Transmission of Correlated Gaussian Samples over a Multiple-Access Channel

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***Abstract*—This paper provides lower bounds on the reconstruction error for transmission of two continuous correlated random vectors sent over a sum channel using the help of two causal feedback links from the decoder to the encoders connected to each sensor. This construction is considered for standard normally distributed sources. Additionally, a two-way retransmission protocol, which is a non-coherent dual-source adaptation of the original work by Yamamoto [1] is introduced for an additive white Gaussian noise channel. Asymptotic optimality of the protocol is analyzed and upper bounds on the distortion level are derived for two-rounds considering two extreme cases of high and low correlation among the two sources. It is shown by both the upper and lower-bounds that collaboration can be achieved through energy accumulation.**

***Index Terms*—Distributed communication, joint source channel coding, detection/estimation, multiple-access channel (MAC)**

I. Introduction

In this work we consider simple transmission strategies for a network of sensors able to measure a physical phenomenon from different locations. To illustrate this more precisely, imagine the simplest scenario of one sensor node tracking a slowly time-varying random sequence and sending its observations to a receiver over a wireless channel. The source is denoted by a random variable U of zero mean and variance $\sigma_u^2 = 1$, representing a single realization of the random sequence at a particular time t. The sensor should be seen as a tiny device with strict energy constraints. The communication channel between the sender and the receiver is an additive white Gaussian noise channel. An important question is how to efficiently encode the random variable U for transmission, and what performance can be achieved upon reconstruction as a function of the energy used to achieve this transmission. We focus our attention on the case where unitary samples of the source are transmitted sporadically due to slow time-variation, and consequently we cannot perform sequence coding. In [2], [3] the best-known lower-bound for the reconstruction fidelity without feedback from the receiver, coherent detection and unlimited channel bandwidth behaves as $e^{-\mathcal{E}/2N_0}$ for uniformly-distributed U where $\mathcal{E}$ is the energy used for transmission of U. Moreover, several schemes can achieve $e^{-\mathcal{E}/3N_0}$ both with and without coherent detection and for both normally and uniformly distributed U. One such scheme based on scalar quantization and orthogonal modulation was previously described in [4]. With feedback, coherent detection and unlimited channel bandwidth, the classical schemes described in [5] can asymptotically achieve $e^{-2\mathcal{E}/N_0}$ for normally distributed U which coincides with the lower-bound from Goblick [6].

Here we consider a multi-sensor scenario as in [7] and [8] which is an important generalization, where two correlated normally-distributed random variables are transmitted over a Gaussian multiple-access channel. The key element being to exploit the correlation, which is assumed to be known, both at the transmitter and the receiver. Moreover, we aim to determine the operating regimes for such a multiple-access system in terms of the role correlation plays in determining the energy efficiency. In a similar vein, the authors in [9] and [10] derive a threshold signal-to-noise ratio (SNR) through the correlation between the sources so that below this threshold, minimum distortion is attained by uncoded transmission in a Gaussian multiple access channel with and without feedback, respectively. In these works, the authors consider transmission of a bi-variate normal source and the distortion can be characterized by two regimes as a function of the relationship between the channel SNR and the source SNR. Through a different approach lower bounds for transmission of correlated sources over Gaussian multiple-access channels is considered in [11].

The main results of the paper are summarized in two main sections. Section II consists of the model description of the addressed problem along with the information theoretic results. In the first two subsections II-A and II-B, we describe the channel and source models, respectively whereas in the last part subsection II-C, we introduce three different lower bounds on the reconstruction error of estimating the source vectors based on different ranges in which the correlation coefficient between the two sources is defined. Section III is focused on the analysis of a two-way retransmission protocol for transmitting correlated information of two Gaussian sources. Here, we provide an upper bound on the reconstruction error for estimating the source messages considering two extremes levels of high and low correlation. Finally, we conclude the paper in Section IV with the comparison of the theoretical lower bounds and upper bounds from the achievable scheme. The derivations of the information theoretic part can be found in detail in Section V.

II. Lower Bounds for the Gaussian MAC

A. Channel Model

Let us begin with the definition of the system model used to analyze the addressed problem.

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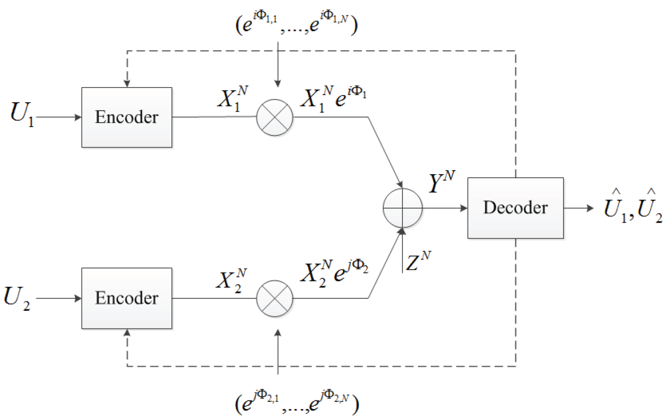

Fig. 1. Transmission of correlated Gaussian samples over GMAC with feedback.

The considered system for the multiple-access is depicted in Figure 1 where we note that the encoders can make use of an ideal feedback link. The received signal $\mathbf{Y} = \{Y_i; i = 1, ..., N\}$ and the energy constraints are given as

$$Y_i = X_{1,i}e^{i\phi_{1,i}} + X_{2,i}e^{j\phi_{2,i}} + Z_{1,i} + Z_{2,i} \tag{1}$$

$$\frac{1}{K}\sum_{i=1}^{N} E[|X_{m,i}|^2] \leq \mathcal{E}_m \tag{2}$$

for $m = 1, 2$ and $i, j = 1, ..., N$, respectively. K is the dimensionality of the source vectors and is assumed to be finite (i.e. it cannot grow without bound with N). The criteria to satisfy is chosen as the squared-error distortion measure, which is defined by $d(u_m, \hat{u}_m) = (u_m - \hat{u}_m)^2$. $\phi_m = \{\phi_{m,i}; i = 1, ..., N\}$ denotes the random phases which are assumed to be unknown both to the transmitter and the receiver. The encoding functions are arbitrary mappings, $(U_m, Y_1, Y_2, \cdots, Y_{i-1}) \to X_{m,i}$ for each channel input in the case of causal feedback, and $U_m \to X_{m,i}$ without feedback.

B. Source Model

The correlational relationship between the sources $\mathbf{U}_1, \mathbf{U}_2$ dimension of K is defined through the following expression

$$\mathbf{U}_2 = \rho\mathbf{U}_1 + \sqrt{1-\rho^2}\mathbf{U}_2' \tag{3}$$

where we denote the first source with $\mathbf{U}_1$ and the second source with $\mathbf{U}_2$. $\mathbf{U}_2'$ here is an auxiliary random vector, which is independent of the first source $\mathbf{U}_1$. The correlated sources $\mathbf{U}_1$ and $\mathbf{U}_2$ are defined to be standard normal random vectors, guaranteed by the auxiliary random vector $\mathbf{U}_2'$ which is also normally distributed with zero mean and unit variance.

C. Derivation of the Bounds

In order to avoid repetition in the derivations of the outer bounds, we will use the notation m to represent one of the sources and m' will be used to indicate the other source.Furthermore, we note that the bounds are valid for both the use of feedback-based encoders and those without feedback.

1) Lower Bound I: In order to obtain a lower bound on the reconstruction error in estimating $\mathbf{U}_1, \mathbf{U}_2$, we derive a relatively simple mutual information between the m^{th} source U_m and the output signal Y through two different expansions, which depends on the sum energy and turns out to be appropriate for the cases of high correlation between the sources. Two different expansions of $I(\mathbf{U}_m; \mathbf{Y})$ are derived first of which is based on the output signal where the second expansion depends on the sources. The two expansions of $I(\mathbf{U}_m; \mathbf{Y})$ are given by

$$I(\mathbf{U}_m; \mathbf{Y}) \leq N \log\left(1 + \frac{K(\mathcal{E}_m + \mathcal{E}_{m'})}{NN_0}\right), \tag{4}$$

$$I(\mathbf{U}_m; \mathbf{Y}) \geq h(\mathbf{U}_m) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m), \tag{5}$$

respectively. The derivations of (4) and (5) can be found in Appendix V-A together with the source entropies. Combining (28) with (29) and substituting into (27) brings out the second expansion based on the sources. Equating the outcome to (30) provides the following lower bound on distortion level for the m^{th} source

$$D_{I,m} \geq \left(1 + \frac{K(\mathcal{E}_m + \mathcal{E}_{m'})}{NN_0}\right)^{-\frac{2N}{K}} \tag{6}$$

Asymptotically in N, (6) becomes

$$D_{I,m} \geq e^{-\frac{2(\mathcal{E}_m + \mathcal{E}_{m'})}{N_0}}. \tag{7}$$

2) Lower Bound II: The main difference between this case and the previous one treated high correlation is the mutual information term to be used in order to come up with a bound on the distortion level corresponding each source. Hence the mutual information between the source $\mathbf{U}_m$ and the output signal $\mathbf{Y}$ will be expanded through two different ways when the information of the other source $\mathbf{U}_{m'}$ is given, so that the output signal expansion yields dependent on the indiviual energy. The aim is to obtain a different bound when the sources are not strongly correlated since $\mathbf{U}_m$ does not have the information of $\mathbf{U}_{m'}$ due to correlation. The two expansions of $I(\mathbf{U}_m; \mathbf{Y}|\mathbf{U}_{m'})$ are given as

$$I(\mathbf{U}_m; \mathbf{Y}|\mathbf{U}_{m'}) \leq N \log\left(1 + \frac{K\mathcal{E}_m}{NN_0}\right), \tag{8}$$

$$I(\mathbf{U}_m; \mathbf{Y}|\mathbf{U}_{m'}) \geq h(\mathbf{U}_m|\mathbf{U}_{m'}) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m). \tag{9}$$

The derivations of (8) and (9) are given in Appendix V-B. Note that $h(\mathbf{U}_m - \hat{\mathbf{U}}_m)$ which is given by (29) in Appendix V-A is common for both lower bounds I and II. Combining it with (33) and substituting into (32) shapes the second expansion based on the source entropies. Equating the outcome to (31) results in the general form of the lower bound on the distortion as $D_{II,m} \geq (1-\rho^2)\left(1 + \frac{K\mathcal{E}_m}{NN_0}\right)^{-\frac{2N}{K}}$ and asymptotically in N, it becomes

$$D_{II,m} \geq (1-\rho^2)e^{-\frac{2\mathcal{E}_m}{N_0}}. \tag{10}$$

3) Lower Bound III: In addition to the lower bounds I and II, the product distortion term $D_{III} = D_1 D_2$ is bounded as given in the following form

$$D_{III} \geq \left(1-\rho^2\right)\exp\left(-\frac{2(\mathcal{E}_m+\mathcal{E}_{m'})}{N_0}\right). \quad (11)$$

The derivation of the bound (11) given above can be found in Appendix V-C. As done for the lower bounds I and II, D_{III} is achieved through equating the expansions (35) and (36) of the mutual information $I(\mathbf{U}_m, \mathbf{U}_{m'}; \mathbf{Y})$.

Combining all three bounds (7), (10) and (11) introduced above, we obtain the following overall bound

$$D_m \geq \begin{cases} D_{I,m} & \text{if } 1-\rho^2 \leq \min(D_{m'}, e^{-\frac{2\mathcal{E}_{m'}}{N_0}}), \\ D_{II,m} & \text{if } D_{m'} \geq e^{-\frac{2\mathcal{E}_{m'}}{N_0}} \text{ and } 1-\rho^2 \geq e^{-\frac{2\mathcal{E}_{m'}}{N_0}}, \\ D_{III}/D_{m'} & \text{if } 1-\rho^2 \geq \min(D_{m'}, e^{-\frac{2\mathcal{E}_{m'}}{N_0}}). \end{cases} \quad (12)$$

The bounds given above predict that energy accumulation cannot be achieved when the distortion resulting from the estimation of one source realization using the other (i.e. $1-\rho^2$) is more than the point-to-point distortion (Goblick bound $e^{-2\mathcal{E}/N_0}$, [6]) incurred during transmission.

III. Two-Way Protocol with Dual Gaussian Sources

As in the original work [1] and its non-coherent version studied in [7], the protocol comprises a data phase and a control phase, which can be repeated up to two rounds. The structure of the sources is defined as in (3) where U_1 and U_2' are independent of each other and normally distributed with zero mean and unit variance. Here U_2' is used as an auxiliary random variable to define the relationship between the two sources U_1 and U_2 with the joint probability density function given below

$$f(u_1, u_2) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left[-\frac{u_1^2 - 2\rho u_1 u_2 + u_2^2}{2(1-\rho^2)}\right] \quad (13)$$

for $-\infty < u_1 < \infty$ and $\infty < u_2 < \infty$. The definition of U_2 ensures that the covariance between the sources equals the correlation coefficient ρ.

The total energy to be used by protocol is fixed and we will denote the energy used in the data phase of the i^{th} round by the j^{th} source by $\mathcal{E}_{\mathrm{D},i,j}$, where $i, j = 1, 2$. In the same way, $\mathcal{E}_{\mathrm{C},i,j}$ denotes the energy used in the control phase of the i^{th} round by the j^{th} source. The energy in the control and data phases of the i^{th} round ($\mathcal{E}_{\mathrm{C},i}$ and $\mathcal{E}_{\mathrm{D},i}$, respectively) are defined as the sum energy on both sources. The quantized source sample of the j^{th} source is encoded into 2^B messages with dimension N. The messages m_1 and m_2 will be discretized through uniform quantization, i.e. the bins are located equidistantly from each other and for each source the reconstruction points $x_{j,n}$ are the midpoints of the intervals $I_{j,n}$ which define each of the bins for the j^{th} source with $n = 2, ..., 2^B - 1$. The quantization intervals corresponding to the tails of the bell curve ($I_{j,1}$ and $I_{j,2^B}$ for $j = 1, 2$) are considered as one bin for each side. The rest of the partitioning is made for each source as $I_{j,n} = [-\Delta + \frac{\Delta(n)}{2^{B-1}-1}, -\Delta + \frac{\Delta(n+1)}{2^{B-1}-1}[$, with $\Delta = 2\sqrt{B\ln 2}$. Let us set the quantization levels for each source as $x_{j,1} = -\Delta$ and $x_{j,2^B} = \Delta$.

In the data phase, the first source sends its message $\mathbf{m}_1(U_1)$ to the receiver using the energy $\mathcal{E}_{\mathrm{D},1,1}$. The receiver detects $\hat{m}_1$ and feeds it back. And the second source sends $\mathbf{m}_2(U_2)$ with energy $\mathcal{E}_{\mathrm{D},1,2}$. This encoding rule allows the second source to exploit the correlation of its sample with that of its peer and the energy used is chosen according to the likelihood of the estimate fed back from the receiver. After the estimation and feedback of $\hat{m}_2$, data phase of the first round ends and the encoders enter the control phase to inform the receiver about the correctness of its decision. For that, each source sends ACK/NACK signals regarding its own message to the decoder. According to the control signals, either the protocol halts or goes on another round to do the retransmission of the message which were not acknowledged in the control phase. For the second data phase, the destination instructs the sources to retransmit and re-detect their messages. The output signal of the j^{th} source in the data phase is

$$\mathbf{Y}_d = \sqrt{\mathcal{E}_{\mathrm{D},1,j}}\mathrm{e}^{j\Phi_j}\mathbf{S}_{m_j} + \mathbf{Z}_j. \quad (14)$$

We assume the random phases Φ_j to be distributed uniformly on $[0, 2\pi)$, the channel noise $\mathbf{Z}_j$ to have zero mean and equal autocorrelation $N_0\mathbf{I}_{N\times N}$ for $j = 1, 2$ and $\mathbf{S}_{m_j}$ are the N-dimensional messages, where $m = 1, 2, \cdots, 2^B$. We assume a detector of the form for the j^{th} source as $e_j = I\left(|y_{c,j}|^2 > \lambda\mathcal{E}_{\mathrm{C},1,j}\right)$ with $y_{c,j} = \mathbf{Y_{c,j}}^H\mathbf{S}_{c,j}$. Here λ is a threshold value to be optimized and included within the interval $[0, 1)$.

Definition (m, n) is called a compatible pair if $|\rho U_1 - U_2| < \theta$ is satisfied for $\forall u_1, u_2 \in B$ where θ is an arbitrary constant.

This definition assures that, during the quantization process, the correlation between the two sources would not allow the second source to fall in a bin further than a certain distance. J_m represents the set that n is assumed to be contained. Outside of this set, the pair (m, n) becomes incompatible with the corresponding probability of error $(1-\Pr(|U_2'| < \theta\sqrt{1-\rho^2}))$. In this case, the probability of having an error can be composed by three different events; both sources to be detected wrong, $\hat{u}_1$ detected correctly as $\hat{u}_2$ detected wrong or vice versa. These three events are summarized in two cases as only one source to be in error or both. The receiver chooses

$$(\hat{\mathbf{m}}_\mathbf{1}, \hat{\mathbf{m}}_\mathbf{2}) \sim \underset{(\hat{\mathbf{m}}_\mathbf{1}, \hat{\mathbf{m}}_\mathbf{2}) s.t. (\hat{\mathbf{m}}_\mathbf{1}, \hat{\mathbf{m}}_\mathbf{2}) \in J_m}{\mathrm{argmax}} |\mathbf{U}^{(1)}_{\mathbf{m_1}}|^2 + |\mathbf{U}^{(1)}_{\mathbf{m_2}}|^2 \quad (15)$$

in the first round, whereas the detection rule is cumulatively given by

$$(\hat{\mathbf{m}}_\mathbf{1}, \hat{\mathbf{m}}_\mathbf{2}) \sim \underset{(\hat{\mathbf{m}}_\mathbf{1}, \hat{\mathbf{m}}_\mathbf{2}) s.t. (\hat{\mathbf{m}}_\mathbf{1}, \hat{\mathbf{m}}_\mathbf{2}) \in J_m}{\mathrm{argmax}} |\mathbf{U}^{(1)}_{\mathbf{m}}|^2 + |\mathbf{U}^{(2)}_{\mathbf{m}}|^2 \quad (16)$$

for the second round where $|\mathbf{U}^{(1)}_{\mathbf{m}}|^2 = |\mathbf{U}^{(1)}_{\mathbf{m_1}}|^2 + |\mathbf{U}^{(1)}_{\mathbf{m_2}}|^2$ and $|\mathbf{U}^{(2)}_{\mathbf{m}}|^2 = |\mathbf{U}^{(2)}_{\mathbf{m_1}}|^2 + |\mathbf{U}^{(2)}_{\mathbf{m_2}}|^2$. The overall distortion at the end of the second round is defined as $D = D_q(1-P_e) + D_e P_e$ and bounded by (17) as given on the top of the next page where ic and c in the subscripts represent the incompatible and compatible pairs, respectively. $P_{e,ic,j}$ is the error probability of j incompatible sources being in error whereas $P_{e,c,j}$ represents the error probability of those which are compatible. $D_{e,ic,j}$

$$D \leq D_q + (1 - \Pr(|U_2'| > \theta\sqrt{1-\rho^2}))\,(D_{e,c,1}P_{e,c,1} + D_{e,c,2}P_{e,c,2}) + \Pr(|U_2'| > \theta\sqrt{1-\rho^2})D_{e,ic,1}P_{e,ic,1} + \Pr(|U_2'| > \theta\sqrt{1-\rho^2})D_{e,ic,2}P_{e,ic,2}$$
$$\stackrel{(a)}{\leq} D_q + D_{e,c,1}P_{e,c,1} + D_{e,c,2}P_{e,c,2} + \Pr(|U_2'| > \theta\sqrt{1-\rho^2})\,(D_{e,ic,1} + D_{e,ic,2}P_{e,ic,2}) \quad (17)$$

and $D_{e,c,j}$ denote the corresponding distortions for each case, respectively. Note that, error probabilities and the corresponding distortion levels for the case of both sources being in error are assumed to be equivalent, i.e. $P_{e,c,2} = P_{e,ic,2} = P_{e,2}$ and $D_{e,c,2} = D_{e,ic,2} = D_{e,2}$. It should be also noted that in step (a) of eq. (17), the probability of error only one incompatible source to be in error is upper bounded by 1. $P_{e,1}$ and $P_{e,2}$ are defined by

$$P_{e,1} \leq \left\lceil 2^B\theta\sqrt{1-\rho^2}\right\rceil \Pr(E_{e\to c,1})P_2(1, \frac{\mathcal{E}_{\mathrm{D},1}}{2}) + \left\lceil 2^B\theta\sqrt{1-\rho^2}\right\rceil P_2(2, \frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2}}{2}) \quad (18)$$

$$P_{e,2} \leq \left\lceil 2^B\theta\sqrt{1-\rho^2}\right\rceil 2^B \Pr(E_{e\to c,1})^2 P_2(2, \mathcal{E}_{\mathrm{D},1}) + \left\lceil 2^B\theta\sqrt{1-\rho^2}\right\rceil 2^B P_2(4, \mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2}) \quad (19)$$

where

$$P_2(L,\gamma) = \frac{1}{2^{2L-1}}e^{-\gamma}\sum_{n=0}^{L-1}\left(\frac{1}{n!}\sum_{k=0}^{L-1-n}\binom{2L-1}{k}\right)\gamma^n$$

in round L given by the formula [12, eq:12.1-24] where γ represents the SNR. $\Pr(E_{e\to c,1})$, error probability of an uncorrectable error to occur in the first round, is defined as $\sum_{j=1}^{2}\Pr(|\sqrt{\mathcal{E}_{\mathrm{C},1,j}} + z_{c,j}|^2 \leq \lambda\mathcal{E}_{\mathrm{C},1,j})$. Explicitly, in the first round for only one source being in error, the error probability is obtained by $P_2(1)$ whereas $P_2(2)$ gives the probability for both sources being in error. Accordingly $P_2(2)$ and $P_2(4)$ represent the probabilities in the second round.

Quantization distortion D_q is defined by

$$D_q = \sum_{m=1}^{2^B}\sum_{n=1}^{2^B}\int_{I_{1,m}}\int_{I_{2,n}}\left[(u_1-\hat{u}_1(m))^2 + (u_2-\hat{u}_2(n))^2\right] f(u_1,u_2)du_2du_1 \quad (20)$$

which can be upper bounded by $K_1e^{-2B\ln 2}$ through substituting the value of Δ. In order to emphasize the exponential term the rest of the factors are given by the coefficient K_1 which represents $O(B)$. Basically, the range within $[-\Delta,\Delta]$ is uniformly quantized whereas the tails are bounded as Q functions. In the same way, for the distortion term $D_{e,2}$, which is caused by the channel when both sources are in error regardless of being compatible or incompatible, is defined as given below.

$$D_{e,2} < 2\left(4\Delta^2\Pr(|u_j|<\Delta)\right) + 2\left(\int_{\Delta}^{\infty}(u_j+\Delta)^2 f(u_j)du_j + \int_{-\infty}^{-\Delta}(u_j-\Delta)^2 f(u_j)du_j\right) \quad (21)$$

We used uniform quantization for the area between the quantization levels under the bell curve and the tails are bounded using an approriate bound on Q functions. The distortion caused by one source to be in error are given below for compatible and incompatible pairs

$$D_{e,c,1} < \sum_{n=1}^{2^B}\int_{I_{j,n}}(u_j-\hat{u}_j(n))^2 f(u_j)du_j + |2\theta^2\sqrt{1-\rho^2}|^2, \quad (22)$$

$$D_{e,ic,1} < \sum_{n=1}^{2^B}\int_{I_{j,n}}(u_j-\hat{u}_j(n))^2 f(u_j)du_j + \int_{u_2'=\theta}^{\infty}\left(\theta\sqrt{1-\rho^2} + \sqrt{1-\rho^2}u_2'\right)^2 f(u_2'||U_2'| > \theta\sqrt{1-\rho^2})d_{u_2'}, \quad (23)$$

respectively. Due to space constraints, the derivations of the distortions defined above cannot be given explicitly. But basically, regardless of being compatible both $D_{e,c,1}$ and $D_{e,ic,1}$ contain one source which is correctly decoded. Therefore both distortion terms include D_q for one of the sources. The inner part under the bell-curve, i.e. the range between the quantization levels, and the tails are treated separately also for the case of 1 source being in error conditioned to be inside (for $D_{e,c,1}$) or outside (for $D_{e,ic,1}$) of the compatible zone ($|\rho U_1 - U_2| < \theta$).

The overall distortion at the end of the second round (17) is obtained by substituting error probabilities (18) and (19) with corresponding distortion terms and given in the explicit form as given on the top of the next page by (24) where $K_2 = 1/2$, $K_3, K_4, K_5, K_6, K_{11}$ and , K_{12} are $O(\mathcal{E}_{\mathrm{D},1})$ and the rest of the factors are $O((\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2})^3)$ with $\epsilon(\rho) \in [0,1)$. For simplification in calculations, the energy used by a source on a particular phase is assumed to be half of the energy on the corresponding round, e.g. $\mathcal{E}_{D,1} = 2\mathcal{E}_{D,1,1} = 2\mathcal{E}_{D,1,2}$. Equating the order of the exponentials for the case of low correlation, i.e. $\theta > 2\sqrt{\frac{B\ln 2}{(1-\rho^2)}}$, we can set the relations of the energies as $\mathcal{E}_{\mathrm{C},1} = \frac{\mathcal{E}_{\mathrm{D},2}}{2(\sqrt{\lambda}-1)^2}$ and $\mathcal{E}_{D,2} = (2-\mu)\mathcal{E}_{D,1}$ where μ is an arbitrary constant within the interval $(0,2)$.

Under this condition, the distortion (24) yields the bound (25) where γ, ω and ϑ are functions of $\mathcal{E}_{\mathrm{D},1}$ and ρ and arose from K_3, K_4, K_5, K_6, K_9, K_10 and K_7, K_8, respectively. On the other hand for the highly correlated sources, we set the relations of the energies as $\mathcal{E}_{C,1} = \frac{\mathcal{E}_{D,2}}{(1-\sqrt{\lambda})^2}$ and $\mathcal{E}_{D,2} = (2-\mu)\mathcal{E}_{D,1}$ where μ is an arbitrary constant satisfying $\mu \in (0,2)$ and the final bound becomes as given by (26) whereas α is a function of $\mathcal{E}_{\mathrm{D},1}$ which arose from K_4, K_8, K_{12}, K_{14} together with the distortion terms and given by $\alpha(\mathcal{E}_{\mathrm{D},1},\rho) = \left(4\sqrt{\frac{\mathcal{E}_{\mathrm{D},1}}{\pi N_0}} + 16\frac{\mathcal{E}_{\mathrm{D},1}}{N_0}\right)^{-2/3}$. The amount of energy used by the protocol is arbitrarily close to the energy

$$
\begin{aligned}
D \leq K_1 D_q + K_2 D_{e,ic,1} e^{-\frac{\theta^2(1-\rho^2)}{2}} + \left(K_3\theta\sqrt{1-\rho^2}e^{B\ln 2} + K_4\epsilon(\rho)\right) D_{e,2} e^{(B-3)\ln 2 - \frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{C},1}(\sqrt{\lambda}-1)^2}{2N_0}} \\
+ \left(K_5\theta\sqrt{1-\rho^2}e^{B\ln 2} + K_6\epsilon(\rho)\right) D_{e,c,1} e^{-\frac{\mathcal{E}_{\mathrm{D},1}+2\mathcal{E}_{\mathrm{C},1}(\sqrt{\lambda}-1)^2}{4N_0}} + \left(K_7\theta\sqrt{1-\rho^2}e^{B\ln 2} + K_8\epsilon(\rho)\right) D_{e,2} e^{(B-7)\ln 2 - \frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2}}{2N_0}} \\
+ \left(K_9\theta\sqrt{1-\rho^2}e^{B\ln 2} + K_{10}\epsilon(\rho)\right) D_{e,c,1} e^{-\frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2}}{4N_0}} + \left(K_{11}\theta\sqrt{1-\rho^2}e^{B\ln 2} + K_{12}\epsilon(\rho)\right) D_{e,2} e^{B\ln 2 - \frac{\theta^2(1-\rho^2)}{2} - \frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{C},1}(\sqrt{\lambda}-1)^2}{2N_0}} \\
+ \left(K_{13}\theta\sqrt{1-\rho^2}e^{B\ln 2} + K_{14}\epsilon(\rho)\right) D_{e,2} e^{B\ln 2 - \frac{\theta^2(1-\rho^2)}{2} - \frac{\mathcal{E}_{\mathrm{D},1}+\mathcal{E}_{\mathrm{D},2}}{2N_0}}
\end{aligned}
\tag{24}
$$

$$
D_{low} \leq e^{-\frac{\mathcal{E}_{\mathrm{D},1}(1-\mu/4)}{2N_0}} \gamma(\mathcal{E}_{\mathrm{D},1},\rho) + e^{-\frac{\mathcal{E}_{\mathrm{D},1}(1-\mu/3)}{2N_0}} \delta(\mathcal{E}_{\mathrm{D},1},\rho) + e^{-\frac{\mathcal{E}_{\mathrm{D},1}(3-\mu)}{4N_0}} \vartheta(\mathcal{E}_{\mathrm{D},1},\rho) \tag{25}
$$

$$
D_{high} \leq e^{-\frac{\mathcal{E}_{\mathrm{D},1}(1-\mu/3)}{N_0}} \alpha(\mathcal{E}_{\mathrm{D},1}) + K_6 e^{-\frac{\mathcal{E}_{\mathrm{D},1}(9-2\mu)}{4N_0}} + K_{10} e^{-\frac{\mathcal{E}_{\mathrm{D},1}(7-\mu)}{4N_0}} \tag{26}
$$

consumed by the first data phase assured by vanishing error probability in this round. The exponential behaviour observed in (26) is the same with a single source yields in [7]. Note that there is a difference of factor 1/2 in the exponentials of the significant term in (26) and the information theoretic bound given by (7) where both upper and lower bounds represent the case of highly correlated sources.

IV. Conclusion

We derived lower bounds on the reconstruction error for the transmission of two correlated analog sources in the presence of causal feedback. The bounds are specialized to the case of wideband channels. We obtained improvement with respect to the performance achieved in [6] in terms of the asymptotic behaviour of the derived bounds on distortion with additional feedback. We then introduced a low-latency two-way protocol for the transmission of two correlated random variables over a wideband channel and analyze its asymptotic behaviour with non-coherent detection for Gaussian distribution. The two extremes considered here (25) and (26) show the effect of correlation on the reconstruction fidelity at the receiver. The high correlation case yields the exponential behaviour of the single-source case [7] and benefits from energy accumulation, or the collaboration of the two sources. Low-correlation results insignificantly reduced energy-efficiency. In a large network scenario as in [13], nodes with highly-correlated samples (in the above sense) would collaborate through joint detection at the receiver in order to optimize the energy efficiency of the network.

V. Appendix

A. Appendix I- Lower Bound I

The mutual information $I(\mathbf{U}_m;\mathbf{Y})$ is derived through two different expansions where the first expansion is

$$
\begin{aligned}
I(\mathbf{U}_m;\mathbf{Y}) &= h(\mathbf{U}_m) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m|\mathbf{Y}) \\
&\geq h(\mathbf{U}_m) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m).
\end{aligned}
\tag{27}
$$

The entropies of the two sources are given by

$$
h(\mathbf{U}_m) = \frac{K}{2}\log 2\pi e. \tag{28}
$$

The final term required to derive the first expansion of (27) is given by

$$
\begin{aligned}
h(\mathbf{U}_m - \hat{\mathbf{U}}_m) &\leq \sum_{j=1}^{K} h(U_{m,j} - \hat{U}_{m,j}) \\
&\leq \frac{K}{2}\log\left(\frac{2\pi e}{K}\sum_{j=1}^{K}\mathbb{E}[(U_{m,j}-\hat{U}_{m,j})^2]\right) \\
&\leq K\log\left(\sqrt{2\pi e D_m}\right)
\end{aligned}
\tag{29}
$$

The expansion of $I(\mathbf{U}_m;\mathbf{Y})$ given above is independent of the source number. On the other hand, for the second expansion of the same mutual information we have

$$
\begin{aligned}
& I(\mathbf{U}_m;\mathbf{Y}) \leq I(\mathbf{U}_m;\mathbf{Y},\Phi_m,\Phi_{m'}) \\
& = h(\mathbf{Y}|\Phi_m,\Phi_{m'}) - h(\mathbf{Y}|\mathbf{U}_m,\Phi_m,\Phi_{m'}) \\
& = \sum_{i=1}^{N} h(Y_i|Y^{i-1},\Phi_m,\Phi_{m'}) - \sum_{i=1}^{N} h(Y_i|Y^{i-1},\mathbf{U}_m,\Phi_m,\Phi_{m'}) \\
& \leq \sum_{i=1}^{N} h(Y_i|Y^{i-1},\Phi_m,\Phi_{m'}) \\
& \overset{(a)}{-} \sum_{i=1}^{N} h(Y_i|Y^{i-1},\mathbf{U}_m,\mathbf{X}_m e^{j\phi_m},\mathbf{X}_{m'} e^{i\phi_{m'}},\Phi_m,\Phi_{m'}) \\
& = \sum_{i=1}^{N} h(Y_i|Y^{i-1},\Phi_m,\Phi_{m'}) - \sum_{i=1}^{N} h(Z_i) \\
& \leq \sum_{i=1}^{N} \log\left(1 + \frac{\mathcal{E}_{m,i}+\mathcal{E}_{m',i}}{N_0}\right) \\
& \leq N\log\left(1 + \frac{\sum_{i=1}^{N}(\mathcal{E}_{m,i}+\mathcal{E}_{m',i})}{NN_0}\right) \\
& \leq N\log\left(1 + \frac{K(\mathcal{E}_{m,i}+\mathcal{E}_{m',i})}{NN_0}\right).
\end{aligned}
\tag{30}
$$

where in step (a), $\mathbf{X}_m e^{j\phi_m}$ is introduced due conditioning on $(Y^{i-1},\mathbf{U}_m,\Phi_m)$ in the case of a feedback link between the decoder and the encoder and simply $(\mathbf{U}_m,\Phi_m)$ when no feedback is present. $\mathbf{X}_{m'}e^{i\phi_{m'}}$ can be added since conditioning reduces differential entropy.

B. Appendix II- Lower Bound II

The first expansion is

$$
\begin{aligned}
&I(\mathbf{U}_m;\mathbf{Y}|\mathbf{U}_{m'}) \le I(\mathbf{U}_m;\mathbf{Y}|\mathbf{U}_{m'},\Phi_m,\Phi_{m'})\\
&= h(\mathbf{Y}|\mathbf{U}_{m'},\Phi_m,\Phi_{m'}) - h(\mathbf{Y}|\mathbf{U}_m,\mathbf{U}_{m'},\Phi_m,\Phi_{m'})\\
&= \sum_{i=1}^{N} h(Y_i|Y^{i-1},\mathbf{U}_{m'},\Phi_m,\Phi_{m'})\\
&- \sum_{i=1}^{N} h(Y_i|Y^{i-1},\mathbf{U}_m,\mathbf{U}_{m'},\Phi_m,\Phi_{m'})\\
&\overset{(b)}{=} \sum_{i=1}^{N} h(Y_i|Y^{i-1},\mathbf{U}_{m'},\mathbf{X}_{m'}e^{i\phi_{m'}},\Phi_m,\Phi_{m'})\\
&\overset{(c)}{-} \sum_{i=1}^{N} h(Y_i|Y^{i-1},\mathbf{U}_m,\mathbf{U}_{m'},\mathbf{X}_m e^{j\phi_m},\mathbf{X}_{m'}e^{i\phi_{m'}},\Phi_m,\Phi_{m'})\\
&\overset{(d)}{=} \sum_{i=1}^{N} h(X_{m,i}e^{j\phi_{m,i}}+Z_i|Y^{i-1},\mathbf{U}_{m'},\Phi_m,\Phi_{m'}) - \sum_{i=1}^{N} h(Z_i)\\
&\le \sum_{i=1}^{N} h(X_{m,i}e^{j\phi_{m,i}}+Z_i) - \sum_{i=1}^{N} h(Z_i)\\
&\le N\log\left(\sum_{i=1}^{N}\log(Var(X_{m,i}e^{j\phi_{m,i}}+Z_i)) - \log(Var(\mathbf{Z}))\right)\\
&= N\log\left(1+\frac{K\mathcal{E}_m}{NN_0}\right).
\end{aligned}
\tag{31}
$$

In step (b), $\mathbf{X}_{m'}e^{i\phi_{m'}}$ comes from conditioning on $(Y^{i-1},\mathbf{U}_{m'},\Phi_{m'})$ in the case of feedback and from conditioning on $(\mathbf{U}_{m'},\Phi_{m'})$ when no feedback is present. Similary step (c) stems from conditioning on $(Y^{i-1},\mathbf{U}_m,\mathbf{U}_{m'},\Phi_m,\Phi_{m'})$. And in (d), $\mathbf{X}_{m'}e^{i\phi_{m'}}$ is subtracted from the output signal, which provides $\mathbf{X}_m e^{j\phi_m}$ together with the noise term in the next step. For the second expansion based on the sources, we have

$$
\begin{aligned}
&I(\mathbf{U}_m;\mathbf{Y}|\mathbf{U}_{m'}) = h(\mathbf{U}_m|\mathbf{U}_{m'}) - h(\mathbf{U}_m|\mathbf{U}_{m'},\mathbf{Y})\\
&= h(\mathbf{U}_m|\mathbf{U}_{m'}) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m|\mathbf{U}_{m'},\mathbf{Y})\\
&\ge h(\mathbf{U}_m|\mathbf{U}_{m'}) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m).
\end{aligned}
\tag{32}
$$

The conditional entropy of one source given the other is obtained as

$$
\begin{aligned}
h(\mathbf{U}_1|\mathbf{U}_2) &= -I(\mathbf{U}_1;\mathbf{U}_2) + h(\mathbf{U}_1)\\
&\overset{(e)}{=} -h(\mathbf{U}_2) + h(\mathbf{U}_2|\mathbf{U}_1) + h(\mathbf{U}_1)\\
&= h(\mathbf{U}_2|\mathbf{U}_1)\\
&= \frac{K}{2}\log(1-\rho^2)2\pi e
\end{aligned}
\tag{33}
$$

where in the step (e), we used the equality of the entropies between two standard normal random vectors.

C. Appendix III- Lower Bound III

The mutual information $I(\mathbf{U}_m,\mathbf{U}_{m'};\mathbf{Y})$ is obtained as

$$
\begin{aligned}
&I(\mathbf{U}_m,\mathbf{U}_{m'};\mathbf{Y}) \le I(\mathbf{U}_m,\mathbf{U}_{m'};\mathbf{Y}|\mathbf{\Phi})\\
&= h(\mathbf{Y}|\mathbf{\Phi}) - h(\mathbf{Y}|\mathbf{U}_m,\mathbf{U}_{m'},\mathbf{\Phi})\\
&= h(\mathbf{Y}|\mathbf{\Phi}) - \sum_{i=1}^{N} h(Y_i|Y^{i-1},\mathbf{U}_m,\mathbf{U}_{m'},\mathbf{\Phi})\\
&\overset{(f)}{\le} \sum_{i=1}^{N} h(Y_i|\mathbf{\Phi}) - \sum_{i=1}^{N} h(Y_i|Y^{i-1},\mathbf{U}_m,\mathbf{X}_m e^{j\phi_m},\mathbf{U}_{m'},\mathbf{X}_{m'}e^{i\phi_{m'}},\mathbf{\Phi})\\
&= \sum_{i=1}^{N} h(Y_i|\mathbf{\Phi}) - \sum_{i=1}^{N} h(Z_i).
\end{aligned}
\tag{34}
$$

Note that, in (f) the additional terms in the second differential entropy stem from conditioning on $(Y^{i-1},\mathbf{U}_m,\mathbf{U}_{m'},\Phi_m,\Phi_{m'})$ in the case of feedback and $(\mathbf{U}_m,\mathbf{U}_{m'},\Phi_m,\Phi_{m'})$ when no feedback is present. The variance of the received signal Y_i becomes $\sum_{i=1}^{N} Var(Y_i) = K(\mathcal{E}_m + \mathcal{E}_{m'}) + NN_0$ and the desired mutual information is obtained as

$$
I(\mathbf{U}_m,\mathbf{U}_{m'};\mathbf{Y}|\mathbf{\Phi}) \le N\log(1+\frac{K(\mathcal{E}_m+\mathcal{E}_{m'})}{NN_0}). \tag{35}
$$

And for the second expansion of $I(\mathbf{U}_m,\mathbf{U}_{m'};\mathbf{Y})$, we have

$$
\begin{aligned}
&I(\mathbf{U}_m,\mathbf{U}_{m'};\mathbf{Y}) \ge I(\mathbf{U}_m,\mathbf{U}_{m'};\hat{\mathbf{U}}_m,\hat{\mathbf{U}}_{m'})\\
&\ge h(\mathbf{U}_m,\mathbf{U}_{m'}) - h(\mathbf{U}_m - \hat{\mathbf{U}}_m) - h(\mathbf{U}_{m'} - \hat{\mathbf{U}}_{m'})\\
&\ge \frac{K}{2}\log(2\pi e)^2(1-\rho^2) - \frac{K}{2}\log(2\pi e)^2 D_m D_{m'}\\
&= \frac{K}{2}\log\left(\frac{(1-\rho^2)}{D_p}\right).
\end{aligned}
\tag{36}
$$

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