from $I_\emptyset$ and the term $I\left(x_0;\mathbf{y}_{\rm r}|\hat{\mathbf{y}}_{\rm r},y_{\rm d},\mathbf{x}_{\rm r},\mathbf{s}_{\rm r}\right) \le \log(1+\sigma^2)$ from $I_{\{1\}}$, with σ^2 being the variance of the quantization noise. We let $\sigma^2 = 1$ for simplicity. Note also that the expressions for $I_\emptyset$ and $I_{\{1\}}$ should be optimized with respect to the power allocation across the relay states, which makes the optimization problem non-linear in $\lambda_s, s \in [0:1]^{m_{\rm r}}$. In order to apply [7, Theorem 1] and hence Theorem 1, we must further bound the two expressions so that to obtain a new optimization problem with constant powers across the relay states, i.e., we need to obtain a linear program in $\{\lambda_s\}$. In Appendix we show (see also [8, Appendix C] for more details) $\mathcal{C}_{\text{case (i)}} \le \mathsf{GAP} + R_{\text{case (i)}}$ where $R_{\text{case (i)}}$ is defined in (4) at the top of this page and where $\mathsf{GAP} \le 8$ bits to account for deterministic switch, independent inputs at the source and at the relay, constant power allocation across the states and NNC transmission strategy.

B. Case (ii): same use of the relay antennas

In this case the $m_{\rm r} = 2$ antennas at the relay are used for the same purpose so it suffices to set $\lambda_{01} = \lambda_{10} = 0$ in (4) and optimize over $\lambda_{00} = 1-\lambda_{11} = \lambda \in [0,1]$. With this we get that $\mathcal{C}_{\text{case (ii)}} \le \mathsf{GAP} + R_{\text{case (ii)}}$ where $R_{\text{case (ii)}}$ is defined in (5) at the top of this page and where again $\mathsf{GAP} \le 8$ bits. The optimal λ for $R_{\text{case (ii)}}$ in (5) was found by equating the two expressions within the $\max\min$.

C. Specific comparisons

We now consider three different channel configurations for the network in Fig. 1 to show that not only $R_{\text{case (i)}} \ge R_{\text{case (ii)}}$ in general, but that independently switching the $m_{\rm r} = 2$ antennas at the relay can obtain a strictly larger pre-log factor / multiplexing gain.

Example 1: we let $|h_{\rm ds}| = 0$, $|h_{\rm rs,2}| = |h_{\rm dr,1}| = 0$ and $|h_{\rm rs,1}|^2 = |h_{\rm dr,2}|^2 = \gamma > 0$ in Fig. 1. With this choice of the channel parameters we get $R_{\text{case (i)}} = \log(1+\gamma)$, with the choice $\lambda_{00} = \lambda_{10} = \lambda_{11} = 0$ and $\lambda_{01} = 1$, i.e., there is $1 < N+1 = 2$ active state, and $R_{\text{case (ii)}} = \frac{\log(1+\gamma)}{2}$. From the two expressions above not only we have $R_{\text{case (i)}} > R_{\text{case (ii)}}, \forall\gamma > 0$, but independently switching the $m_{\rm r} = 2$ antennas also provides a pre-log factor that is twice the one provided by using the antennas for the same purpose.

Example 2: we let $|h_{\rm ds}| = 0$, $|h_{\rm rs,1}|^2 = |h_{\rm rs,2}|^2 = |h_{\rm dr,1}|^2 = |h_{\rm dr,2}|^2 = \gamma > 0$ in Fig. 1. With this choice of the channel parameters we get

$$R_{\text{case (i)}} = \begin{cases} \log(1+\gamma) & \text{if } \gamma \ge 0.752 \\ \frac{\log(1+2\gamma)\log(1+4\gamma)}{\log(1+2\gamma)+\log(1+4\gamma)} & \text{otherwise} \end{cases},$$

with the choice $\lambda_s = [0,0,1,0]$ ($1 < N+1 = 2$ active state) if $\gamma \ge 0.752$ and $\lambda_s = [\lambda,0,0,1-\lambda]$ ($2 = N+1$ active states), with $\lambda = \frac{\log(1+4\gamma)}{\log(1+2\gamma)+\log(1+4\gamma)}$ otherwise, and

$$R_{\text{case (ii)}} = \frac{\log(1+2\gamma)\log(1+4\gamma)}{\log(1+2\gamma)+\log(1+4\gamma)}.$$

It hence follows that $R_{\text{case (i)}} > R_{\text{case (ii)}}, \forall\gamma \ge 0.752$. Moreover, in the high-SNR regime, the pre-log factor for $R_{\text{case (i)}} = \log(1+\gamma)$ is again twice of the one of $R_{\text{case (ii)}} \approx \frac{1}{2}\log(1+\gamma)$. This example (as also Example 1) highlights the importance of smartly switching the relay antennas in order to fully exploit the available system resources.

Example 3: we consider the case of Rayleigh fading, where $h_{\rm ds} \sim \mathcal{N}\left(0,\sigma^2_{\rm ds}\right)$, $h_{{\rm rs},i} \sim \mathcal{N}\left(0,\sigma^2_{\rm rs}\right)$ and $h_{{\rm dr},i} \sim \mathcal{N}\left(0,\sigma^2_{\rm dr}\right)$ with $i \in [1:2]$ are assumed to be constant over the whole slot (block-fading model) and we let $\sigma^2_{\rm ds} = \mathbb{E}\left[|h_{\rm ds}|^2\right] = \frac{\mathsf{c}}{1^\alpha}$, $\sigma^2_{\rm rs} = \mathbb{E}\left[|h_{{\rm rs},i}|^2\right] = \frac{\mathsf{c}}{d^\alpha}$ and $\sigma^2_{\rm dr} = \mathbb{E}\left[|h_{{\rm dr},i}|^2\right] = \frac{\mathsf{c}}{(1-d)^\alpha}$, where c is a constant, $d \in [0,1]$ is the distance between the source and the relay and $(1-d)$ is the distance between the relay and the destination, and $\alpha \ge 2$ is the path loss exponent. Fig. 2 shows the average $R_{\text{case (i)}}$ in (4) (solid curve) and the average $R_{\text{case (ii)}}$ in (5) (dashed curve) versus $d \in [0,1]$, with fixed $\alpha = 3$ and $\mathsf{c} = 1$. The average was taken over $5 \cdot 10^4$ different realizations of the channel gains for each value of $d \in [0,1]$. From Fig. 2 we observe again that in general $\mathbb{E}\left[R_{\text{case (i)}}\right] > \mathbb{E}\left[R_{\text{case (ii)}}\right]$, with a maximum difference of around 0.6 bits at $d = 0.5$. Note, in fact, that for $d = 0.5$ we have $\sigma^2_{\rm ds} = 1$ and $\sigma^2_{\rm rs} = \sigma^2_{\rm dr} = 8$. Under these channel

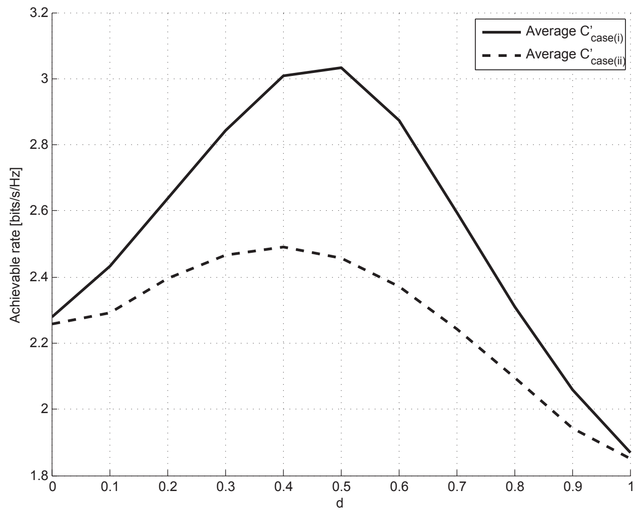

Fig. 2: $\mathbb{E}\left[R_{\text{case (i)}} \text{ in (4)}\right]$ (solid curve) and $\mathbb{E}\left[R_{\text{case (ii)}} \text{ in (5)}\right]$ (dashed curve) versus different values of $d \in [0, 1]$.

conditions, by independently switching the $m_{\rm r} = 2$ antennas at the relay we (approximately) achieve the full-duplex performance, i.e., $\mathbb{E}\left[R_{\text{case (i)}}\right] \approx \log\left(\sigma_{\rm rs}^2\right) = 3$ bits/s/Hz, while by using the $m_{\rm r} = 2$ antennas for the same purpose the rate performance reduces to the capacity of a single-antenna HD relay channel, i.e., from (5) we have $\mathbb{E}\left[R_{\text{case (ii)}}\right] \approx \log\left(\sigma_{\rm ds}^2\right) + \frac{\log\left(\frac{4\sigma_{\rm dr}^2}{\sigma_{\rm ds}^2}\right)\log\left(\frac{2\sigma_{\rm rs}^2}{\sigma_{\rm ds}^2}\right)}{\log\left(\frac{4\sigma_{\rm dr}^2}{\sigma_{\rm ds}^2}\right)+\log\left(\frac{2\sigma_{\rm rs}^2}{\sigma_{\rm ds}^2}\right)} = \frac{\log(32)\log(16)}{\log(32)+\log(16)} \approx 2.2$ bits/s/Hz.

V. Conclusions

We studied Gaussian networks with multiple-antenna nodes where the communication between a source and a destination is assisted by N relays operating in half-duplex. We proposed a novel antenna switching policy where the antennas at the relays are switched between receive and transmit state depending on the channel conditions but independently of one another. For such networks, we showed that the rate achieved by the noisy network coding strategy is a constant number of bits away from the cut-set outer bound and that the gap depends on the total number of antennas. Moreover, we proved that an approximately optimal relay schedule is simple, where the number of active states only depends on the number of relays and not on the total number of antennas. Finally, through an example we showed that the proposed scheme with a dynamic switching of the antennas based on the channel conditions can provide a strictly larger pre-log factor / multiplexing gain compared to using the antennas for the same purpose.

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Appendix

In (3) we assume, without loss of optimality that $P_{1|00} = P_{1|01} = 0$ (resp. $P_{2|00} = P_{2|10} = 0$), since for the HD constraint when the first (resp. second) antenna at the relay is receiving the relay's transmit power on that antenna is zero. With this, we let

$$P_{0|00} = \frac{\alpha_0}{\lambda_{00}},\ P_{0|01} = \frac{\beta_0}{\lambda_{01}},\ P_{0|10} = \frac{\gamma_0}{\lambda_{10}},\ P_{0|11} = \frac{\delta_0}{\lambda_{11}},$$
$$P_{2|01} = \frac{\alpha_1}{\lambda_{01}},\ P_{1|10} = \frac{\beta_1}{\lambda_{10}},\ P_{1|11} = \frac{\gamma_1}{\lambda_{11}},\ P_{2|11} = \frac{\delta_1}{\lambda_{11}},$$

where $\alpha_i + \beta_i + \gamma_i + \delta_i \leq 1, i \in [0:1]$ in order to meet the power constraints in (3). In order to further upper bound $I_\emptyset$ in (6) and $I_{\{1\}}$ in (7) we used the following steps: (i) we upper bounded the entropy of the discrete state random variable by the logarithm of the size of its support; (ii) we upper bounded the power splits by setting $\alpha_i = \beta_i = \gamma_i = \delta_i = 1$, $i \in [0:1]$; (iii) we upper bounded all the $\lambda_s, s \in [0:1]^2$ inside the logarithms by one. With this we got

$$\mathcal{C}_{\text{case (i)}} \leq R_{\text{case (i)}} + m_{\rm d}\log(2) + 2\log(2),$$

where $R_{\text{case (i)}}$ is defined in (4). In order to further lower bound $I_\emptyset$ in (6) and $I_{\{1\}}$ in (7) we used the following steps: (i) we set $\alpha_1 = \lambda_{01}$, $\beta_1 = \lambda_{10}$, $\gamma_1 = \delta_1 = \frac{\lambda_{11}}{2}$, $\alpha_0 = \lambda_{00}$, $\beta_0 = \lambda_{01}$, $\gamma_0 = \lambda_{10}$ and $\delta_0 = \lambda_{11}$ (note that with these power splits the power constraints in (3) are satisfied) and (ii) we used the further bound $\log\left(1 + \left(\sqrt{\frac{a}{2}} + \sqrt{\frac{c}{2}}\right)^2\right) \geq \log\left(1 + \left(\sqrt{a} + \sqrt{c}\right)^2\right) - \log(2)$. With this we got

$$\mathcal{C}_{\text{case (i)}} \geq R_{\text{case (i)}} - \log(2),$$

where $R_{\text{case (i)}}$ is defined in (4). This concludes the proof.

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