On the Gaussian Half-Duplex Relay Channel

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***Abstract*—This paper considers the Gaussian half-duplex relay channel (G-HD-RC): a channel model where a source transmits a message to a destination with the help of a relay that can not transmit and receive at the same time. It is shown that the cut-set upper bound on the capacity can be achieved to within a constant gap, regardless of the actual value of the channel parameters, by either Partial-Decode-and-Forward or Compress-and-Forward. The performance of these coding strategies is evaluated with both random and deterministic switch at the relay. Numerical evaluations show that the actual gap is less than what analytically obtained, and that random switch achieves higher rates than deterministic switch. As a result of this analysis, the generalized Degrees-of-Freedom of the G-HD-RC is exactly characterized for this channel. In order to get insights into practical schemes for the G-HD-RC that are less complex than Partial-Decode-and-Forward or Compress-and-Forward, the exact capacity of the Linear Deterministic Approximation (LDA) of the G-HD-RC at high-SNR is determined. It is shown that random switch and correlated non-uniform inputs bits are optimal for the LDA. It is then demonstrated that deterministic switch is to within one bit from the capacity. This latter scheme is translated into a coding strategy for the original G-HD-RC and its optimality to within a constant gap is proved. The gap attained by this scheme is larger than that of Partial-Decode-and-Forward, thereby pointing to an interesting practical tradeoff between gap to capacity and complexity.**

***Index Terms*—Capacity to within a constant gap, Gaussian relay channel, generalized degrees-of-freedom, half-duplex, inner bound, outer bound.**

I. INTRODUCTION

The performance of wireless systems can be enhanced by enabling cooperation between the wireless nodes. The simplest form of cooperation is modeled by the Relay Channel (RC) where a source terminal communicates to a destination with the help of a relay node. In this multi-hop system the relay helps to increase the coverage and the throughput of the network. Relays employed in practical wireless networks can be classified into two categories: Full-Duplex (FD) and Half-Duplex (HD). The relay is said to operate in FD mode if it can receive and transmit simultaneously over the same time-frequency resource, and in HD mode otherwise. There are some relatively expensive relay devices which work in FD, normally used in military communications. However, FD relaying in commercial wireless networks has practical restrictions such as self-interference, which make the implementation of the decoding algorithm challenging. As a result HD relaying proves to be a more practical technology with its relatively simple signal processing. Thus, it is more realistic to assume that the relay operates in HD mode, either in Frequency Division Duplexing (FDD) or Time Division Duplexing (TDD). In FDD, the relay uses one frequency band to transmit and another one to receive, while in TDD, the relay listens for a fraction of time and then transmits in the remaining time. From an application point of view, the HD model fits future 4G networks with relays [3], where the relay communicates over-the-air with the source base station connected to a network infrastructure. We keep our focus on deployment scenarios where the relay works in TDD HD mode.

HD relaying has received considerable attention lately. The main results on this channel model are summarized next.

A. *Related work*

The RC was first introduced by van der Meulen [4] and then thoroughly studied by Cover and El Gamal [5]. In [5], the authors studied the general memoryless FD RC, derived inner and outer bounds on the capacity and established the capacity for some classes of channels. The proposed outer bound is now known as the *max-flow min-cut outer bound*, or cut-set for short, which can be extended to more general memoryless networks [6]. Two relaying strategies were proposed in [5], whose combination is still the largest known achievable rate for a general RC, namely Decode-and-Forward (DF) and Compress-and-Forward (CF). In DF, the relay fully decodes the message sent by the source and then coherently cooperates with the source to communicate this information to the destination. In CF, the relay does not attempt to recover the source message, but it just compresses the received signal and then sends it to the destination. The capacity of the general memoryless RC is known for some special classes of RCs. For example, the cut-set upper bound is known to be tight for the degraded RC, the reversely degraded RC and the semi-deterministic RC [5]. Despite these results, it is known that the cut-set upper bound is not tight in general [7].

The Gaussian half-duplex relay channel (G-HD-RC) was studied in [8] where the author derived an upper and a lower bound on the capacity. The former is based on a cut-set argument, the latter uses Partial-Decode-and-Forward (PDF) strategy[1]. In [8], the transmit- and receive-phases of the relay were assumed fixed a priori and therefore known to all nodes. We shall refer to this specific HD mode of operation as

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[1] PDF is a generalization of DF where the relay only decodes part of the message sent by the source.

deterministic switch [9]. Other recent works on the G-HD-RC with deterministic switch are: [10] where the authors investigated the effect of noise correlation at the destination and at the relay (both FD and HD); [11], where the authors proved that the diversity multiplexing tradeoff of the G-HD-RC meets the 2×1-MISO bound; [12] where the problem of minimizing the network energy consumption was considered.

In [9], Kramer showed that larger rates can be achieved by randomly switching between the transmit- and receive-phases at the relay. In this way, the randomness that lies into the switch can be harnessed to transmit (at most one bit per channel use of) information to the destination. We shall refer to this specific HD mode of operation as *random switch* [9]. An important observation is that there is no need to develop a separate theory for memoryless networks with HD nodes as the HD constraints can be incorporated into the memoryless FD framework [9]. In this work we shall adopt this approach when specializing the FD-RC bounds of [6] to the HD-RC.

The exact characterization of the capacity region of a general memoryless network is challenging. Recently, it has been advocated that progress can be made towards understanding the capacity region by showing that achievable strategies are provably close to outer bounds [13]. In [13], the binary-valued Linear Deterministic Approximation (LDA) of the Gaussian noise channel at high Signal-to-Noise-Ratio (SNR) was proposed. The LDA captures, in a simple deterministic way, the interaction between interfering signals and neglects the noise. In [13] it was shown that Quantize-Map-and-Forward (QMF) is at most $5(N+2)$ bits away from the cut-set upper bound of a FD relay network, where N denotes the number of relays. In [14] it was shown that this $5(N+2)$ bits gap can be extended to deterministic switch HD and/or fading relay networks. In [15] it was shown that lattice codes are optimal to within a gap of $5N$ for HD relay networks with random switch.

The gap result for the multi-relay network [15] gives a gap of 5 bits for the G-HD-RC considered here. The goal of this work is to show that this gap of 5 bits is too pessimistic and that it can be reduced to 1 bit with PDF with random switch [5]. In a companion paper we showed that Noisy Network Coding (NNC), proposed in [16] to reduce the gap of wireless FD multi-relay networks from $5(N+2)$ [14] to $1.26(N+2)$, can be used in HD multi-relay networks with random switch to reduce the gap from $5N$ [15] to $1.96(N+2)$ [17].

B. Contributions

In this work we focus on the G-HD-RC whose exact capacity is unknown. We make progress toward determining its capacity by characterizing its *generalized Degrees-of-Freedom* (gDoF) in closed-form and proving a constant gap result. We also propose an achievable scheme inspired by the LDA, which is provably asymptotically optimal. Our main contribution can be summarized as follows:

1) We determine the exact capacity of the LDA channel: we show that random switch and correlated non-uniform input bits at the relay are optimal. To the best of our knowledge, this is the first closed-form result where the exactly optimal random switch policy is determined. We also show that deterministic switch is at most 1 bit from optimal.
2) We determine the gDoF of the G-HD-RC: we show that both PDF and CF are gDoF optimal, either with deterministic or random switch. We also show that a scheme inspired by the LDA with deterministic switch is gDoF optimal.
3) For the G-HD-RC, we prove that the above achievable schemes are optimal to within a constant gap, uniformly over all channel parameters. In particular, PDF is optimal to within 1 bit, CF to within 1.61 bits, and the scheme inspired by the LDA to within 3 bits. In all cases, the gap is smaller than that available in the literature for the case of one relay.
4) For the three coding schemes, we obtain a closed-form expression for the approximately optimal schedule with deterministic switch, i.e., duration of the transmit- and receive-phases at the relay. This result sheds light on the design of a HD relay node in future wireless networks.
5) We prove that PDF with random switch is exactly optimal for the Gaussian line network, i.e., a G-HD-RC without a direct link between the source and the destination. A closed-form expression for the optimal input distribution with random switch policy is however not available.

C. Paper organization

The rest of the paper is organized as follows. Section II describes the channel model and summarizes our main results. Section III derives the gDoF in closed-form by adapting known FD upper and lower bounds to the HD case. Section IV characterizes exactly the capacity of the LDA and proposes an achievable scheme with deterministic switch for the G-HD-RC that is gDoF optimal (whose achievable rate has a simple closed-form expression). Section V is devoted to the proof that the capacity for the G-HD-RC is achievable to within a constant gap, where more complex schemes achieve, in general, lower gaps. In Section VI, several schemes are numerically compared in terms of rates and gaps. Section VII concludes the paper.

II. System model and main results

In the rest of the paper we adopt the notation convention of [6]. We also use the subscript s for source, r for relay, and d for destination.

A. General memoryless RC

A RC consists of two input alphabets $(\mathcal{X}_s, \mathcal{X}_r)$, two output alphabets $(\mathcal{Y}_r, \mathcal{Y}_d)$ and a transition probability $P_{Y_r,Y_d|X_s,X_r}$. The source has a message $W \in [1 : 2^{NR}]$ for the destination where N denotes the codeword length and R the transmission rate in bits per channel use[2]. At time i, $i \in [1 : N]$, the source maps its message W into a channel input symbol $X_{s,i}(W)$ and the relay maps its past channel observations into a channel input symbol $X_{r,i}(Y_r^{i-1})$. The channel is assumed

[2]Logarithms are in base 2.

to be memoryless, that is, the following Markov chain holds for all $i \in [1:N]$

$$(W, Y_r^{i-1}, Y_d^{i-1}, X_s^{i-1}, X_r^{i-1}) \to (X_{s,i}, X_{r,i}) \to (Y_{r,i}, Y_{d,i}).$$

At time N, the destination makes an estimate of the message W based on all its channel observations Y_d^N as $\widehat{W}(Y_d^N)$. A rate R is said to be ϵ-achievable if, for some block length N, there exists a code such that $\mathbb{P}[\widehat{W} \neq W] \leq \epsilon$ for any $\epsilon > 0$. The capacity is the largest nonnegative rate that is ϵ-achievable.

In this general memoryless framework, the relay can listen and transmit at the same time, i.e., it is a FD node. HD channels are a special case of the memoryless FD framework in the following sense [9]. With a slight abuse of notation compared to the previous paragraph, we let the channel input of the relay be the pair (X_r, S_r), where $X_r \in \mathcal{X}_r$ as before and $S_r \in \{0,1\}$ is the *state* random variable that indicates whether the relay is in receive-mode ($S_r = 0$) or in transmit-mode ($S_r = 1$). The memoryless HD channel transition probability is defined by

$$P_{Y_r,Y_d|X_s,X_r,S_r=0} := P^{(0)}_{Y_r,Y_d|X_s,S_r=0}$$
$$P_{Y_r,Y_d|X_s,X_r,S_r=1} := P^{(1)}_{Y_d|X_s,X_r,S_r=1} P^{(1)}_{Y_r|S_r=1},$$

that is, when the relay is in receive-mode ($S_r = 0$) the outputs Y_r, Y_d are independent of X_r and when the relay is in transmit-mode ($S_r = 1$) the relay output Y_r is independent of everything else. In other words, the (still memoryless) channel is now specified by the two transition probabilities one for each mode of operation [9].

B. The Gaussian half-duplex RC (G-HD-RC)

The single-antenna complex-valued power-constrained G-HD-RC is described by the input/output relationship

$$Y_r = \sqrt{C} X_s \ (1 - S_r) + Z_r \in \mathbb{C}, \tag{1a}$$
$$Y_d = \sqrt{S} X_s + \mathrm{e}^{\mathrm{j}\theta} \sqrt{I} X_r \ S_r + Z_d \in \mathbb{C}, \tag{1b}$$

where the real-valued and non-negative channel power gains C, S, I and the phase θ are constant and therefore known to all terminals. Since a node can compensate for the phase of one of its channel gains, we can assume without loss of generality that the channel gains from the source to the other two terminals are real-valued and nonnegative. The channel inputs are subject to unitary average power constraints without loss of generality, i.e., $\mathbb{E}[|X_u|^2] \leq 1, \ u \in \{s, r\}$. The *switch* random variable S_r is binary. In our model, both X_r and S_r at any given time, are functions of the past received channel outputs. The noise (Z_d, Z_r) is a zero-mean proper-complex Gaussian random vector with, without loss of generality, unit entries on the main diagonal of the covariance matrix. In particular, but not without loss of generality [10], in this work we assume that Z_d and Z_r are independent. In the following we will only consider the G-HD-RC for which $C > 0$ and $I > 0$, since for either $C = 0$ or $I = 0$ the relay is disconnected from either the source or the destination, respectively, so the channel reduces to a point-to-point channel with capacity equal to the direct-link capacity $\log(1 + S)$.

C. The linear deterministic approximation (LDA) of the G-HD-RC at high SNR

The LDA approximates the G-HD-RC in (1) at high SNR. It is a deterministic channel with input-output relationship

$$Y_r = \mathbf{S}^{n-\beta_{\rm sr}} X_s \ (1 - S_r), \tag{2a}$$
$$Y_d = \mathbf{S}^{n-\beta_{\rm sd}} X_s + \mathbf{S}^{n-\beta_{\rm rd}} X_r \ S_r, \tag{2b}$$

for some non-negative integers $\beta_{\rm sr}, \beta_{\rm sd}, \beta_{\rm rd}$, where the vectors Y_r, Y_d, X_r, X_s are of length $n := \max\{\beta_{\rm sr}, \beta_{\rm sd}, \beta_{\rm rd}\}$, $\mathbf{S}$ is the $n \times n$ shift matrix [13], and S_r is the relay binary-valued state random variable.

D. Overview of main results

The capacity of the channel in (1) is unknown. Here we make progress toward determining its capacity by first establishing its gDoF, i.e., an exact "pre-log" capacity characterization in the limit for high SNR, and then by characterizing its capacity to within a constant gap at any finite SNR. Consider $\mathsf{SNR} > 0$ and the parameterization

$$S := \mathsf{SNR}^{\beta_{\rm sd}}, \text{source-destination link}, \tag{3a}$$
$$I := \mathsf{SNR}^{\beta_{\rm rd}}, \text{relay-destination link}, \tag{3b}$$
$$C := \mathsf{SNR}^{\beta_{\rm sr}}, \text{source-relay link}, \tag{3c}$$

for some non-negative real-valued triplet $(\beta_{\rm sd}, \beta_{\rm rd}, \beta_{\rm sr})$[3]. We define:

Definition 1. *The gDoF of the G-HD-RC is*

$$\mathsf{d}^{(\rm HD-RC)} := \lim_{\mathsf{SNR} \to +\infty} \frac{C^{(\rm HD-RC)}}{\log(1 + \mathsf{SNR})},$$

where $C^{(\rm HD-RC)}$ is the capacity of the G-HD-RC.

Definition 2. *The capacity $C^{(\rm HD-RC)}$ is said to be known to within* GAP *bits if one can show achievable rates $R^{(\rm in)}$ and outer bound $R^{(\rm out)}$ such that*

$$R^{(\rm in)} \leq C^{(\rm HD-RC)} \leq R^{(\rm out)} \leq R^{(\rm in)} + \mathsf{GAP}.$$

Our main results are summarized as follows:

Theorem 1. *The gDoF of the G-HD-RC in* (1) *is given by* (4) *at the top of next page. Moreover, the cut-set upper bound is achieved to within the following number of bits*

Achievable scheme	*LDAi*	*CF*	*PDF*
analytical gap	*3*	*1.61*	*1*
numerical gap	*1.32*	*1.16*	*1*

where LDAi is an achievable scheme inspired by the LDA, PDF is Partial-Decode-and-Forward and CF is Compress-and-Forward.

Sections III and V are devoted to the proof of Theorem 1, with an interlude in Section IV where the LDAi scheme is

[3] We use the symbols $(\beta_{\rm sd}, \beta_{\rm rd}, \beta_{\rm sr})$ for both the LDA in (2) and the SNR parameterization in (3) for the channel power gains of the G-HD-RC in (1). In the former case $(\beta_{\rm sd}, \beta_{\rm rd}, \beta_{\rm sr}) \in \mathbb{N}^3$, while in the latter case $(\beta_{\rm sd}, \beta_{\rm rd}, \beta_{\rm sr}) \in \mathbb{R}^3_+$. This choice will not be confusing for the reader. We decided so since the capacity of the LDA is related to the gDoF of the G-HD-RC.

$$\mathsf{d}^{(\mathrm{HD-RC})} = \begin{cases} \beta_{\mathrm{sd}} + \frac{(\beta_{\mathrm{rd}}-\beta_{\mathrm{sd}})(\beta_{\mathrm{sr}}-\beta_{\mathrm{sd}})}{(\beta_{\mathrm{rd}}-\beta_{\mathrm{sd}})+(\beta_{\mathrm{sr}}-\beta_{\mathrm{sd}})} & \text{for } \beta_{\mathrm{sr}} > \beta_{\mathrm{sd}},\ \beta_{\mathrm{rd}} > \beta_{\mathrm{sd}} \\ \beta_{\mathrm{sd}} & \text{otherwise} \end{cases} \tag{4}$$

$$C^{(\mathrm{HD})} = \begin{cases} \beta_{\mathrm{sd}} + \max_{\gamma\in[0,1]} \min\left\{\mathsf{A}\left(\gamma\right), \gamma(\beta_{\mathrm{sr}} - \beta_{\mathrm{sd}})\right\} & \text{for } \beta_{\mathrm{sr}} > \beta_{\mathrm{sd}},\ \beta_{\mathrm{rd}} > \beta_{\mathrm{sd}} \\ \beta_{\mathrm{sd}} & \text{otherwise} \end{cases} \tag{5}$$

derived by mimicking a deterministic switch scheme that is optimal to within 1 bit for the LDA.

The result of Theorem 1 should be compared to a similar result for the FD case. The gDoF of the G-FD-RC is

$$\mathsf{d}^{(\mathrm{FD-RC})} = \beta_{\mathrm{sd}} + \min\{[\beta_{\mathrm{sr}} - \beta_{\mathrm{sd}}]^+, [\beta_{\mathrm{rd}} - \beta_{\mathrm{sd}}]^+\}, \tag{6}$$

and its capacity $C^{(\mathrm{FD-RC})}$ is achievable to within 1 bit by either DF or CF [13]. We notice that HD achieves the same gDoF of FD if $\min\{\beta_{\mathrm{rd}}, \beta_{\mathrm{sr}}\} \le \beta_{\mathrm{sd}}$, in which case the RC behaves gDoF-wise like a point-to-point channel from the source to the destination with gDoF given by β_{sd}. In both FD and HD the gDoF has a *routing* interpretation [13]: if the weakest link from the source to the destination through the relay is smaller than the direct link from the source to the destination, then direct transmission is optimal and the relay can be kept silent, otherwise it is optimal to communicate with the help of the relay, i.e., route part of the information through the relay.

Regarding gaps, we notice that Theorem 1 improves on the 5 bits gap of [15]. Moreover, we notice a tradeoff between the coding scheme complexity and the gap, with lower gaps for more complex schemes (for example, compare the gap of PDF with that of LDAi).

In an attempt to design simple and asymptotically optimal achievable schemes for the G-HD-RC, by following the footsteps of [13], we study the capacity of the LDA. We show:

Theorem 2. *The capacity of the LDA in* (2) *is given by* (5) *at the top of the page, where*

$$\mathsf{A}\left(\gamma\right) := \left(1-\theta^*\left(\gamma\right)\right)\log\frac{1}{1-\theta^*\left(\gamma\right)} + \theta^*\left(\gamma\right)\log\frac{L-1}{\theta^*\left(\gamma\right)},$$
$$\theta^*\left(\gamma\right) := 1 - \max\left\{\frac{1}{L}, \gamma\right\},$$
$$L := 2^{(\beta_{\mathrm{rd}}-\beta_{\mathrm{sd}})},$$

and is achieved with random switch and correlated non-uniform input bits at the relay. Moreover, a scheme with deterministic switch and i.i.d. Bernoulli$(1/2)$ *bits at the relay is at most 1 bit from the capacity in* (5)*.*

Section IV is devoted to the proof of Theorem 2. To the best of our knowledge, Theorem 2 is the first exact capacity result for a HD-RC where the random switch policy and the input distribution have been determined.

III. gDoF for the G-HD-RC

In this section we derive the gDoF of the G-HD-RC in (1). This is accomplished by adapting known bounds for the general memoryless FD-RC [6] to the G-HD-RC by following the methodology introduced by [9].

A. Cut-set upper bounds

This subsection is devoted to the proof of a number of upper bounds that we shall use for the converse part of Theorem 1. From the cut-set bound we have:

Proposition 1. *The capacity of the G-HD-RC is upper bounded as* (7), (8) *and* (9) *at the top of next page, where:*
• *In* (7)*: the distribution $P^*_{X_s,X_r,S_r}$ is the one that maximizes the cut-set upper bound, i.e.,*

$$P^*_{X_s,X_r,S_r} := \arg\max_{P_{X_s,X_r,S_r}} \min\Big\{I(X_s,X_r,S_r;Y_d), \\ I(X_s;Y_r,Y_d|X_r,S_r)\Big\}.$$

• *In* (8)*: the parameter $\gamma := \mathbb{P}[S_r = 0] \in [0,1]$ represents the fraction of time the relay node listens, $\mathcal{H}(\gamma)$ is the binary entropy function defined as*

$$\mathcal{H}(\gamma) := -\gamma\log(\gamma) - (1-\gamma)\log(1-\gamma), \tag{13}$$

the maximization is over the set

$$\gamma \in [0,1], \tag{14}$$
$$|\alpha_1| \le 1, \tag{15}$$
$$(P_{s,0}, P_{s,1}, P_{r,0}, P_{r,1}) \in \mathbb{R}^4_+ \\ : \gamma P_{u,0} + (1-\gamma)P_{u,1} \le 1,\ u \in \{s,r\}, \tag{16}$$

and the mutual information terms $I_1, \ldots, I_4$ are defined as

$$I_1 := \log\left(1 + S\ P_{s,0}\right), \tag{17}$$
$$I_2 := \log\left(1 + SP_{s,1} + IP_{r,1} + 2|\alpha_1|\sqrt{SP_{s,1}\ IP_{r,1}}\right), \tag{18}$$
$$I_3 := \log\left(1 + (C+S)P_{s,0}\right), \tag{19}$$
$$I_4 := \log\left(1 + (1-|\alpha_1|^2)S\ P_{s,1}\right). \tag{20}$$

• *In* (9)*: the terms b_1 and b_2 are defined as*

$$b_1 := \frac{\log\left(1 + (\sqrt{I}+\sqrt{S})^2\right)}{\log\left(1+S\right)} > 1 \text{ since } I > 0, \tag{21}$$
$$b_2 := \frac{\log\left(1+C+S\right)}{\log\left(1+S\right)} > 1 \text{ since } C > 0. \tag{22}$$

Proof: The proof can be found in Appendix A. ■

The upper bound in (7) will be used to prove that PDF with random switch achieves capacity to within 1 bit, the one in (8) to prove that PDF with deterministic switch also achieves capacity to within 1 bit and for numerical evaluations (since we do not know the distribution $P^*_{X_s,X_r,S_r}$ that maximizes the cut-set upper bound in (7)), and the one in (9) for analytical computations such as the derivation of the gDoF.

With the upper bound in Proposition 1 we can show:

$$C^{(\mathrm{HD-RC})} \le \min\left\{I(X_s, X_r, S_r; Y_d), I(X_s; Y_r, Y_d|X_r, S_r)\right\}\Big|_{(X_s,X_r,S_r)\sim P^*_{X_s,X_r,S_r}} \tag{7}$$

$$\le \max\min\left\{\mathcal{H}(\gamma) + \gamma I_1 + (1-\gamma)I_2, \gamma I_3 + (1-\gamma)I_4\right\} =: r^{(\mathrm{CS-HD})} \tag{8}$$

$$\le 2 + \log(1+S)\left(1 + \frac{(b_1-1)(b_2-1)}{(b_1-1)+(b_2-1)}\right) \tag{9}$$

$$C^{(\mathrm{HD-RC})} \ge \min\left\{I(U; Y_r|X_r, S_r) + I(X_s; Y_d|X_r, S_r, U),\ I(X_s, X_r, S_r; Y_d)\right\}\Big|_{\substack{(X_s,X_r,S_r)\sim P^*_{X_s,X_r,S_r} \\ \text{and } U = X_r \text{ or } U = X_r S_r + X_s(1-S_r)}} \tag{10}$$

$$\ge \max\min\left\{I_0^{(\mathrm{PDF})} + \gamma I_5 + (1-\gamma)I_6, \gamma I_7 + (1-\gamma)I_8\right\} =: r^{(\mathrm{PDF-HD})} \tag{11}$$

$$\ge \log(1+S)\left(1 + \frac{(c_1-1)(c_2-1)}{(c_1-1)+(c_2-1)}\right) \tag{12}$$

Proposition 2. *The gDoF of the G-HD-RC is upper bounded by the right hand side of* (4).

Proof: The proof can be found in Appendix B. ∎

B. PDF lower bounds

This subsection is devoted to the proof of a number of lower bounds that we shall use for the direct part of Theorem 1. From the achievable rate with PDF we have:

Proposition 3. *The capacity of the G-HD-RC is lower bounded as* (10), (11) *and* (12) *at the top of the page, where:*
• *In* (10)*: we fix the input P_{U,X_s,X_r,S_r} to evaluate the PDF lower bound; in particular we set P_{X_s,X_r,S_r} to be the same distribution that maximizes the cut-set upper bound in* (7) *and we choose either $U = X_r$ or $U = X_r S_r + X_s(1-S_r)$.*
• *In* (11)*: the parameter $\gamma := \mathbb{P}[S_r = 0] \in [0,1]$ represents the fraction of time the relay node listens, the maximization is over the set* (14)-(16) *as for the cut-set upper bound in* (8)*, the mutual information terms $I_5, \ldots, I_8$ are*

$$I_5 := I_1 \text{ given in } (17), \tag{23}$$

$$I_6 := I_2 \text{ given in } (18), \tag{24}$$

$$I_7 := \log(1 + \max\{C, S\}P_{s,0}) \le I_3 \text{ given in } (19), \tag{25}$$

$$I_8 := I_4 \text{ given in } (20), \tag{26}$$

and $I_0^{(\mathrm{PDF})} := I(S_r; Y_d)$ is computed from the density

$$f_{Y_d}(t) = \frac{\gamma}{\pi v_0}\mathrm{e}^{-|t|^2/v_0} + \frac{1-\gamma}{\pi v_1}\mathrm{e}^{-|t|^2/v_1},\ t \in \mathbb{C}, \tag{27}$$

with $v_0 = 2^{I_5}$ where I_5 is given in (23)*, and $v_1 = 2^{I_6}$ where I_6 is given in* (24)*.*
• *In* (12)*: the terms c_1 and c_2 are*

$$c_1 := \frac{\log(1+I+S)}{\log(1+S)} > 1 \text{ since } I > 0, \tag{28}$$

$$c_2 := \frac{\log(1+\max\{C,S\})}{\log(1+S)} > 1 \text{ since } C > 0. \tag{29}$$

Proof: The proof can be found in Appendix C. ∎

The lower bound in (10) will be compared to the upper bound in (7) to prove that PDF with random switch achieves capacity to within 1 bit, the one in (11) with the one in (8) to prove that PDF with deterministic switch also achieves capacity to within 1 bit and for numerical evaluations, and the one in (12) will be used for analytical computations such as the evaluation of the achievable gDoF.

Remark 1. *In Appendix C, we found that the approximately optimal schedule for PDF with deterministic switch is*

$$\gamma^*_{\mathrm{PDF}} := \frac{(c_1-1)}{(c_1-1)+(c_2-1)} \ \in\ [0,1],$$

where c_1 is given in (28) *and c_2 is given in* (29)*. The expression for γ^*_{PDF} can be understood as follows. Suppose that $\min\{C,I\} \ge S$, otherwise the relay is not exploited in the transmission and hence setting either $\gamma^*_{\mathrm{PDF}} = 0$ or $\gamma^*_{\mathrm{PDF}} = 1$ is approximately optimal. Notice that γ^*_{PDF} is a decreasing function in C and increasing in I. This implies that the stronger C compared to I the lesser the time the relay needs to listen to the channel to (partially) decode the source message – and thus the relay should allocate more time to forward the message to the destination. On the other hand, if $C < I$, more time is needed to learn the message and less time to convey the message to the destination.*

With the lower bound in Proposition 3 we can show:

Proposition 4. *The gDoF of the G-HD-RC is lower bounded by the right hand side of* (4).

Proof: The proof can be found in Appendix D. ∎

Propositions 2 and 4 prove that the gDoF of the G-HD-RC is given by (4) and that PDF achieves the gDoF.

Fig. 1 shows the difference between the gDoF of the G-FD-RC in (6) and that of the G-HD-RC in (4) as a function of β_{sr} and β_{rd}, where without loss of generality we fixed $\beta_{\mathrm{sd}} = 1$. This difference is zero when $\min\{\beta_{\mathrm{rd}}, \beta_{\mathrm{sr}}\} \le \beta_{\mathrm{sd}} = 1$, in which case both the FD and the HD channels are gDoF-wide equivalent to a point-to-point channel without relay. When $\min\{\beta_{\mathrm{rd}}, \beta_{\mathrm{sr}}\} > \beta_{\mathrm{sd}} = 1$, the point-to-point communication channel is outperformed by the RC since now using the relay to convey the information is optimal. Moreover, as expected, the difference is always greater than or equal to zero because in the G-FD-RC the relay can simultaneously listen and transmit; therefore, the G-FD-RC represents an outer bound for the G-HD-RC. The largest difference occurs when

$$C^{(\mathrm{HD})} := \max_{P_{X_s,X_r,S_r}} \min\left\{I(X_s,X_r,S_r;Y_d), I(X_s;Y_r,Y_d|X_r,S_r)\right\} = \max_{P_{X_s,X_r,S_r}} \min\left\{H(Y_d), H(Y_r,Y_d|X_r,S_r)\right\} \quad (30)$$

$$C^{(\mathrm{FD})} := \max_{P_{X_s,X_r}} \min\left\{I(X_s,X_r;Y_d), I(X_s;Y_r,Y_d|X_r)\right\} = \beta_{\mathrm{sd}} + \min\{[\beta_{\mathrm{rd}}-\beta_{\mathrm{sd}}]^+, [\beta_{\mathrm{sr}}-\beta_{\mathrm{sd}}]^+\} \quad (31)$$

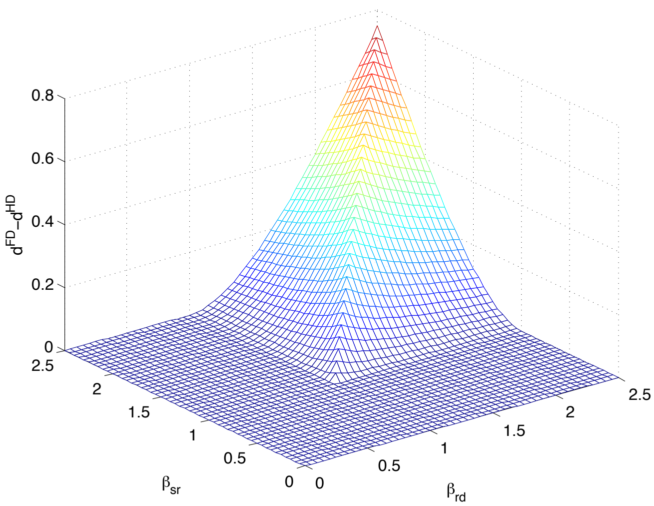


Fig. 1: Difference between the gDoF of the G-FD-RC and of the G-HD-RC, for $\beta_{\mathrm{sd}} = 1$, as a function of β_{sr} and β_{rd}.

$\beta_{\mathrm{rd}} = \beta_{\mathrm{sr}} := \beta_{\mathrm{sd}}\alpha$ in which case

$$\frac{\mathsf{d}^{(\mathrm{FD-RC})}}{\beta_{\mathrm{sd}}} = \max\{1, \alpha\},$$

while

$$\frac{\mathsf{d}^{(\mathrm{HD-RC})}}{\beta_{\mathrm{sd}}} = \max\left\{1, \frac{1+\alpha}{2}\right\},$$

in other words, for $\alpha > 1$ the rate difference between FD and HD grows unboundedly as SNR increases. This might motivate the use of more expensive FD relays in future wireless networks in this regime.

IV. Capacity of the LDA and a simple achievable strategy for the Gaussian noise channel

In the previous section we showed that PDF achieves the gDoF of the G-HD-RC. PDF is based on block Markov encoding and joint decoding [6], which can be too complex to realize in practical systems. For this reason we seek now schemes that are simpler than PDF and that are still gDoF optimal. In order to do so, we consider the LDA in (2). Based on the many recent success stories, such as [13], we first determine the capacity achieving scheme for the LDA and we then try to 'translate' it into a gDoF-optimal scheme for the G-HD-RC. The rational is the "folk's theorem" that the capacity of the LDA gives the gDoF of the corresponding Gaussian noise channel.

In Section IV-A we derive the exact capacity of the LDA and then show an achievable strategy with deterministic switch that is optimal to within 1 bit. In Section IV-B we then mimic the latter to derive an achievable rate for the G-HD-RC at any SNR; this achievable scheme will be referred to as the LDAi.

A. Capacity of the LDA

The capacity of the general memoryless deterministic RC is given by the cut-set bound [5]. For the LDA the cut-set bound evaluates to (5) in Theorem 2, which is proved next.

Proof: The capacity of a HD channel is upper bounded by the capacity of the corresponding FD channel. Therefore for the capacity of the LDA we have $C^{(\mathrm{HD})} \leq C^{(\mathrm{FD})}$ where $C^{(\mathrm{HD})}$ and $C^{(\mathrm{FD})}$ are defined in (30) and (31), respectively, at the top of the page and where $C^{(\mathrm{FD})}$ in (31) is achieved by i.i.d. Bernoulli$(1/2)$ input bits for both the source and the relay [13]. In order to evaluate $C^{(\mathrm{HD})}$ we distinguish two cases:

Regime 1: $\beta_{\mathrm{rd}} \leq \beta_{\mathrm{sd}}$ or $\beta_{\mathrm{sr}} \leq \beta_{\mathrm{sd}}$ in which case $C^{(\mathrm{HD})} \leq C^{(\mathrm{FD})} = \beta_{\mathrm{sd}}$. Since the rate $C^{(\mathrm{HD})} = \beta_{\mathrm{sd}}$ can be achieved by silencing the relay and using i.i.d. Bernoulli$(1/2)$ input bits for the source, we conclude that $C^{(\mathrm{HD})} = C^{(\mathrm{FD})} = \beta_{\mathrm{sd}}$ in this regime.

Regime 2: $\beta_{\mathrm{rd}} > \beta_{\mathrm{sd}}$ and $\beta_{\mathrm{sr}} > \beta_{\mathrm{sd}}$. Here we need to evaluate the expression in (30), for which we need to determine the optimal $H(Y_d)$ and

$$\begin{aligned} H(Y_r, Y_d|X_r, S_r) &= \mathbb{P}[S_r = 0] H(Y_r, Y_d|X_r, S_r = 0) \\ &\quad + \mathbb{P}[S_r = 1] H(Y_r, Y_d|X_r, S_r = 1) \\ &\leq \gamma \max\{\beta_{\mathrm{sr}}, \beta_{\mathrm{sd}}\} + (1-\gamma)\beta_{\mathrm{sd}}. \end{aligned}$$

To upper bound $H(Y_d)$, we write $Y_d = [Y_{d,u}, Y_{d,l}]$, where

• $Y_{d,l}$ contains the lower β_{sd} bits of Y_d. These bits are a combination of the bits of X_s and the lower bits of X_r. The lower bits of X_r are indicated as $X_{r,l}$. With reference to Fig 2(b), $Y_{d,l}$ corresponds to the portion of Y_d containing the "orange bits" labeled $b_1[2]$.

• $Y_{d,u}$ contains the upper $\beta_{\mathrm{rd}} - \beta_{\mathrm{sd}}$ bits of Y_d. These bits only depend on the upper bits of X_r. The upper bits of X_r are indicated as $X_{r,u}$. With reference to Fig 2(b), $Y_{d,u}$ corresponds to the portion of Y_d containing the "green bits" labeled a.

Hence we have

$$\begin{aligned} H(Y_d) &= H(Y_{d,u}, Y_{d,l}) \\ &\leq H(Y_{d,u}) + H(Y_{d,l}) \\ &\leq H(Y_{d,u}) + \beta_{\mathrm{sd}}, \end{aligned}$$

since $Y_{d,l}$ contains β_{sd} bits and where $H(Y_{d,u})$ is computed from the distribution

$$\begin{aligned} \mathbb{P}[Y_{d,u} = y] &= \mathbb{P}[S_r = 0]\mathbb{P}[Y_{d,u} = y|S_r = 0] \\ &\quad + \mathbb{P}[S_r = 1]\mathbb{P}[Y_{d,u} = y|S_r = 1] \\ &= \gamma\delta[y] + (1-\gamma)\mathbb{P}[X_{r,u} = y|S_r = 1] \end{aligned}$$

for $y \in [0 : L-1]$, $L := 2^{\beta_{\mathrm{rd}} - \beta_{\mathrm{sd}}} > 1$, where $\delta[y] = 1$ if $y = 0$ and zero otherwise, and where $\gamma := \mathbb{P}[S_r = 0]$. Let $\mathbb{P}[X_{r,u} = y|S_r = 1] = p_y \in [0,1] : \sum_y p_y = 1$. Then, we

$$
\begin{aligned}
H(Y_{d,u}) &= H\Big(\big[\gamma + (1-\gamma)p_0, (1-\gamma)p_1, \ldots, (1-\gamma)p_{L-1}\big]\Big) \\
&\leq H\left(\left[\gamma + (1-\gamma)p_0,\ \underbrace{(1-\gamma)\frac{1-p_0}{L-1}, \ldots, (1-\gamma)\frac{1-p_0}{L-1}}_{L-1 \text{ times}}\right]\right) \\
&= (1-\theta)\log\frac{1}{1-\theta} + \theta\log\frac{L-1}{\theta}\Big|_{\theta:=(1-\gamma)(1-p_0)\in[0,1-\gamma]}
\end{aligned}
\tag{32}
$$

$$
C^{(\mathrm{HD})} \leq \beta_{\mathrm{sd}} + \max_{\gamma\in[0,1]} \min\left\{(1-\theta^*)\log\frac{1}{1-\theta^*} + \theta^*\log\frac{L-1}{\theta^*}, \gamma(\beta_{\mathrm{sr}} - \beta_{\mathrm{sd}})\right\}
\tag{33}
$$

$$
\begin{aligned}
C^{(\mathrm{HD})} &= \max_{P_{X_s,X_r,S_r}} \min\Big\{H(Y_d), H(Y_r, Y_d|X_r, S_r)\Big\} \\
&\leq \max_{P_{X_s,X_r,S_r}} \min\Big\{H(Y_d|S_r), H(Y_r, Y_d|X_r, S_r)\Big\} + H(S_r) \\
&\leq \max_{\gamma\in[0,1]} \min\{\gamma\beta_{\mathrm{sd}} + (1-\gamma)\max\{\beta_{\mathrm{sd}}, \beta_{\mathrm{rd}}\}, \gamma\max\{\beta_{\mathrm{sd}}, \beta_{\mathrm{sr}}\} + (1-\gamma)\beta_{\mathrm{sd}}\} + 1 \\
&= \beta_{\mathrm{sd}} + \gamma^*_{\mathrm{LDA}}[\beta_{\mathrm{sr}} - \beta_{\mathrm{sd}}]^+ + 1
\end{aligned}
\tag{34}
$$

have that (32), at the top of the page, holds. The upper bound on the entropy $H(Y_{d,u})$ in (32) is maximized by

$$
\theta^* = 1 - \max\{1/L, \gamma\} \iff p_0^* = \frac{[1/L-\gamma]^+}{1-\gamma}. \tag{35}
$$

Therefore, collecting all the bounds, we have that $C^{(\mathrm{HD})}$ in (30) is upper bounded as (33) at the top of the page.

In order to show the achievability of (33) consider the following inputs: the state S_r is Bernoulli$(1-\gamma)$ independent of any other random variable, and X_s and X_r are independent. The source uses i.i.d. Bernoulli$(1/2)$ bits. The relay uses i.i.d. Bernoulli(0) bits for $X_{r,l}$ and $\mathbb{P}[X_{r,u} = y] = p_0^*$ if $y = 0$ and $\mathbb{P}[X_{r,u} = y] = (1-p_0^*)/(L-1)$ otherwise, for p_0^* in (35), i.e., the components of $X_{r,u}$ are neither independent nor uniformly distributed. Notice that the distribution of $X_{r,u}$ in state $S_r = 0$ is irrelevant because its contribution at the destination is zero anyway, so we can assume that the input distribution for X_r is independent of the state S_r. It is straightforward to verify that this choice of input distribution achieves the upper bound in (33) thereby showing capacity in this regime. ∎

Our motivation to determine the capacity of the LDA was to get 'inspiration' to design a simple achievable scheme for the G-HD-RC. Unfortunately, while proving Theorem 2 we found that the capacity achieving distribution of the LDA has two fundamental features that can not be straightforwardly translated into a strategy for the G-HD-RC, namely: (i) the relay employs random switch, and (ii) correlated non-uniform inputs at the relay are optimal. Therefore next we further upper bound the capacity in (33) in the hope to get finally 'inspired'. Consider (34) at the top of the page, where γ^*_{LDA} is the optimal $\gamma := \mathbb{P}[S_r = 0] \in [0,1]$ obtained by equating the two arguments within the min and is given by

$$
\gamma^*_{\mathrm{LDA}} := \begin{cases} \frac{(\beta_{\mathrm{rd}}-\beta_{\mathrm{sd}})}{(\beta_{\mathrm{rd}}-\beta_{\mathrm{sd}})+(\beta_{\mathrm{sr}}-\beta_{\mathrm{sd}})} & \text{if } \beta_{\mathrm{rd}} > \beta_{\mathrm{sd}}, \beta_{\mathrm{sr}} > \beta_{\mathrm{sd}} \\ 0 & \text{otherwise.} \end{cases} \tag{36}
$$

Next we show that the upper bound in (34) is achievable to within 1 bit. This 1 bit represents the maximum amount of information $I(S_r; Y_d)$ that could be conveyed to the destination through a random switch at the relay. If we neglect this 1 bit we can achieve the upper bound in (34) with the scheme shown in Figs. 2(a) and 2(b) for the case $\min\{\beta_{\mathrm{sr}}, \beta_{\mathrm{rd}}\} > \beta_{\mathrm{sd}}$, which is the case where the upper bound differs from direct transmission and for which $X_r \neq 0$. In Phase I / Fig. 2(a) the relay listens and the source sends b_1 (of length β_{sd} bits) directly to the destination and b_2 (of length $\beta_{\mathrm{sr}} - \beta_{\mathrm{sd}}$ bits) to the relay; note that b_2 is below the noise floor at the destination; the duration of Phase I is γ, hence the relay has accumulated $\gamma(\beta_{\mathrm{sr}} - \beta_{\mathrm{sd}})$ bits to forward to the destination. In Phase II / Fig. 2(b) the relay forwards the bits learnt in Phase I to the destination by 'repackaging' them into a (of length $\beta_{\mathrm{rd}} - \beta_{\mathrm{sd}}$ bits); the source keeps sending a new b_1 (of length β_{sd} bits) directly to the destination; note that a does not interfere with b_2 at the destination; the duration of Phase II is such that all the bits accumulated by the relay in Phase I can be delivered to the destination, that is, γ must solve

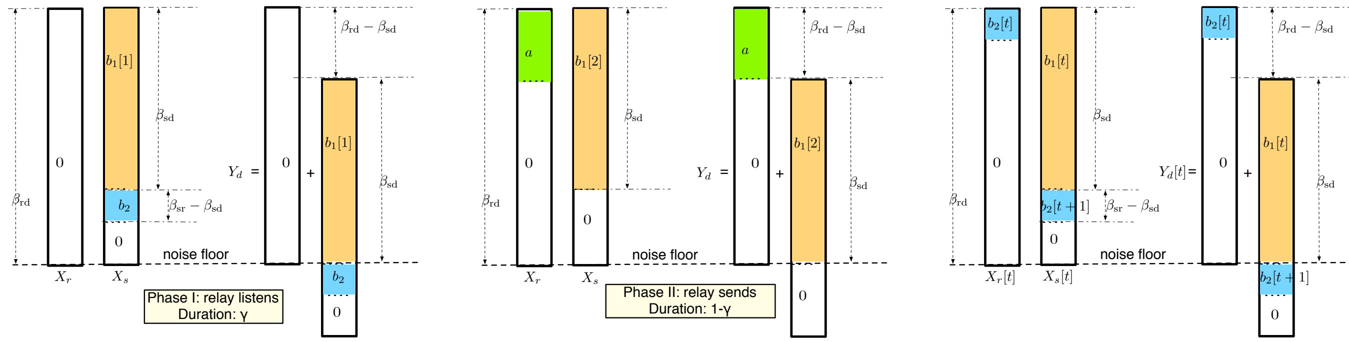
$$
\gamma(\beta_{\mathrm{sr}} - \beta_{\mathrm{sd}}) = (1-\gamma)(\beta_{\mathrm{rd}} - \beta_{\mathrm{sd}}),
$$

giving precisely the optimal γ^*_{LDA} in (36). The total number of bits decoded at the destination is

$$
1 \cdot \beta_{\mathrm{sd}} + \gamma^*_{\mathrm{LDA}} \cdot (\beta_{\mathrm{sr}} - \beta_{\mathrm{sd}}),
$$

which shows that the rate in (34) is achievable to within 1 bit. Notice that the LDA-rate in (34), besides the 1 bit term, looks formally the same as the gDoF in (4) after straightforward manipulations.

The scheme that is optimal within 1 bit for the LDA uses deterministic switch and i.i.d. Bernoulli$(1/2)$ input bits, similarly to the FD optimal scheme in [13]; therefore, similarly to the FD case, we are now in the position to obtain a scheme for the original G-HD-RC. Before we describe the scheme for



(a) HD Phase I ($S_r = 0$).

(b) HD Phase II ($S_r = 1$).

(c) FD.

Fig. 2: An example of achievable strategy for the LDA with $\beta_{\rm sd} \le \beta_{\rm sr} \le \beta_{\rm rd}$. This strategy inspires the LDAi scheme for the G-HD-RC. The HD case is in Figs. 2(a) and 2(b): the scheme uses deterministic switch and it is to within 1 bit of the channel capacity. The FD case is in Fig. 2(c).

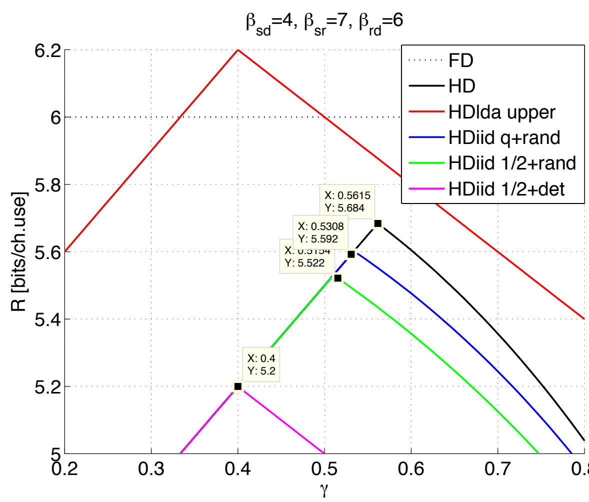


Fig. 3: Comparison of the capacities of the LDA for both HD and FD modes of operation at the relay.

the Gaussian noise channel, let us compare the results obtained for the LDA. The HD optimal strategy in Figs. 2(a) and 2(b) should be compared with the FD optimal strategy in Fig. 2(c). In Fig. 2(c), in a given time slot t, the source sends $b_1[t]$ (of length $\beta_{\rm sd}$ bits) directly to the destination and $b_2[t+1]$ (of length at most $\beta_{\rm sr}-\beta_{\rm sd}$ bits) to the relay; the relay decodes both $b_1[t]$ and $b_2[t+1]$ and forwards $b_2[t+1]$ in the next slot; in slot t the relay sends $b_2[t]$ (of length at most $\beta_{\rm rd}-\beta_{\rm sd}$ bits) to the destination; the number of bits the relay forwards must be the minimum among the number of bits the relay can decode (given by $\beta_{\rm sr}-\beta_{\rm sd}$) and the number of bits that can be decoded at the destination without harming the direct transmission from the source (given by $\beta_{\rm rd}-\beta_{\rm sd}$). Therefore, the total number of bits decoded at the destination is

$$\beta_{\rm sd} + \min\{\beta_{\rm rd}-\beta_{\rm sd}, \beta_{\rm sr}-\beta_{\rm sd}\},$$

which formally looks exactly as the optimal gDoF for the G-FD-RC in (6) in the case the relay is actually used.

Fig. 3 compares the capacities of the FD and HD LDA channels; it also shows some achievable rates for the HD LDA channel. In particular, the capacity of the FD channel is given by (6) (dotted black curve labeled "FD"), the capacity of the HD channel is given by (5) (solid black curve labeled "HD" obtained with the optimal p_0^* in (35)) and its upper bound by (34) (red curve labeled "HDlda upper"). For comparison we also show the performance when the source uses i.i.d. Bernoulli$(1/2)$ bits and the relay uses one of the following strategies: i.i.d. Bernoulli(q) bits and random switch (blue curve labeled "HDiid q+rand" obtained by numerically optimizing $q \in [0,1]$), i.i.d. Bernoulli$(1/2)$ bits and random switch (green curve labeled "HDiid 1/2+rand" obtained with $p_0 = 1/L$ in (32)), and i.i.d. Bernoulli$(1/2)$ bits and deterministic switch (magenta curve labeled "HDiid 1/2+det" and given by $\beta_{\rm sd} + \min\{\gamma[\beta_{\rm sr}-\beta_{\rm sd}]^+, (1-\gamma)[\beta_{\rm rd}-\beta_{\rm sd}]^+\}$). We can draw conclusions from Fig. 3:

- With deterministic switch: i.i.d. Bernoulli$(1/2)$ bits for the relay are optimal but this choice is quite far from capacity (magenta curve vs. solid black curve); this choice however is at most 1 bit from optimal (magenta curve vs. red curve).
- With random switch: the optimal input distribution for the relay is not i.i.d. bits; i.i.d. inputs incur a rate loss (blue curve vs. solid black curve); if in addition we insist on i.i.d. Bernoulli$(1/2)$ bits for the relay we incur a further loss (green curve vs. blue curve).

This shows that for optimal performance the relay inputs are correlated and that random switch must be used.

B. *LDAi: an achievable strategy for the G-HD-RC inspired by the LDA*

We mimic the LDA strategy with deterministic switch from Section IV-A so as to get an achievable rate for the G-HD-RC. We assume $S < C$, otherwise we use direct transmission to achieve $R = \log(1+S)$. The transmission is divided into two phases (it might help to refer to Figs. 2(a) and 2(b)):

- Phase I of duration γ: the transmit signals are

$$X_s[1] = \sqrt{1-\delta} X_{b_1[1]} + \sqrt{\delta} X_{b_2}, \ \delta := \frac{1}{1+S},$$
$$X_r[1] = 0.$$

$$r^{(\mathrm{LDAi-HD})} = \log(1+S) + \frac{\log\left(1+\frac{I}{1+S}\right)\left[\log\left(1+\frac{C}{1+S}\right) - \log\left(1+\frac{S}{1+S}\right)\right]^+}{\log\left(1+\frac{I}{1+S}\right) + \left[\log\left(1+\frac{C}{1+S}\right) - \log\left(1+\frac{S}{1+S}\right)\right]^+} \tag{37}$$

The relay applies successive decoding of $X_{b_1[1]}$ followed by X_{b_2} from

$$Y_r[1] = \sqrt{C}\ \sqrt{1-\delta} X_{b_1[1]} + \sqrt{C}\ \sqrt{\delta} X_{b_2} + Z_r[1],$$

which is possible if (rates are normalized by the total duration of the two phases)

$$\begin{aligned} R_{b_1[1]} &\leq \gamma \log(1+C) - \gamma \log\left(1 + C\frac{1}{1+S}\right) \\ R_{b_2} &\leq \gamma \log\left(1 + C\frac{1}{1+S}\right). \end{aligned} \tag{38}$$

The destination decodes $X_{b_1[1]}$ treating X_{b_2} as noise from

$$Y_d[1] = \sqrt{S}\ \sqrt{1-\delta} X_{b_1[1]} + \sqrt{S}\ \sqrt{\delta} X_{b_2} + Z_d[1],$$

which is possible if

$$R_{b_1[1]} \leq \gamma \log(1+S) - \gamma \log\left(1 + S\frac{1}{1+S}\right). \tag{39}$$

Finally, since we assume $S < C$, Phase I is successful if (38) and (39) are satisfied.

• Phase II of duration $1-\gamma$: the transmit signals are

$$\begin{aligned} X_s[2] &= X_{b_1[2]} \\ X_r[2] &= X_{b_2}, \end{aligned}$$

recall that the bits in a in Fig. 2(b) are the exact same bits in b_2 in Fig. 2(a) just 'repacked' to form a vector with different length, which we mimic here by setting $X_r[2] = X_{b_2}$.

The destination applies successive decoding of X_{b_2} (by exploiting also the information about b_2 that it gathered in the first phase) followed by $X_{b_1[2]}$ from

$$Y_d[2] = \sqrt{S} X_{b_1[2]} + \mathrm{e}^{+\mathrm{j}\theta}\sqrt{I} X_{b_2} + Z_d[2],$$

which is possible if

$$R_{b_2} \leq (1-\gamma)\log\left(1+\frac{I}{1+S}\right) + \gamma \log\left(1+\frac{S}{1+S}\right) \tag{40}$$

$$R_{b_1[2]} \leq (1-\gamma)\log(1+S). \tag{41}$$

• By imposing that the rate R_{b_2} is the same in both phases, that is, that (38) and (40) are equal, we get that γ should be chosen equal to γ^*

$$\gamma^* = \frac{\log\left(1+\frac{I}{1+S}\right)}{\log\left(1+\frac{I}{1+S}\right) + \log\left(1+\frac{C}{1+S}\right) - \log\left(1+\frac{S}{1+S}\right)}. \tag{42}$$

Note that γ^* in (42) tends to γ^*_{LDA} in (36) as SNR increases when using the parameterization in (3). Moreover we give here an explicit closed form expression for the optimal duration of the time the relay listens to the channel.

The rate sent directly from the source to the destination, that is, the sum of (39) and (41), is

$$R_{b_1[1]} + R_{b_1[2]} = \log(1+S) - \underbrace{\gamma^* \log\left(1+\frac{S}{1+S}\right)}_{\in[0,1]}. \tag{43}$$

Therefore the total rate decoded at the destination through the two phases is $r^{(\mathrm{LDAi-HD})} := R_{b_1[1]} + R_{b_1[2]} + R_{b_2}$ as in Proposition 5 below:

Proposition 5. *The capacity of the G-HD-RC is lower bounded as $C^{(\mathrm{HD-RC})} \geq r^{(\mathrm{LDAi-HD})}$, where $r^{(\mathrm{LDAi-HD})}$ is defined in* (37) *at the top of the page.*

We notice that the rate expression for $r^{(\mathrm{LDAi-HD})}$ in (37) (please notice the operator $[\cdot]^+$), which was derived under the assumption $C > S$, is valid for all C since for $C < S$ it reduces to direct transmission from the source to the destination. Moreover we can show that:

Proposition 6. *The LDAi strategy achieves the gDoF upper bound in* (4)*.*

Proof: The proof can be found in Appendix E. ■

Remark 2. *The LDAi scheme can be seen as a specialization of PDF with deterministic switch at the relay combined with the scheduling and power splits inspired by the analysis of the LDA channel. The specialization consists of the classical PDF with sliding window decoding and without coherent codebooks [6]. Thus, the same observations drawn for γ^*_{PDF} in Remark 1 also hold for the LDAi schedule γ^* in* (42)*.*

We can also consider a simple one-shot decoding scheme, where decoding at the destination is performed slot-by-slot without sliding window decoding. Sliding window decoding combines the received signals over two slots. In one such slot, the desired signal is below the noise floor because of our choice of power splits. If this signal is neglected, thereby further simplifying the scheme, the achievable rate R_{b2} in (40) *is without the term $\gamma \log\left(1+\frac{S}{1+S}\right)$. The simplified slot-by-slot decoding scheme still achieves the optimal gDoF.*

Before concluding this section, we point out some important practical aspects of the LDAi that are worth noticing:

1) The proposed scheme is not the classical block Markov encoding scheme with backward decoding; in particular, the destination uses sliding window decoding, which simplifies the decoding procedure and incurs no delay; a further simplification would be to consider a slot-by-slot decoding scheme.
2) The destination uses successive decoding, which is simpler than joint decoding.
3) No power allocation is applied at the source or at the relay across phases; this simplifies the encoding

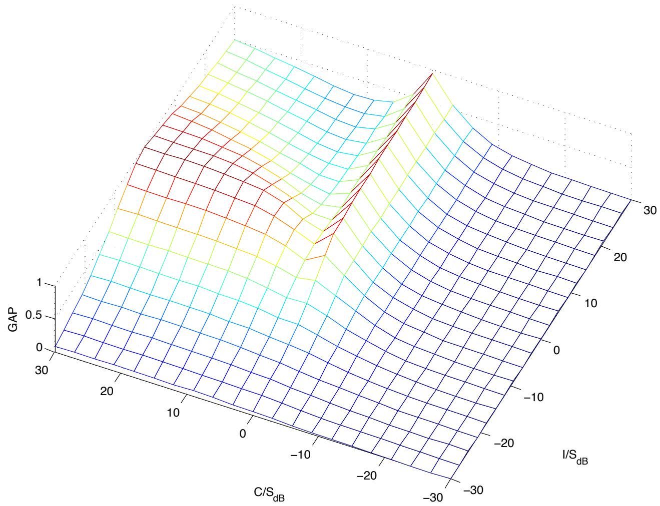


Fig. 4: Gap between the cut-set upper bound and PDF lower bound with deterministic switch for different values of the channel gains between source-relay and relay-destination normalized by the source-destination link in dB scale.

procedure and can be used for time-varying channels as well. The source uses superposition coding, i.e., power split, only to 'route' part of its data through the relay.

V. Analytical Gaps

In Sections III and IV we described upper and lower bounds to determine the gDoF of the G-HD-RC. Moreover in Section IV we proposed a scheme inspired by the analysis of the LDA channel that also achieves the optimal gDoF. We now show that the same upper and lower bounds are to within a constant gap of one another thereby concluding the proof of Theorem 1. We consider both the case of random switch and of deterministic switch for the relay. For completeness we also consider the CF lower bound.

Proposition 7. *PDF with random switch is optimal to within 1 bit.*

Proof: The proof can be found in Appendix G. ■

Proposition 8. *PDF with deterministic switch is optimal to within 1 bit.*

Proof: The proof can be found in Appendix H. ■

The intuition of why the gap does not improve with random switch is that there exist channel parameters for which direct transmission is approximately optimal (when $\min\{C, I\} \leq S$); in the case of direct transmission there are no benefits to use the relay at all and silencing the relay is a case of deterministic switch. This agrees with the PDF result for the G-FD-RC [13] and it is also supported by Fig. 4. Fig. 4 shows the gap between the cut-set upper bound and PDF with deterministic switch as a function of $\frac{C}{S}\big|_{\mathrm{dB}}$ and $\frac{I}{S}\big|_{\mathrm{dB}}$. We see in Fig. 4 that the 1 bit gap is attained only when either the link from the source to the relay or the link from the relay to the destination have the same strength as the direct link between the source and the destination. For this set of parameters, the gDoF is that of a point-to-point channel without a relay. Hence, for this set of parameters the relay can only provide a power gain, which is bounded by 1 bit.

Proposition 9. *LDAi is optimal to within 3 bits.*

Proof: The proof can be found in Appendix I. ■

For completeness, we conclude this section with a discussion on the gap that can be obtained with CF. For the FD-RC, it is known that CF represents a good alternative to PDF in the case when the link between the source and the relay is weaker than the direct link [6]. The CF achievable rate is presented in Appendix F. By using Remark 5 in Appendix F we have:

Proposition 10. *CF with deterministic switch is optimal to within 1.61 bits.*

Proof: The proof can be found in Appendix J. ■

Remark 3. *In Appendix F, we found that the approximately optimal schedule with CF and deterministic switch is given by*

$$\gamma^*_{\mathrm{CF}} := \frac{(c_5-1)}{(c_5-1)+(c_6-1)} \in [0,1],$$
$$c_5 := \frac{\log(1+I+S)}{\log(1+S)} > 1 \text{ since } I > 0,$$
$$c_6 := \frac{\log\left(1+\frac{C}{1+\sigma_0^2}+S\right)}{\log(1+S)} > 1 \text{ since } C > 0.$$

Suppose, as in Remark 1 for PDF, that $\min\{C, I\} \geq S$, otherwise setting either $\gamma^*_{\mathrm{CF}} = 0$ or $\gamma^*_{\mathrm{CF}} = 1$ is approximately optimal. Notice that, although the same observations drawn from the analysis of γ^*_{PDF} in Remark 1 hold, γ^*_{CF} here also depends on the variance of the quantization noise at the relay, i.e., σ_0^2. The schedule γ^*_{CF} is an increasing function of σ_0^2, meaning that the higher σ_0^2 the longer the time the relay should listen to the channel. Therefore, differently from PDF, the approximately optimal schedule does not only depend on the channel gains, but also on the level at which the signal at the relay is quantized.

Proposition 11. *CF with random switch is optimal to within 1.61 bits.*

Proof: Random switch improves on deterministic switch, since at most 1 bit of further information may be conveyed to the destination by randomly switching between the transmit- and receive-phases. Thus, it follows that any rate achievable with deterministic switch is also achievable with random switch, i.e., random switch can not increase the gap. ■

Fig. 5 shows the gap between the cut-set upper bound and CF with deterministic switch as a function of $\frac{C}{S}\big|_{\mathrm{dB}}$ and $\frac{I}{S}\big|_{\mathrm{dB}}$. Fig. 5 shows that the worst gap (of around 1.16 bits) is attained when using the relay gives a rate gain with respect to the point-to-point communication by using the direct link only. This behavior is different from what we observed for PDF. This suggests that if CF is employed the design of an achievable scheme with random switch is more critical.

$$r^{(\mathrm{CS-HD})}|_{S=0} = \max_{\gamma\in[0,1]} \min\left\{\mathcal{H}(\gamma) + (1-\gamma)\log\left(1+\frac{I}{1-\gamma}\right),\ \gamma\log\left(1+\frac{C}{\gamma}\right)\right\} \tag{44}$$

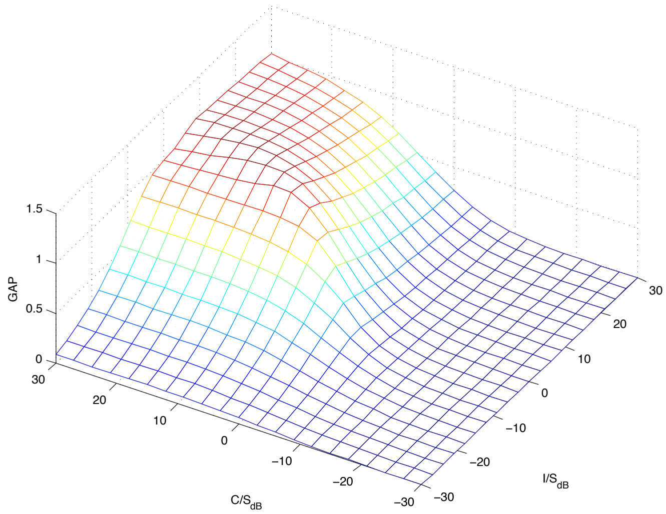


Fig. 5: Gap between the cut-set upper bound and the CF lower bound with deterministic switch for different values of the channel gains between source-relay and relay-destination normalized by the source-destination link in dB scale.

VI. Numerical gaps

In this section we show that the gap results obtained in Section V are pessimistic and are due to crude bounding of the upper and lower bounds, which was necessary in order to obtain rate expressions that can be handled analytically. In order to illustrate our point, we first consider a relay network without the source-destination link, that is, with $S = 0$, in Section VI-A and then we show that the same observations are valid for any network in Section VI-B.

A. G-HD-RC without a source-destination link (single-relay line network)

a) Upper Bound: We start by showing that the (upper bound on the) cut-set upper bound in (8) can be improved upon. Note that we were not able to evaluate the actual cut-set upper bound in (7) so we further bounded it as in (8), which for $S = 0$ reduces to (44) at the top of the page.

The capacity of the G-FD-RC for $S = 0$ is known exactly and is given by the cut-set upper bound, i.e., $C^{(\mathrm{FD})}|_{S=0} = \log(1+\min\{C, I\})$. $C^{(\mathrm{FD})}$ is a trivial upper bound for the capacity of the G-HD-RC. Now we show that our upper bound $r^{(\mathrm{CS-HD})}|_{S=0}$ can be larger than $C^{(\mathrm{FD})}|_{S=0}$. For the case $C = 15/2 > I = 3/2$ we have

$$\begin{aligned} r^{(\mathrm{CS-HD})}|_{S=0} &\geq \min\left\{\mathcal{H}\left(\frac{1}{2}\right) + \frac{1}{2}\log(1+2I),\right.\\ &\qquad\left.\frac{1}{2}\log(1+2C)\right\}\\ &= \log(4) > C^{(\mathrm{FD})}|_{S=0} = \log(2.5). \end{aligned}$$

The reason why the capacity of the FD channel can be smaller than our upper bound $r^{(\mathrm{CS-HD})}|_{S=0}$ is the crude bound $I(S_r; Y_d) \leq H(S_r) = \mathcal{H}(\gamma)$. As mentioned earlier, we needed this bound in order to have an analytical expression for the upper bound. Actually for $S = 0$ the cut-set upper bound in (7) is tight, as we show next.

b) Exact capacity with PDF:

Theorem 3. *In absence of direct link between the source and the destination PDF with random switch achieves the cut-set upper bound.*

Proof: With $S = 0$, the cut-set upper bound in (7) and the PDF lower bound in (10) are the same (see also Appendix G with $S = 0$). ■

c) Improved gap for the LDAi Lower Bound: Despite knowing the capacity expression for $S = 0$ from Theorem 3, its actual evaluation is elusive as it is not clear what the optimal input distribution $P^*_{X_s,X_r,S_r}$ in (7) is. For this reason we next specialize the LDAi strategy to the case $S = 0$ and evaluate its gap from the (upper bound on the) cut-set upper bound in (8).

The LDAi achievable rate in (37) with $S = 0$ is

$$\begin{aligned} r^{(\mathrm{LDAi-HD})}|_{S=0} = \max_{\gamma\in[0,1]} \min\{&\gamma\log(1+C),\\ &(1-\gamma)\log(1+I)\}, \end{aligned}$$

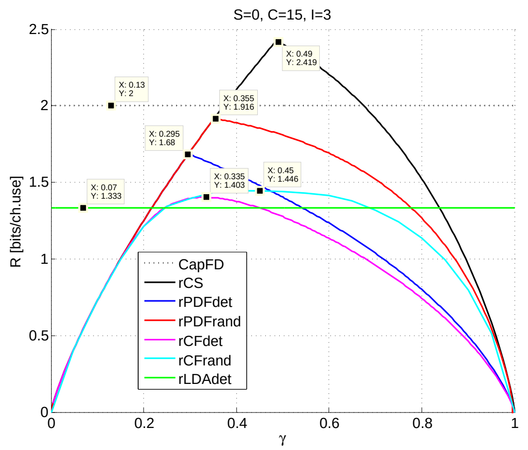
where we left intentionally explicit the optimization with respect to γ, and where we note that $r^{(\mathrm{LDAi-HD})}|_{S=0}$ coincides with the PDF lower bound with deterministic switch at the relay and without optimizing the powers between the relay transmit- and receive-phases.

The gap between the outer bound and $r^{(\mathrm{LDAi-HD})}|_{S=0}$ is less than 3 bits since

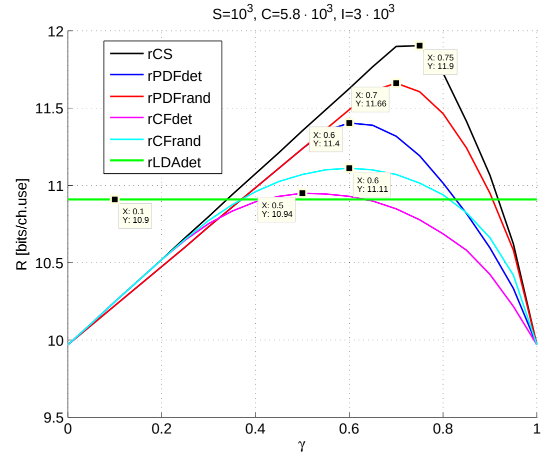
$$\begin{aligned} &\mathsf{GAP} \leq r^{(\mathrm{CS-HD})}|_{S=0} - r^{(\mathrm{LDAi-HD})}|_{S=0}\\ &\leq \max_{\gamma\in[0,1]}\left\{\gamma\log\left(1+\frac{C}{\gamma}\right) - \gamma\log(1+C),\right.\\ &\quad\left.\mathcal{H}(\gamma) + (1-\gamma)\log\left(1+\frac{I}{1-\gamma}\right) - (1-\gamma)\log(1+I)\right\}\\ &\leq \max_{\gamma\in[0,1]}\left\{\gamma\log\left(\frac{1}{\gamma}\right), \mathcal{H}(\gamma) + (1-\gamma)\log\left(\frac{1}{1-\gamma}\right)\right\}\\ &= \max_{\gamma\in[0,1]}\left\{\mathcal{H}(\gamma) + (1-\gamma)\log\left(\frac{1}{1-\gamma}\right)\right\} = 1.5112 \text{ bits.} \end{aligned}$$

Note that the actual gap is even less than 1.5 bits. In fact, by numerically evaluating the difference between the improved upper bound $\min\{C^{(\mathrm{FD})}, r^{(\mathrm{CS-HD})}\}|_{S=0}$ and $r^{(\mathrm{LDAi-HD})}|_{S=0}$ one can found that the gap is at most 1.11 bits.

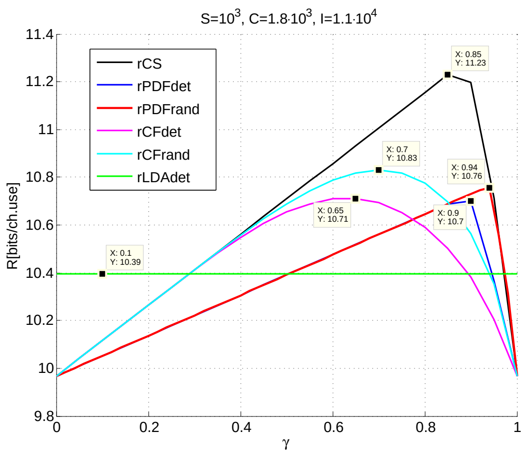
d) Numerical gaps with deterministic switch: Similarly to what done for the LDAi, by numerical evaluations one can find that the PDF strategy with deterministic switch in Remark 4-Appendix C and the CF strategy with deterministic switch in Remark 5-Appendix F are to within 0.80 bits



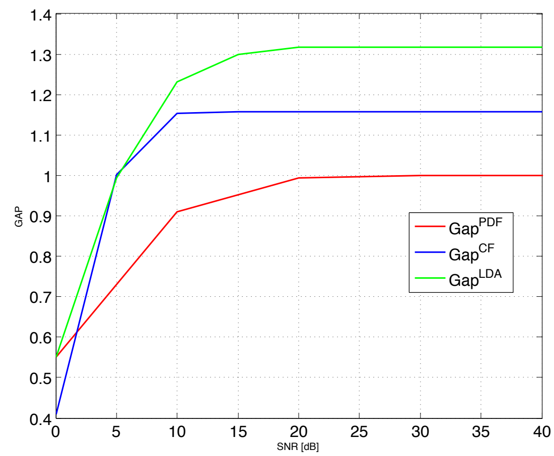
(a) Comparison of the rates of the G-HD-RC without a direct link ($S = 0$) with $C = 15,\ I = 3$.



(b) Comparison of the rates of the G-HD-RC for $S = 30$ dB, $\mathrm{C} = 37.63$ dB, $\mathrm{I} = 34.77$ dB.



(c) Comparison of the rates of the G-HD-RC for $S = 30$ dB, $\mathrm{C} = 32.55$ dB, $\mathrm{I} = 40.41$ dB.



(d) Numerical evaluation of the maximum gap when varying the SNR for $\beta_{\rm sd} = 1$ and $(\beta_{\rm rd}, \beta_{\rm sr}) \in [0, 2.4]$ with deterministic switch.

Fig. 6: Numerical evaluation of the various achievable schemes.

and 1.01 bits, respectively, of the improved upper bound $\min\{C^{(\rm FD)}, r^{(\rm CS-HD)}\}|_{S=0}$. Notice that in these cases there is no information conveyed by the relay to the destination through the switch. Further reductions in the gap with random switch are discussed next for a general network with a direct source-destination link.

Fig. 6(a) shows different upper an lower bounds for the G-HD-RC for $S = 0,\ C = 15,\ I = 3$ versus $\gamma = \mathbb{P}[S_r = 0]$. We see that the cut-set upper bound (solid black curve) exceeds the capacity of the G-FD-RC (dashed black curve). Different achievable strategies are also shown, whose order from the most performing to the least performing is: PDF with random switch (red curve with maximum rate 1.916 bits/ch.use), PDF with deterministic switch (blue curve with maximum rate 1.68 bits/ch.use), CF with random switch (cyan curve with maximum rate 1.446 bits/ch.use), CF with deterministic switch (magenta curve with maximum rate 1.403 bits/ch.use), and LDAi (green curve with maximum rate 1.333 bits/ch.use). In this particular setting, the maximum rate using the CF strategy with random switch (cyan curve with maximum rate 1.446 bits/ch.use) is achieved for $\mathbb{P}[Q = 0, S_r = 0] = 0, \mathbb{P}[Q = 0, S_r = 1] \approx 0.33, \mathbb{P}[Q = 1, S_r = 0] \approx 0.45, \mathbb{P}[Q = 1, S_r = 1] \approx 0.22$. This is due to the absence of the direct link ($S = 0$) between the source and the destination. Actually, since the source can communicate with the destination only through the relay, it is necessary a coordination between the transmissions of the source and those of the relay. This coordination is possible thanks to the time-sharing random variable Q, i.e., when $Q = 0$ the source stays silent while when $Q = 1$ the source transmits.

B. *G-HD-RC with a source-destination link*

Fig. 6(b) and Fig. 6(c) show the rates achieved by using the different achievable schemes presented in the previous sections for a channel with $S > 0$. In Fig. 6(b) the channel conditions are such that PDF outperforms CF, while in Fig. 6(c) the opposite holds. In Fig. 6(b) the PDF strategy with random

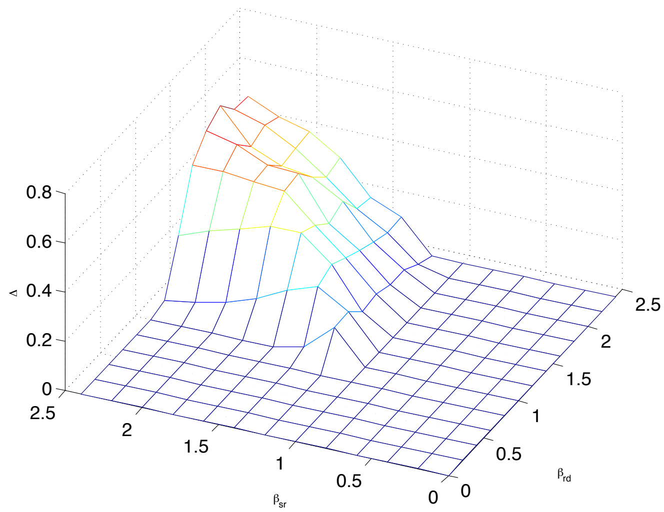


Fig. 7: $\Delta = r^{(\mathrm{PDF-HD})}|_{I_0^{\mathrm{PDF}}=I_0^{\mathrm{opt}}} - r^{(\mathrm{PDF-HD})}|_{I_0^{\mathrm{PDF}}=0}$ at $\mathrm{SNR} = 20\mathrm{dB}$ for $\beta_{\mathrm{sd}} = 1$ as a function of $(\beta_{\mathrm{rd}}, \beta_{\mathrm{sr}}) \in [0, 2.4]$.

switch (red curve with maximum rate 11.66 bits/ch.use) outperforms both the CF with random switch (cyan curve with maximum rate 11.11 bits/ch.use) and the PDF with deterministic switch (blue curve with maximum rate 11.4 bits/ch.use); then the PDF with deterministic switch outperforms the CF with deterministic switch (magenta curve with maximum rate 10.94 bits/ch.use), which is also encompassed by the CF with random switch. Differently from the case without direct link, we observe that the maximum CF rates both in Fig. 6(b) and in Fig. 6(c) are achieved with the choice $Q = \emptyset$, i.e., the time-sharing random variable Q is a constant. This is due to the fact that the source is always heard by the destination even when the relay transmits so there is no need for the source to remain silent when the relay sends.

Fig. 6(d) shows, as a function of SNR and for $\beta_{\mathrm{sd}} = 1, (\beta_{\mathrm{rd}}, \beta_{\mathrm{sr}}) \in [0, 2.4]$, the maximum gap between the cut-set upper bound $r^{(\mathrm{CS-HD})}$ in (8) and the following lower bounds with deterministic switch: the PDF lower bound obtained from $r^{(\mathrm{PDF-HD})}$ in (11) with $I_0^{(\mathrm{PDF})} = 0$, the CF lower bound in Remark 5 in Appendix F, and the LDAi lower bound in (37). From Fig. 6(d) we observe that the maximum gap with PDF is 1 bit as in Proposition 8, but with CF the gap is around 1.16 bits and with LDAi around 1.32 bits, which are lower than the analytical gaps found in Propositions 10 and 9, respectively.

The lower bounds can be improved upon by considering that information can be transmitted through a random switch. However, this improvement depends on the channel gains. If the information can not be routed through the relay because $\min\{C, I\} \leq S$, then the system can not exploit the randomness of the switch, and so $I_0^{\mathrm{PDF}} = 0$ and $I_0^{\mathrm{CF}} = 0$ are approximately optimal (in this case the relay can remain silent). This behavior for the PDF strategy is represented in Fig. 7. In this figure we numerically evaluate the difference between the analytical gap, i.e., the one computed with $I_0^{\mathrm{PDF}} = 0$, and the numerical one, i.e., computed with the optimal I_0^{PDF} indicated as I_0^{opt} (i.e., I_0^{opt} is the actual value of I_0^{PDF}), at a fix $\mathrm{SNR} = 20$ dB and by varying $(\beta_{\mathrm{rd}}, \beta_{\mathrm{sr}})$. We observe that when the information can not be conveyed through the relay, i.e., $\min\{\beta_{\mathrm{rd}}, \beta_{\mathrm{sr}}\} \leq 1$, then $I_0^{\mathrm{PDF}} = 0$ is optimal, since the information only flows through the direct link. On the other hand, when $\min\{\beta_{\mathrm{rd}}, \beta_{\mathrm{sr}}\} > 1$, random switch outperforms deterministic switch. Moreover, from Fig. 7 we observe that, the stronger the channel gains along the path through the relay the larger the amount of information conveyed by random switch.

In Fig. 8 the channel gains are set such that the use of the relay increases the gDoF of the channel ($\beta_{\mathrm{sd}} = 1$ and $(\beta_{\mathrm{rd}}, \beta_{\mathrm{sr}}) \in [1.2, 2.4]$). Here the relay uses PDF. We observe that we have a further improvement in terms of gap by using a random switch (blue curve) instead of using a deterministic switch (red curve). We notice that at high SNR, where the gap is maximum, this improvement is around 0.1 bits. As mentioned earlier, the rate advantage of random switch over deterministic switch depends on the channel gains.

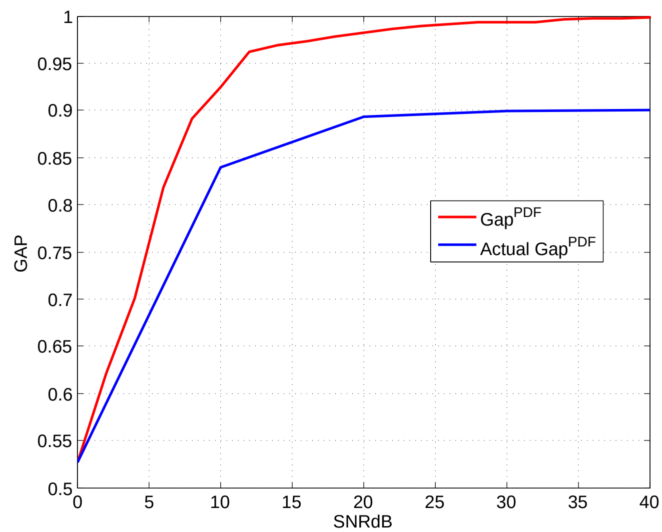


Fig. 8: Numerical evaluation of the maximum gap varying the SNR for $\beta_{\mathrm{sd}} = 1$ and $(\beta_{\mathrm{rd}}, \beta_{\mathrm{sr}}) \in [1.2, 2.4]$ with deterministic (red curve) and random switch (blue curve).

VII. Conclusions

In this work we considered a system where a source communicates with a destination across a Gaussian channel with the help of a half-duplex relay node. We determined the capacity of the linear deterministic approximation of the Gaussian noise channel at high SNR, by showing that random switch and correlated non-uniform input bits at the relay are optimal. We then analyzed the Gaussian noise channel at finite SNR; we derived its gDoF and showed several schemes that achieve the cut-set upper bound on the capacity to within a constant finite gap, uniformly for all channel parameters. We considered both the case of deterministic switch and of random switch at the relay. We showed that random switch is optimal and for the case without a direct link it achieves the exact

$$
\begin{aligned}
C^{(\mathrm{HD-RC})} &\leq \max_{P_{X_s,[X_r,S_r]}} \min\Big\{ I(X_s,[X_r,S_r];Y_d), I(X_s;Y_r,Y_d|[X_r,S_r]) \Big\} && \text{(45a)}\\
&= \max_{P_{X_s,X_r,S_r}} \min\Big\{ I(S_r;Y_d) + I(X_s,X_r;Y_d|S_r), I(X_s;Y_r,Y_d|X_r,S_r) \Big\} && \text{(45b)}\\
&\leq \max_{P_{X_s,X_r,S_r}} \min\Big\{ H(S_r) + I(X_s,X_r;Y_d|S_r), I(X_s;Y_r,Y_d|X_r,S_r) \Big\} && \text{(45c)}\\
&\leq \max\min\Big\{ \mathcal{H}(\gamma) + \gamma I_1 + (1-\gamma)I_2,\ \gamma I_3 + (1-\gamma)I_4 \Big\} =: r^{(\mathrm{CS-HD})} && \text{(45d)}
\end{aligned}
$$

$$
\begin{aligned}
&I(X_s,X_r;Y_d|S_r=0) \leq \log\left(1+SP_{s,0}\right) =: I_1, && \text{(46a)}\\
&I(X_s,X_r;Y_d|S_r=1) \leq \log\Big(1+SP_{s,1}+IP_{r,1}+2|\alpha_1|\sqrt{SP_{s,1}\ IP_{r,1}}\Big) =: I_2, && \text{(46b)}\\
&I(X_s;Y_r,Y_d|X_r,S_r=0) \leq \log\left(1+(C+S)(1-|\alpha_0|^2)P_{s,0}\right) \leq \log\left(1+(C+S)P_{s,0}\right) =: I_3, && \text{(46c)}\\
&I(X_s;Y_r,Y_d|X_r,S_r=1) \leq \log\left(1+S(1-|\alpha_1|^2)P_{s,1}\right) =: I_4, && \text{(46d)}
\end{aligned}
$$

capacity. In general random switch increases the achievable rate at the expense of more complex coding and decoding schemes. For each scheme, we determined in closed form the approximately optimal schedule, i.e., duration of the transmit- and receive-phases at the relay, to shed light into practical HD relays for future wireless networks.

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Appendix A
Proof of Proposition 1

An outer bound on the capacity of the memoryless RC is given by the cut-set outer bound [6, Thm.16.1] that specialized to our G-HD-RC channel gives (45) at the top of the page, where the different steps follow since:

- We indicate the (unknown) distribution that maximizes (45a) as $P^*_{X_s,X_r,S_r}$ in order to get the bound in (7).
- In order to obtain the bound in (45c) we used the fact that, for a discrete binary-valued random variable S_r, we have

$$I(S_r;Y_d) = H(S_r) - H(S_r|Y_d) \leq H(S_r) = \mathcal{H}(\gamma)$$

for some $\gamma := \mathbb{P}[S_r=0] \in [0,1]$ that represents the fraction of time the relay listens and where $\mathcal{H}(\gamma)$ is the binary entropy function in (13). In (45d) the maximization is over the set defined by (14)-(16) and is obtained as an application of the 'Gaussian maximizes entropy' principle as follows. Given any input distribution P_{X_s,X_r,S_r}, the covariance matrix of (X_s,X_r) conditioned on S_r can be written as

$$\mathrm{Cov}\begin{bmatrix}X_s\\X_r\end{bmatrix}\Bigg|_{S_r=\ell} = \begin{bmatrix} P_{s,\ell} & \alpha_\ell\ \sqrt{P_{s,\ell}P_{r,\ell}} \\ \alpha_\ell^*\ \sqrt{P_{s,\ell}P_{r,\ell}} & P_{r,\ell}\end{bmatrix},$$

with $|\alpha_\ell| \leq 1$ for some $(P_{s,0},P_{s,1},P_{r,0},P_{r,1}) \in \mathbb{R}^4_+$ satisfying the average power constraint in (16). Then, a zero-mean jointly Gaussian input with the above covariance matrix maximizes the different mutual information terms in (45c). In particular, we obtain (46) at the top of the page, as defined in (17)-(20) thereby proving the upper bound in (8), which is the same as $r^{(\mathrm{CS-HD})}$ in (45d).

- Regarding (9), the average power constraints at the source and at the relay given in (16) can be expressed as follows. Since the source transmits in both phases we define, for some $\beta \in [0,1]$, the power split

$$P_{s,0} = \frac{\beta}{\gamma},\ P_{s,1} = \frac{1-\beta}{1-\gamma}.$$

Since the relay transmission only affects the destination output for a fraction $(1-\gamma)$ of the time, i.e., when $S_r=1$, the relay must exploit all its available power when $S_r=1$; we therefore split the relay power as

$$P_{r,0} = 0,\ P_{r,1} = \frac{1}{1-\gamma}.$$

With this, the cut-set upper bound $r^{(\mathrm{CS-HD})}$ in (45d) can be rewritten as (47) at the top of next page, where we defined b_1 and b_2 as in (21)-(22), namely

$$
\begin{aligned}
b_1 &:= \frac{\log\Big(1+(\sqrt{I}+\sqrt{S})^2\Big)}{\log\left(1+S\right)} > 1 \text{ since } I>0,\\
b_2 &:= \frac{\log\left(1+C+S\right)}{\log\left(1+S\right)} > 1 \text{ since } C>0.
\end{aligned}
$$

$$
\begin{aligned}
r^{(\mathrm{CS-HD})} = & \max_{(\gamma,|\alpha_1|,\beta)\in[0,1]^3} \min\Bigg\{ \mathcal{H}(\gamma)+\gamma\log\left(1+\frac{S\beta}{\gamma}\right)+(1-\gamma)\log\left(1+\frac{I}{1-\gamma}+\frac{S(1-\beta)}{1-\gamma}+2|\alpha_1|\sqrt{\frac{I}{1-\gamma}\frac{S(1-\beta)}{1-\gamma}}\right), \\
& \gamma\log\left(1+\frac{C\beta}{\gamma}+\frac{S\beta}{\gamma}\right)+(1-\gamma)\log\left(1+(1-|\alpha_1|^2)\frac{S(1-\beta)}{1-\gamma}\right)\Bigg\} \\
\le & \max_{\gamma\in[0,1]} \min\Bigg\{ \mathcal{H}(\gamma)+\gamma\log\left(1+\frac{S}{\gamma}\right)+(1-\gamma)\log\left(1+\left(\sqrt{\frac{I}{1-\gamma}}+\sqrt{\frac{S}{1-\gamma}}\right)^2\right), \\
& \gamma\log\left(1+\frac{C}{\gamma}+\frac{S}{\gamma}\right)+(1-\gamma)\log\left(1+\frac{S}{1-\gamma}\right)\Bigg\} \\
= & \max_{\gamma\in[0,1]} \min\left\{2\mathcal{H}(\gamma)+\gamma\log(\gamma+S)+(1-\gamma)\log\left(1-\gamma+\left(\sqrt{I}+\sqrt{S}\right)^2\right),\right. \\
& \left.\mathcal{H}(\gamma)+\gamma\log(\gamma+C+S)+(1-\gamma)\log(1-\gamma+S)\right\} \\
\le & 2+\max_{\gamma\in[0,1]} \min\left\{\gamma\log(1+S)+(1-\gamma)\log\left(1+\left(\sqrt{I}+\sqrt{S}\right)^2\right), \gamma\log(1+C+S)+(1-\gamma)\log(1+S)\right\} \\
= & 2+\log(1+S)\max_{\gamma\in[0,1]}\min\{\gamma+(1-\gamma)b_1, \gamma b_2+(1-\gamma)\} \\
= & 2+\log(1+S)\left(1+\max_{\gamma\in[0,1]}\min\{(1-\gamma)(b_1-1),\gamma(b_2-1)\}\right) \\
= & 2+\log(1+S)\left(1+\frac{(b_1-1)(b_2-1)}{(b_1-1)+(b_2-1)}\right)
\end{aligned} \tag{47}
$$

Note that the optimal γ is found by equating the two arguments of the $\min$ and is given by

$$\gamma^*_{\mathrm{CS}} := \frac{(b_1-1)}{(b_1-1)+(b_2-1)}.$$

APPENDIX B
PROOF OF PROPOSITION 2

The upper bound in (9) implies

$$
\begin{aligned}
& \mathsf{d}^{(\mathrm{HD-RC})} \\
& \le \lim_{\mathsf{SNR}\to+\infty} \frac{\log(1+S)}{\log(1+\mathsf{SNR})}\left(1+\frac{(b_1-1)(b_2-1)}{(b_1-1)+(b_2-1)}\right) \\
& =\beta_{\mathrm{sd}}\left(1+\frac{[\beta_{\mathrm{rd}}/\beta_{\mathrm{sd}}-1]^+\ [\beta_{\mathrm{sr}}/\beta_{\mathrm{sd}}-1]^+}{[\beta_{\mathrm{rd}}/\beta_{\mathrm{sd}}-1]^+ + [\beta_{\mathrm{sr}}/\beta_{\mathrm{sd}}-1]^+}\right) \\
& =\beta_{\mathrm{sd}} + \frac{[\beta_{\mathrm{rd}}-\beta_{\mathrm{sd}}]^+\ [\beta_{\mathrm{sr}}-\beta_{\mathrm{sd}}]^+}{[\beta_{\mathrm{rd}}-\beta_{\mathrm{sd}}]^+ + [\beta_{\mathrm{sr}}-\beta_{\mathrm{sd}}]^+},
\end{aligned}
$$

since $b_1 \to \max\{\beta_{\mathrm{sd}},\beta_{\mathrm{rd}}\}/\beta_{\mathrm{sd}}$ and $b_2 \to \max\{\beta_{\mathrm{sd}},\beta_{\mathrm{sr}}\}/\beta_{\mathrm{sd}}$ at high SNR, which is equivalent to the right hand side of (4) after straightforward manipulations.

APPENDIX C
PROOF OF PROPOSITION 3

The PDF scheme in [6, Thm.16.3] adapted to the HD model gives the following rate lower bound

$$
\begin{aligned}
& C^{(\mathrm{HD-RC})} \\
& \ge \max_{P_{U,X_s,X_r,S_r}} \min\Big\{I(S_r;Y_d)+I(X_s,X_r;Y_d|S_r), \\
& \qquad I(U;Y_r|X_r,S_r)+I(X_s;Y_d|U,X_r,S_r)\Big\} \\
& \ge \max\min\left\{I_0^{(\mathrm{PDF})}+\gamma I_5+(1-\gamma)I_6, \gamma I_7+(1-\gamma)I_8\right\} \\
& = r^{(\mathrm{PDF-HD})} \text{ in (11)},
\end{aligned}
$$

where for the last inequality we let $\gamma := \mathbb{P}[S_r=0]\in[0,1]$ be the fraction of time the relay listens and, conditioned on $S_r=\ell$, $\ell\in\{0,1\}$, we consider the following jointly Gaussian inputs

$$
\left.\begin{pmatrix} U \\ \frac{X_s}{\sqrt{P_{s,\ell}}} \\ \frac{X_r}{\sqrt{P_{r,\ell}}} \end{pmatrix}\right|_{S_r=\ell} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} 1 & \rho_{s|\ell} & \rho_{r|\ell} \\ \rho^*_{s|\ell} & 1 & \alpha_\ell \\ \rho^*_{r|\ell} & \alpha^*_\ell & 1 \end{bmatrix}\right)
$$

$$
: \begin{bmatrix} 1 & \rho_{s|\ell} & \rho_{r|\ell} \\ \rho^*_{s|\ell} & 1 & \alpha_\ell \\ \rho^*_{r|\ell} & \alpha^*_\ell & 1 \end{bmatrix} \succeq \mathbf{0}.
$$

In particular, we use specific values for the parameters $\{\rho_{s|\ell},\rho_{r|\ell},\alpha_\ell\}_{\ell\in\{0,1\}}$, namely

$$\angle\alpha_1+\theta=0, \tag{48a}$$

$$
\begin{aligned}
& \alpha_0=0 \text{ and either } |\rho_{s|0}|^2 = 1-|\rho_{r|0}|^2=0 \\
& \qquad \text{or } |\rho_{r|0}|^2=1-|\rho_{s|0}|^2=0,
\end{aligned} \tag{48b}
$$

$$\rho_{s|1}=\alpha_1^*,\ \rho_{r|1}=1. \tag{48c}$$

With these definitions, the mutual information terms $I_0^{(\mathrm{PDF})}, I_5,\ldots,I_8$ in (11) are

$$
\begin{aligned}
& I(X_s,X_r;Y_d|S_r=0)=\log(1+SP_{s,0})=:I_5, \\
& I(X_s,X_r;Y_d|S_r=1) \\
& \quad =\log\left(1+SP_{s,1}+IP_{r,1}+2|\alpha_1|\sqrt{SP_{s,1}\ IP_{r,1}}\right)=:I_6,
\end{aligned}
$$

(note $I_5=I_1$ and $I_6=I_2$ because of the assumption in (48a)); next, by using the assumption in (48b), that is, in state $S_r=0$ the inputs X_s and X_r are independent, and that either $U=X_s$

$$
\begin{aligned}
r^{(\mathrm{PDF-HD})} &= \max_{\gamma\in[0,1],|\alpha|\leq 1,\beta\in[0,1]} \min\Big\{ I_0^{(\mathrm{PDF})} + \ \gamma\log\left(1+\frac{\beta S}{\gamma}\right) + \\
&\quad +(1-\gamma)\log\left(1+\frac{S(1-\beta)}{1-\gamma}+\frac{I}{1-\gamma}+2|\alpha|\sqrt{\frac{S(1-\beta)}{1-\gamma}\frac{I}{1-\gamma}}\right), \\
&\quad \gamma\log\left(1+\frac{1}{\gamma}\max\{C\beta, S\beta\}\right)+(1-\gamma)\log\left(1+(1-|\alpha|^2)\frac{S(1-\beta)}{1-\gamma}\right)\Big\} \\
&\geq \max_{\gamma\in[0,1],\beta\in[0,1]} \min\Big\{\gamma\log\left(1+\frac{\beta S}{\gamma}\right)+(1-\gamma)\log\left(1+\frac{S(1-\beta)}{(1-\gamma)}+\frac{I}{1-\gamma}\right), \\
&\qquad \gamma\log\left(1+\frac{1}{\gamma}\max\{\beta C, \beta S\}\right)+(1-\gamma)\log\left(1+\frac{S(1-\beta)}{(1-\gamma)}\right)\Big\} \\
&\geq \max_{\gamma\in[0,1]} \min\{\gamma\log(1+S)+(1-\gamma)\log(1+S+I), \\
&\qquad \gamma\log(1+\max\{C,S\})+(1-\gamma)\log(1+S)\} \\
&= \log(1+S)\max_{\gamma\in[0,1]}\min\{\gamma+(1-\gamma)c_1, \gamma c_2+(1-\gamma)\} \\
&= \log(1+S)\left(1+\max_{\gamma\in[0,1]}\min\{(1-\gamma)(c_1-1), \gamma(c_2-1)\}\right) \\
&= \log(1+S)\left(1+\frac{(c_1-1)(c_2-1)}{(c_1-1)+(c_2-1)}\right) \qquad (49)
\end{aligned}
$$

or $U = X_r$, we have: if $U = X_s$ independent of X_r

$$
\begin{aligned}
&I(U;Y_r|X_r,S_r=0)+I(X_s;Y_d|U,X_r,S_r=0) \\
&= I(X_s;\sqrt{C}X_s+Z_r|X_r,S_r=0) \\
&\quad + I(X_s;\sqrt{S}X_s+Z_d|X_s,X_r,S_r=0) \\
&= \log(1+CP_{s,0}),
\end{aligned}
$$

and if $U = X_r$ independent of X_s

$$
\begin{aligned}
&I(U;Y_r|X_r,S_r=0)+I(X_s;Y_d|U,X_r,S_r=0) \\
&= I(X_r;\sqrt{C}X_s+Z_r|X_r,S_r=0) \\
&\quad + I(X_s;\sqrt{S}X_s+Z_d|X_r,S_r=0) \\
&= \log(1+SP_{s,0});
\end{aligned}
$$

therefore under the assumption in (48b) we have

$$
\begin{aligned}
&I(U;Y_r|X_r,S_r=0)+I(X_s;Y_d|U,X_r,S_r=0) \\
&= \log(1+\max\{C,S\}P_{s,0}) =: I_7;
\end{aligned}
$$

next, by using the assumption in (48c), that is, in state $S_r = 1$ we let $U = X_r$, we have

$$
\begin{aligned}
&I(U;Y_r|X_r,S_r=1)+I(X_s;Y_d|U,X_r,S_r=1) \\
&= I(X_r;Z_r|X_r,S_r=1)+I(X_s;\sqrt{S}X_s+Z_d|X_r,S_r=1) \\
&= I(X_s;\sqrt{S}X_s+Z_d|X_r,S_r=1) \\
&= \log\left(1+S(1-|\alpha_1|^2)P_{s,1}\right) =: I_8,
\end{aligned}
$$

(note $I_7 \leq I_3$ and $I_8 = I_4$); finally

$$
\begin{aligned}
I(S_r;Y_d) &= \mathbb{E}\left[\log\frac{1}{f_{Y_d}(Y_d)}\right] \\
&- [\gamma\log(v_0)+(1-\gamma)\log(v_1)+\log(\pi\mathrm{e})] =: I_0^{(\mathrm{PDF})},
\end{aligned}
$$

where $f_{Y_d}(\cdot)$ is the density of the destination output Y_d, which is a mixture of (proper complex) Gaussian random variables, i.e.,

$$
\begin{aligned}
f_{Y_d}(t) &= \frac{\gamma}{\pi v_0}\mathrm{e}^{-|t|^2/v_0}+\frac{1-\gamma}{\pi v_1}\mathrm{e}^{-|t|^2/v_1},\ t\in\mathbb{C}, \\
v_0 &:= \mathrm{Var}[Y_d|S_r=0] = 2^{I_5}, \\
v_1 &:= \mathrm{Var}[Y_d|S_r=1] = 2^{I_6}.
\end{aligned}
$$

Note that $I_0^{(\mathrm{PDF})} = I(S_r;Y_d) \leq H(S_r) = \mathcal{H}(\gamma)$. This proves the lower bound in (11).

Next we show how to further lower bound the rate in (11) to obtain the rate expression in (12). With the same parameterization of the powers as in Appendix A, namely

$$
P_{s,0} = \frac{\beta}{\gamma},\ P_{s,1} = \frac{1-\beta}{1-\gamma},\ P_{r,0} = 0,\ P_{r,1} = \frac{1}{1-\gamma}.
$$

We have that (49) at the top of the page holds, where we defined c_1 and c_2 as in (28)-(29), namely

$$
\begin{aligned}
c_1 &:= \frac{\log(1+I+S)}{\log(1+S)} > 1 \text{ since } I > 0, \\
c_2 &:= \frac{\log(1+\max\{C,S\})}{\log(1+S)} > 1 \text{ since } C > 0.
\end{aligned}
$$

Notice that $c_i \leq b_i, i = 1,2$, where $b_i, i = 1,2$, are defined in (21)-(22). The optimal γ, indicated by γ^*_{PDF} is given by

$$
\gamma^*_{\mathrm{PDF}} := \frac{(c_1-1)}{(c_1-1)+(c_2-1)} \ \in\ [0,1].
$$

Remark 4. *A further lower bound on the PDF rate $r^{(\mathrm{PDF-HD})}$ in* (11) *can be obtained by trivially lower bounding $I_0^{(\mathrm{PDF})} \geq 0$, which corresponds to a fixed transmit/receive schedule for the relay.*

$$
\begin{aligned}
C^{(\mathrm{HD-RC})} &\geq \max_{P_Q P_{X_s|Q} P_{[X_r,S_r]|Q} P_{\widehat{Y}_r|[X_r,S_r],Y_r,Q}:|Q|\leq 2} \min\Big\{ I(X_s;\widehat{Y}_r,Y_d|[X_r,S_r],Q), \\
&\qquad I(X_s,[X_r,S_r];Y_d|Q) - I(Y_r;\widehat{Y}_r|X_s,[X_r,S_r],Y_d,Q)\Big\} \\
&= \max_{P_Q P_{S_r|Q} P_{X_s|Q} P_{X_r|S_r,Q} P_{\widehat{Y}_r|X_r,Y_r,S_r,Q}:|Q|\leq 2} \min\Big\{ I(X_s;\widehat{Y}_r,Y_d|Q,S_r,X_r), \\
&\qquad I(S_r;Y_d|Q) + I(X_s,X_r;Y_d|S_r,Q) - I(Y_r;\widehat{Y}_r|X_s,X_r,Y_d,S_r,Q)\Big\} \\
&\geq r^{(\mathrm{CF-HD})} \text{ in (51a)} \qquad (50)
\end{aligned}
$$

APPENDIX D
PROOF OF PROPOSITION 4

The lower bound in (12) implies

$$
\begin{aligned}
&\mathsf{d}^{(\mathrm{HD-RC})} \\
&\geq \lim_{\mathsf{SNR}\to+\infty} \frac{\log(1+S)}{\log(1+\mathsf{SNR})}\left(1+\frac{(c_1-1)(c_2-1)}{(c_1-1)+(c_2-1)}\right) \\
&=\beta_{\mathrm{sd}}\left(1+\frac{[\beta_{\mathrm{rd}}/\beta_{\mathrm{sd}}-1]^+\ [\beta_{\mathrm{sr}}/\beta_{\mathrm{sd}}-1]^+}{[\beta_{\mathrm{rd}}/\beta_{\mathrm{sd}}-1]^+ + [\beta_{\mathrm{sr}}/\beta_{\mathrm{sd}}-1]^+}\right) \\
&=\beta_{\mathrm{sd}} + \frac{[\beta_{\mathrm{rd}}-\beta_{\mathrm{sd}}]^+\ [\beta_{\mathrm{sr}}-\beta_{\mathrm{sd}}]^+}{[\beta_{\mathrm{rd}}-\beta_{\mathrm{sd}}]^+ + [\beta_{\mathrm{sr}}-\beta_{\mathrm{sd}}]^+},
\end{aligned}
$$

since $c_1 \to \max\{\beta_{\mathrm{sd}},\beta_{\mathrm{rd}}\}/\beta_{\mathrm{sd}}$ and $c_2 \to \max\{\beta_{\mathrm{sd}},\beta_{\mathrm{sr}}\}/\beta_{\mathrm{sd}}$ at high SNR, which is equivalent to the right hand side of (4) after straightforward manipulations.

APPENDIX E
PROOF OF PROPOSITION 6

The rate in (37) can be further lower bounded as

$$
r^{(\mathrm{LDAi-HD})} \geq -1+\log(1+S)\left(1+\frac{(c_3-1)(c_4-1)}{(c_3-1)+(c_4-1)}\right),
$$

where $c_3 := c_1 = \frac{\log(1+I+S)}{\log(1+S)}$ and $c_4 := b_2 = \frac{\log(1+C+S)}{\log(1+S)}$. The rate above implies

$$
\begin{aligned}
\mathsf{d} &\geq \lim_{\mathsf{SNR}\to+\infty} \frac{\log(1+S)}{\log(1+\mathsf{SNR})}\left(1+\frac{(c_3-1)(c_4-1)}{(c_3-1)+(c_4-1)}\right) \\
&= \beta_{\mathrm{sd}}\left(1+\frac{[\beta_{\mathrm{rd}}/\beta_{\mathrm{sd}}-1]^+\ [\beta_{\mathrm{sr}}/\beta_{\mathrm{sd}}-1]^+}{[\beta_{\mathrm{rd}}/\beta_{\mathrm{sd}}-1]^+ + [\beta_{\mathrm{sr}}/\beta_{\mathrm{sd}}-1]^+}\right) \\
&= \beta_{\mathrm{sd}} + \frac{[\beta_{\mathrm{rd}}-\beta_{\mathrm{sd}}]^+\ [\beta_{\mathrm{sr}}-\beta_{\mathrm{sd}}]^+}{[\beta_{\mathrm{rd}}-\beta_{\mathrm{sd}}]^+ + [\beta_{\mathrm{sr}}-\beta_{\mathrm{sd}}]^+},
\end{aligned}
$$

since $c_3 \to \max\{\beta_{\mathrm{sd}},\beta_{\mathrm{rd}}\}/\beta_{\mathrm{sd}}$ and $c_4 \to \max\{\beta_{\mathrm{sd}},\beta_{\mathrm{sr}}\}/\beta_{\mathrm{sd}}$ at high SNR, which is equivalent to the right hand side of (4) after straightforward manipulations.

APPENDIX F
ACHIEVABLE RATE WITH CF

Proposition 12. *The capacity of the G-HD-RC is lower bounded as*

$$
\begin{aligned}
&C^{(\mathrm{HD-RC})} \geq r^{(\mathrm{CF-HD})} \\
&:= \max\min\Big\{ I_0^{(\mathrm{CF})} + \sum_{(i,j)\in[0:1]^2} \gamma_{ij} I_{9,ij},\ \sum_{(i,j)\in[0:1]^2} \gamma_{ij} I_{10,ij} \Big\},
\end{aligned}
\tag{51a}
$$

where the maximization is over

$$
\gamma_{ij} \in [0,1] : \sum_{(i,j)\in[0:1]^2} \gamma_{ij} = 1, \tag{51b}
$$

$$
P_{s,i} \geq 0 : \sum_{(i,j)\in[0:1]^2} \gamma_{ij}\ P_{s,i} \leq 1, \tag{51c}
$$

$$
P_{r,ij} \geq 0 : \sum_{(i,j)\in[0:1]^2} \gamma_{ij}\ P_{r,ij} \leq 1, \tag{51d}
$$

where the different mutual information terms in (51) *are defined next.*

Proof: The CF scheme in [6, Thm 16.4] adapted to the HD model gives the rate lower bound in (50) at the top of the page, where the mutual information terms $\{I_{9,ij}, I_{10,ij}\}$, $(i,j) \in [0:1]^2$ and $I_0^{(\mathrm{CF})}$ in (51a) are obtained as follows. We consider the following assignment on the inputs and on the auxiliary random variables for each $(i,j) \in [0:1]^2$

$$
\begin{aligned}
&\mathbb{P}[Q=i,S_r=j] = \gamma_{ij} \text{ such that (51b) is satisfied,} \\
&\begin{pmatrix} X_s \\ X_r \end{pmatrix}\Bigg|_{Q=i,S_r=j} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} P_{s,i} & 0 \\ 0 & P_{r,ij} \end{bmatrix}\right) \\
&\text{such that (51c) and (51d) are satisfied,} \\
&\widehat{Y}_r|_{X_r,Y_r,Q=i,S_r=j} = Y_r + \widehat{Z}_{r,ij}, \\
&\widehat{Z}_{r,ij} \sim \mathcal{N}(0,\sigma_{ij}^2) \text{ and independent of everything else,}
\end{aligned}
$$

and in order to meet the constraint that X_s can not depend on S_r conditioned on Q we must impose the constraint that in state $Q=i, S_r=j$ the power of the source only depends on the index i. Then for each $(i,j) \in [0:1]^2$

$$
\begin{aligned}
&I(X_s;\widehat{Y}_r,Y_d|X_r,Q=i,S_r=j) \\
&= \log\left(1+\left(S+\frac{C(1-j)}{1+\sigma_{ij}^2}\right)P_{s,i}\right) =: I_{10,ij},
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
&I(X_s,X_r;Y_d|Q=i,S_r=j)+ \\
&- I(Y_r;\widehat{Y}_r|X_s,X_r,Y_d,Q=i,S_r=j) \\
&= \log(1+SP_{s,i}+IjP_{r,ij}) - \log\left(1+\frac{1}{\sigma_{ij}^2}\right) =: I_{9,ij},
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
I(S_r;Y_d|Q) &= -\sum_{(i,j)} \gamma_{ij}\log(v_{ij}) - \log(\pi \mathrm{e}) \\
&\quad + (\gamma_{00}+\gamma_{01})\ \mathbb{E}\left[\log\frac{1}{f_0(Y)}\Bigg| Q=0\right] \\
&\quad + (\gamma_{10}+\gamma_{11})\ \mathbb{E}\left[\log\frac{1}{f_1(Y)}\Bigg| Q=1\right] =: I_0^{(\mathrm{CF})},
\end{aligned}
$$

$$
\begin{aligned}
r^{(\mathrm{CF-HD})} &\geq \max_{\gamma\in[0,1],\sigma_0^2\geq 0,\beta\in[0,1]} \min\left\{\gamma\log\left(1+\frac{\beta S}{\gamma}\right)-\gamma\log\left(1+\frac{1}{\sigma_0^2}\right)+\right.\\
&\qquad +(1-\gamma)\log\left(1+\frac{(1-\beta)S}{1-\gamma}+\frac{I}{1-\gamma}\right),\\
&\qquad \left.\gamma\log\left(1+\frac{C\beta}{(1+\sigma_0^2)\gamma}+\frac{S\beta}{\gamma}\right)+(1-\gamma)\log\left(1+\frac{(1-\beta)S}{1-\gamma}\right)\right\}\\
&\overset{\beta=\gamma}{\geq} \max_{\gamma\in[0,1],\sigma_0^2\geq 0}\min\left\{\gamma\log(1+S)+(1-\gamma)\log(1+S+I),\right.\\
&\qquad \left.\gamma\log\left(1+\frac{C}{1+\sigma_0^2}+S\right)+(1-\gamma)\log(1+S)\right\}-\gamma\log\left(1+\frac{1}{\sigma_0^2}\right)\\
&= \max_{\gamma\in[0,1],\sigma_0^2\geq 0}\left[\log(1+S)\min\left\{\gamma+(1-\gamma)c_5,\gamma c_6+(1-\gamma)\right\}-\gamma\log\left(1+\frac{1}{\sigma_0^2}\right)\right]\\
&\overset{\gamma=\gamma^*_{\mathrm{CF}}}{\geq} \max_{\sigma_0^2\geq 0}\log(1+S)\left(1+\frac{(c_5-1)(c_6-1)}{(c_5-1)+(c_6-1)}\left(1-\frac{\log\left(1+\frac{1}{\sigma_0^2}\right)}{\log\left(1+\frac{C}{(1+\sigma_0^2)(1+S)}\right)}\right)\right)\\
&\overset{\sigma_0^2=1}{\geq} -1+\log(1+S)\left(1+\frac{(c_5-1)(c_6-1)}{(c_5-1)+(c_6-1)}\right) \qquad (52)
\end{aligned}
$$

where

$$
\begin{aligned}
Y_d|_{Q=0} \sim f_0(t) &:= \frac{\gamma_{00}}{\gamma_{00}+\gamma_{01}}\ \frac{1}{\pi v_{00}}\mathrm{e}^{-|t|^2/v_{00}}\\
&\quad + \frac{\gamma_{01}}{\gamma_{00}+\gamma_{01}}\ \frac{1}{\pi v_{01}}\mathrm{e}^{-|t|^2/v_{01}},\ t\in\mathbb{C},\\
Y_d|_{Q=1} \sim f_1(t) &:= \frac{\gamma_{10}}{\gamma_{10}+\gamma_{11}}\ \frac{1}{\pi v_{10}}\mathrm{e}^{-|t|^2/v_{10}}\\
&\quad + \frac{\gamma_{11}}{\gamma_{10}+\gamma_{11}}\ \frac{1}{\pi v_{11}}\mathrm{e}^{-|t|^2/v_{11}},\ t\in\mathbb{C},\\
v_{ij} &:= \mathrm{Var}[Y_d|Q=i,S_r=j] = 1+S\ P_{s,i}+I\ j\ P_{r,ij}.
\end{aligned}
$$

This proves the lower bound in (51) as a function of $\sigma_{ij}^2, (i,j)\in\{0,1\}^2$.

In order to find the optimal $\sigma_{ij}^2, (i,j)\in\{0,1\}^2$ we reason as follows. $I_{10,ij}$ in (53) is decreasing in σ_{ij}^2 while $I_{9,ij}$ in (54) is increasing. At the optimal point these two rates are the same. Let

$$
C_i := 1+\frac{CP_{s,i}}{1+SP_{s,i}},\ x_i\ := \frac{1}{\sigma_{i0}^2},\ I' := I(S_r,X_r;Y_d|Q),
$$

and rewrite the lower bound in (51) as

$$
\begin{aligned}
r^{(\mathrm{CF-HD})} &= (\gamma_{00}+\gamma_{01})\log(1+SP_{s,0})\\
&\quad + (\gamma_{10}+\gamma_{11})\log(1+SP_{s,1})\\
&\quad - \gamma_{00}\log(1+x_0)-\gamma_{10}\log(1+x_1)\\
&\quad + \min\left\{\gamma_{00}\log(1+x_0C_0)+\gamma_{10}\log(1+x_1C_1),I'\right\}.
\end{aligned}
$$

The solution of

$$
\min_{(x_0,x_1)\in\mathbb{R}_+^2}\left\{\gamma_{00}\log(1+x_0)+\gamma_{10}\log(1+x_1)\right\}
$$

subject to $\quad \gamma_{00}\log(1+x_0C_0)+\gamma_{10}\log(1+x_1C_1)=I'$

can be found to be

$$
x_i = \frac{[\eta C_i-1]^+}{(1-\eta)C_i},\ i\in\{1,2\},
$$

with $\eta\leq 1$ such that

$$
\gamma_{00}\log(1+x_0C_0)+\gamma_{10}\log(1+x_1C_1)=I'.
$$

■

Remark 5. *For the special case of $Q=S_r$, that is, $I_0^{(\mathrm{CF})}=I(S_r;Y_d|Q)=I(Q;Y_d|Q)=0$, the achievable rate in Proposition 12 reduces to*

$$
\begin{aligned}
r^{(\mathrm{CF-HD})} &\geq \max_{(\gamma,\beta)\in[0,1]^2}\min\left\{\gamma I_9+(1-\gamma)I_{10},\right.\\
&\qquad\left.\gamma I_{11}+(1-\gamma)I_{12}\right\}, &&(55a)\\
I_9 &:= \log(1+SP_{s,0})-\log\left(1+\frac{1}{\sigma_0^2}\right), &&(55b)\\
I_{10} &:= \log(1+SP_{s,1}+IP_{r,1})-\log\left(1+\frac{1}{\sigma_1^2}\right), &&(55c)\\
I_{11} &:= \log\left(1+SP_{s,0}+\frac{C}{1+\sigma_0^2}P_{s,0}\right), &&(55d)\\
I_{12} &:= \log(1+SP_{s,1}), &&(55e)\\
\sigma_0^2 &:= \frac{B+1}{(1+A)^{\frac{1}{\gamma}-1}-1},\quad \sigma_1^2=+\infty, &&(55f)\\
A &:= \frac{IP_{r,1}}{1+SP_{s,1}},\quad B := \frac{CP_{s,0}}{1+SP_{s,0}}, &&(55g)\\
P_{s,0} &= \frac{\beta}{\gamma},\quad P_{s,1}=\frac{1-\beta}{1-\gamma},\quad P_{r,1}=\frac{1}{1-\gamma}, &&(55h)
\end{aligned}
$$

where the optimal value for σ_0^2 in (55f) *is obtained by equating the two expressions within the* min *in* (55a).

Proposition 13. *CF with deterministic switch achieves the gDoF upper bound in* (4).

Proof: With the achievable rate in Remark 5 (where here we explicitly write the optimization with respect to σ_0^2) we have that (52) at the top of the page holds, where we defined

c_5 and c_6 as

$$c_5 = c_1 := \frac{\log(1+I+S)}{\log(1+S)} > 1 \text{ since } I > 0 \text{ and as in (28),}$$

$$c_6 := \frac{\log\left(1+\frac{C}{1+\sigma_0^2}+S\right)}{\log(1+S)} > 1 \text{ since } C > 0,$$

and where

$$\gamma^*_{\rm CF} := \frac{(c_5-1)}{(c_5-1)+(c_6-1)} \in [0,1].$$

By reasoning as for the PDF in Appendix D, it follows from the last rate bound that CF also achieves the gDoF in (4). ■

Remark 6. *For the special case of $Q=\emptyset$, i.e., the time-sharing variable Q is a constant, the achievable rate in Proposition 12 reduces to*

$$\begin{aligned}
&r^{(\rm CF-HD)} \geq \max_{P_{X_s}P_{X_r,S_r}P_{\widehat{Y}_r|[X_r,S_r],Y_r}} \min\Big\{ I\left(X_s;\widehat{Y}_r,Y_d|S_r,X_r\right),\\
&I\left(X_s,X_r,S_r;Y_d\right) - I\left(Y_r;\widehat{Y}_r|S_r,X_r,X_s,Y_d\right)\Big\}\\
&\geq \max_{\gamma\in[0,1],\sigma^2} \min\left\{\gamma\log\left(1+S+\frac{C}{1+\sigma^2}\right)+(1-\gamma)\log(1+S),\right.\\
&\quad I\left(S_r;Y_d\right)+\gamma\log(1+S)-\gamma\log\left(1+\frac{1}{\sigma^2}\right)\\
&\quad \left.+(1-\gamma)\log\left(1+S+\frac{I}{1-\gamma}\right)\right\}.
\end{aligned}$$

Note that with $Q=\emptyset$ the source always transmits with constant power, regardless of the state of the relay, while the relay sends only when in transmitting mode. Thus in this particular setting there is no coordination between the source and the relay.

Appendix G
Proof of Proposition 7

Consider the upper bound in (7) and the lower bound in (10). Since the term $I(X_s,X_r,S_r;Y_d)$ is the same in the upper and lower bounds, the gap is given by [4]

$$\begin{aligned}
\mathsf{GAP} \leq & I(X_s;Y_r,Y_d|X_r,S_r) - I(U;Y_r|X_r,S_r)\\
& - I(X_s;Y_d|X_r,S_r,U).
\end{aligned}$$

Next we consider two different choices for U:

[4]Let a lower bound be $\min_{\mathcal{A}}\{f_l(\mathcal{A})\}$ and an upper bound be $\min_{\mathcal{A}}\{f_u(\mathcal{A})\}$. With the definition

$$\mathcal{A}_{u,\min} := \arg\min_{\mathcal{A}}\{f_u(\mathcal{A})\},\ \mathcal{A}_{l,\min} := \arg\min_{\mathcal{A}}\{f_l(\mathcal{A})\},$$

we have $f_u(\mathcal{A}_{u,\min}) \leq f_u(\mathcal{A}_{l,\min})$. This fact implies that the gap is upper bounded as

$$\begin{aligned}
\mathsf{GAP} &\leq \min_{\mathcal{A}}\{f_u(\mathcal{A})\} - \min_{\mathcal{A}}\{f_l(\mathcal{A})\} = f_u(\mathcal{A}_{u,\min}) - f_l(\mathcal{A}_{l,\min})\\
&\leq f_u(\mathcal{A}_{l,\min}) - f_l(\mathcal{A}_{l,\min}) \leq \max_{\mathcal{A}}\{f_u(\mathcal{A}) - f_l(\mathcal{A})\}.
\end{aligned}$$

- For $C \leq S$ we choose $U = X_r$ and

$$\begin{aligned}
\mathsf{GAP} &\leq I(X_s;Y_r,Y_d|X_r,S_r) - I(X_s;Y_d|X_r,S_r)\\
&= I(X_s;Y_r|X_r,S_r,Y_d)\\
&= \mathbb{P}[S_r=0] I(X_s;\sqrt{C}X_s+Z_r|X_r,S_r=0,\sqrt{S}X_s+Z_d)\\
&\quad + \mathbb{P}[S_r=1] I(X_s;Z_r|X_r,S_r=1,\ \sqrt{S}X_s+Z_d)\\
&= \mathbb{P}[S_r=0]\log\left(1+\frac{CP_{s,0}}{1+SP_{s,0}}\right) + \mathbb{P}[S_r=1]\cdot 0\\
&\leq 1\cdot\log\left(1+\frac{SP_{s,0}}{1+SP_{s,0}}\right)\\
&\leq 1 \text{ bit.}
\end{aligned}$$

- For $C > S$ we choose $U = X_rS_r + X_s(1-S_r)$ and

$$\begin{aligned}
\mathsf{GAP} &\leq I(X_s;Y_r,Y_d|X_r,S_r)\\
&\quad - I(X_rS_r+X_s(1-S_r);Y_r|X_r,S_r)\\
&\quad - I(X_s;Y_d|X_r,S_r,\ X_rS_r+X_s(1-S_r))\\
&= \mathbb{P}[S_r=0]\Big(I(X_s;Y_r,Y_d|X_r,S_r=0)\\
&\quad - I(X_s;Y_r|X_r,S_r=0)\Big)\\
&\quad + \mathbb{P}[S_r=1]\Big(I(X_s;Y_r,Y_d|X_r,S_r=1)\\
&\quad - I(X_s;Y_d|X_r,S_r=1)\Big)\\
&= \mathbb{P}[S_r=0]\ I(X_s;Y_d|X_r,S_r=0,Y_r)\\
&\quad + \mathbb{P}[S_r=1]\ I(X_s;Y_r|X_r,S_r=1,Y_d)\\
&= \mathbb{P}[S_r=0] I(X_s;\sqrt{S}X_s+Z_d|X_r,S_r=0,\sqrt{C}X_s+Z_r)\\
&\quad + \mathbb{P}[S_r=1]\ I(X_s;Z_r|X_r,S_r=1,\ \sqrt{S}X_s+Z_d)\\
&= \mathbb{P}[S_r=0]\ \log\left(1+\frac{SP_{s,0}}{1+CP_{s,0}}\right) + \mathbb{P}[S_r=1]\cdot 0\\
&\leq 1\cdot\log\left(1+\frac{CP_{s,0}}{1+CP_{s,0}}\right)\\
&\leq 1 \text{ bit.}
\end{aligned}$$

Appendix H
Proof of Proposition 8

Consider the upper bound in (8) and the lower bound in (11). Recall that $I_1 = I_5$ $I_2 = I_6$ $I_3 \geq I_7$ $I_4 = I_8$ and therefore

$$\begin{aligned}
\mathsf{GAP} &\leq \max\Big\{\mathcal{H}(\gamma) + \gamma I_1 + (1-\gamma)I_2 - \gamma I_5 - (1-\gamma)I_6,\\
&\qquad \gamma I_3 + (1-\gamma)I_4 - \gamma I_7 - (1-\gamma)I_8\Big\}\\
&= \max\Big\{\mathcal{H}(\gamma), \gamma(I_3 - I_7)\Big\}\\
&\leq \max\left\{1, \log\left(\frac{1+CP_{s,0}+SP_{s,0}}{1+\max\{C,S\}P_{s,0}}\right)\right\}\\
&\leq \max\left\{1, \log\left(\frac{1+2\max\{C,S\}P_{s,0}}{1+\max\{C,S\}P_{s,0}}\right)\right\}\\
&\leq \max\Big\{1,1\Big\} = 1 \text{ bit.}
\end{aligned}$$

Appendix I
Proof of Proposition 9

Consider the upper bound in (9) and the lower bound in (37). We distinguish two cases:

$$
\begin{aligned}
\mathsf{GAP} &\le \max\Big\{\mathcal{H}(\gamma)+\gamma I_1+(1-\gamma)I_2-\gamma I_9-(1-\gamma)I_{10},\gamma I_3+(1-\gamma)I_4-\gamma I_{11}-(1-\gamma)I_{12}\Big\}\\
&\le \max\Big\{\mathcal{H}(\gamma)+\gamma\log\left(1+SP_{s,0}\right)+\gamma\log\left(1+\frac{1}{\sigma_0^2}\right)-\gamma\log\left(1+SP_{s,0}\right)\\
&\qquad +(1-\gamma)\log\left(1+(\sqrt{SP_{s,1}}+\sqrt{IP_{r,1}})^2\right)-(1-\gamma)\log\left(1+SP_{s,1}+IP_{r,1}\right),\\
&\qquad \gamma\log\left(1+(C+S)P_{s,0}\right)+(1-\gamma)\log\left(1+SP_{s,1}\right)+\\
&\qquad -\gamma\log\left(1+SP_{s,0}+\frac{CP_{s,0}}{1+\sigma_0^2}\right)-(1-\gamma)\log\left(1+SP_{s,1}\right)\Big\}\\
&\le \max\left\{\mathcal{H}(\gamma)+(1-\gamma)+\gamma\log\left(1+\frac{1}{\sigma_0^2}\right),\gamma\log\left(1+\frac{\frac{\sigma_0^2}{1+\sigma_0^2}CP_{s,0}}{1+SP_{s,0}+\frac{1}{1+\sigma_0^2}CP_{s,0}}\right)\right\}\\
&\le \max\left\{\mathcal{H}(\gamma)+(1-\gamma)+\gamma\log\left(1+\frac{1}{\sigma_0^2}\right),\gamma\log\left(1+\sigma_0^2\right)\right\}\\
&\le 1.6081 \text{ bits}
\end{aligned}
\tag{56}
$$

- Case 1: $S > C$. In this case $r^{(\mathrm{LDAi-HD})} = \log(1+S)$. The gap is

$$
\begin{aligned}
\mathsf{GAP} &\le r^{(\mathrm{CS-HD})} - r^{(\mathrm{LDAi-HD})}\\
&\le 2+\log\left(1+S\right)\frac{(b_1-1)(b_2-1)}{(b_1-1)+(b_2-1)}\\
&\le 2+\log\left(1+S\right)(b_2-1)\\
&= 2+\log\left(1+\frac{C}{1+S}\right)\\
&\le 2+\log\left(1+\frac{S}{1+S}\right)\\
&\le 3 \text{ bits.}
\end{aligned}
$$

- Case 2: $S \le C$. First, by noticing that $\log\left(1+(\sqrt{I}+\sqrt{S})^2\right) \le \log\left(1+I+S\right)+1$, we further upper bound the expression in (9) as

$$
r^{(\mathrm{CS-HD})} \le 2+\log\left(1+S\right) + \frac{\left(\log\left(1+\frac{I}{1+S}\right)+1\right)\log\left(1+\frac{C}{1+S}\right)}{\log\left(1+\frac{I}{1+S}\right)+1+\log\left(1+\frac{C}{1+S}\right)}.
$$

Next we further lower bound $r^{(\mathrm{LDAi-HD})}$ in (37) as

$$
r^{(\mathrm{LDAi-HD})} \ge \log\left(1+S\right) + \frac{\log\left(1+\frac{I}{1+S}\right)\left(\log\left(1+\frac{C}{1+S}\right)-1\right)}{\log\left(1+\frac{I}{1+S}\right)+\log\left(1+\frac{C}{1+S}\right)}.
$$

Hence, with $x = \log\left(1+\frac{I}{1+S}\right), y = \log\left(1+\frac{C}{1+S}\right)$, we have

$$
\begin{aligned}
\mathsf{GAP} &\le r^{(\mathrm{CS-HD})} - r^{(\mathrm{LDAi-HD})}\\
&\le 2+\frac{(x+1)y}{x+1+y}-\frac{x(y-1)}{x+y}\\
&= 2+\frac{x^2+y^2+xy+x}{x^2+y^2+2xy+x+y}\\
&\le 3 \text{ bits.}
\end{aligned}
$$

Appendix J
Proof of Proposition 10

With CF we have that (56) at the top of the page holds, where for σ_0^2 we chose the value

$$
\sigma_0^2 = 2^{\frac{\mathcal{H}(\gamma)+(1-\gamma)}{\gamma}}
$$

by equating the two arguments of the $\max$ (this is so because $\mathcal{H}(\gamma)+(1-\gamma)+\gamma\log\left(1+\frac{1}{\sigma_0^2}\right)$ is decreasing in σ_0^2, while $\log\left(1+\sigma_0^2\right)$ is increasing in σ_0^2). Numerically one can find that with the chosen σ_0^2 the maximum over $\gamma \in [0,1]$ is 1.6081 for $\gamma = 0.3855$.

Note that by choosing $\sigma_0^2 = 1$ the gap would be upper bounded by 2 bits.

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