Cramer-Rao Bounds for Power Delay Profile Fingerprinting based Positioning

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***Abstract*— Power Delay Profile-Fingerprinting (PDP-F) allows to do positioning in multipath and even in NLOS environments. Although many algorithms for position fingerprinting have been developed, analytical investigation in this area is still not matured. In this paper, we derive Cramér-Rao bounds (CRBs) for location dependent parameters (LDPs) when they are finite and perform local identifiability analysis under different path amplitude assumptions. We show that local identifiability of the position vector can be accomplished if a condition for the pulse shape is satisfied even with one path under the assumption that path amplitude is a genuine function of position (anisotropic path attenuation). On the other hand at least two paths are required to achieve local identifiability for a distance dependent attenuation model (isotropic path attenuation) for path amplitudes. In order to simplify the analysis we assume that pulses from different paths are non-overlapping. Fisher Information Matrix (FIM) for LDPs and the position vector is derived to prove the statements.**

***Index Terms*— fingerprinting, local identifiability, localization, Fisher Information Matrix, Cramér-Rao bound**

I. Introduction

Conventional localization techniques such as TOA (Time of Arrival) based algorithms depend on LOS conditions. Moreover more than one Base Station (BS)-Mobile Terminal (MT) links should satisfy LOS conditions to locate the MT. It is not always the case that multiple links satisfy the LOS conditions simultaneously. On the contrary location fingerprinting (LF) (introduced by U.S. Wireless Corp. of San Ramon, Calif.) relies on signal structure characteristics. It exploits the multipath nature of the channel hence the NLOS conditions. By using multipath propagation pattern, the LF creates a signature unique to a given location. The position of the mobile is determined by matching measured signal characteristics from the BS-MT link to an entry of the database. The location corresponding to the highest match of the database entry is considered as the location of the mobile. For LF, it is enough to have only one BS-MT link (multiple BSs are not required) to determine the location of the mobile. Also LF is classified among Direct Location Estimation (DLE) techniques. Ahonen and Eskelinen suggest using the measured Power Delay Profiles (PDPs) in the database [1] for fingerprints. In [2], authors provide deterministic and Bayesian methods for PDP-F based localization. The Gaussian Maximum Likelihood (GML) based PDP-F revealed in this article is also important for the work we have developed in this article in the Rayleigh fading section. It is well known that at least 3 BSs are required for a TOA based localization system to uniquely identify (global identifiability) the location of the MT in 2D. Local identifiability is a similar issue in the sense that the position of the mobile must be uniquely identified around a local neighborhood of the MT. Hence if only signals from 2BSs are available, the intersection of two circles will result in two possible candidates for the MT position. In this case it is clear that there is no global identifiability. However local identifiability is present. To summarize no global identifiability in presence of local identifiability means that there are discrete (not continuous) ambiguities left. No local identifiability means that there are continuous ambiguities left.

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Notations: upper-case and lower-case boldface letters denote matrices and vectors, respectively. $(.)^T$ and $(.)^H$ represent the transpose and the transpose-conjugate operators. $E\{.\}$ is the statistical expectation, $\Re\{.\}$ is the real part and $tr\{.\}$ is the trace operator defined for square matrices.

II. Modeling of the Path Amplitudes and the Channel Model

We begin with the channel model. The time varying channel impulse response (CIR) between the BS and MT can be written as:

$$h(t,\tau) = \sum_{i=1}^{L} A_i(t)\ p(\tau - \tau_i(t)) \tag{1}$$

where L denotes the number of paths (rays), $p(t)$ is the convolution of the transmit and receive filters (pulse shape), $\tau_i(t)$, $A_i(t)$ denote delay and complex attenuation coefficient (amplitude and phase of the ray) of the i^{th} path respectively. It is reasonable to assume that path delays and amplitudes vary slowly with the position. Let us now consider sampling CIR with a sampling period of τ_s leading to N_τ samples and stacking them in a vector as follows:

$$\mathbf{h}(t)=\begin{bmatrix} h(\tau_s,t) \\ h(2\tau_s,t) \\ \vdots \\ h(N_\tau\tau_s,t) \end{bmatrix}=\sum_{i=1}^{L} A_i(t)\ \mathbf{p}_{\tau_i} \tag{2}$$

where $\mathbf{p}_\tau$ is defined as: $\mathbf{p}_\tau=\begin{bmatrix} p(\tau_s-\tau) \\ p(2\tau_s-\tau) \\ \vdots \\ p(N_\tau\tau_s-\tau) \end{bmatrix}$ which is the sampled complex pulse shape vector having a delay equal to the delay of the path in samples and has N nonzero samples. We implicitly assumed that paths are resolvable (system bandwidth W is sufficiently large). If we write Eq. (2) in matrix notation and include the channel estimation noise, we obtain the estimated CIR vector as:

$$\hat{\mathbf{h}}(t)=\underbrace{[\mathbf{p}_{\tau_1}\cdots\mathbf{p}_{\tau_L}]}_{\mathbf{P}_\tau}\underbrace{\begin{bmatrix} A_1(t) \\ \vdots \\ A_L(t) \end{bmatrix}}_{\mathbf{b}(t)}+\mathbf{v}(t). \tag{3}$$

Two possible models can now be considered for the path amplitudes:

- Gaussian model: $A_i(t)$ Gaussian with zero mean, characterized by a power (variance) i.e. $var(A_i)=\sigma_i^2$, which corresponds to Rayleigh fading case for the magnitude
- deterministic model: $A_i(t)$ deterministic unknowns

We will investigate the local identifiability issues for both cases. In general, local identifiability of a parameter vector $\mathbf{r}$ can be achieved when the Fisher Information Matrix (FIM) is nonsingular [3]. Moreover we will investigate a special case of the CIR which will make the derivation of the FIM easier. The assumption is that pulse contributions corresponding to different path delays do not overlap with each other. This makes the pulse matrix $\mathbf{P}_\tau$ an orthogonal matrix, i.e. $\mathbf{P}_\tau^H\mathbf{P}_\tau=e_p\mathbf{I}$ where $e_p=\|\mathbf{p}(\tau)\|^2$ is the pulse energy. This assumption can be valid for high bandwidth systems where the pulse durations are quite short.

III. CRB Analysis for the Rayleigh Fading Case

Let θ represent the vector of LDPs. If we just consider the delays and the variances of the complex path amplitudes as LDPs, θ is given as: $\theta=\left[\tau_1,\ \tau_2,\ \cdots\ \tau_L,\ \sigma_1^2,\ \sigma_2^2,\ \cdots\ \sigma_L^2\right]^T$, where τ_i and σ_i^2 represent the delay and the amplitude variance of the i^{th} path respectively. The log-likelihood of the data vector for complex white Gaussian noise is given as:

$$\mathcal{LL}\propto-\ln\left(\det\left(\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}\right)\right)-\left(\hat{\mathbf{h}}-\mu\right)^H\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}^{-1}\left(\hat{\mathbf{h}}-\mu\right) \tag{4}$$

Hence from Eq. (4), the elements of the FIM $\mathbf{J}_\theta$ for a general complex Gaussian scenario is given by [4]

$$[\mathbf{J}_\theta]_{ij}=tr\left(\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}^{-1}\frac{\partial\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}}{\partial\theta_{\mathrm{i}}}\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}^{-1}\frac{\partial\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}}{\partial\theta_{\mathrm{j}}}\right)+2\Re\left(\left[\frac{\partial\mu}{\partial\theta_i}\right]^H\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}^{-1}\left[\frac{\partial\mu}{\partial\theta_{\mathrm{j}}}\right]\right). \tag{5}$$

Note that we are computing the FIM in the true position. The covariance matrix and the mean vector which were computed offline according to the parameters of a database entry (each entry in the database correspond to a different position with unique parameters such as path delays, amplitudes, etc.) belong to the same position of the measured channel estimates. And also our main interest is the local identifiability of the position vector $\mathbf{r}=[x,y]$ which denotes the coordinates of the mobile position. Hence there will be a FIM transformation of parameters from θ to $\mathbf{r}$. We can easily obtain the transformation from $\mathbf{J}_\theta$ to $\mathbf{J}_\mathbf{r}$ by the following formula [5]:

$$\mathbf{J}_\mathbf{r}=\mathbf{F}\mathbf{J}_\theta\mathbf{F}^H \tag{6}$$

where $\mathbf{F}=\frac{\partial\theta}{\partial\mathbf{r}}\Big|_{\mathbf{r}=\mathbf{r}_0}$ ($\mathbf{r}_0=[x_0,y_0]^T$ being the true position of the mobile) is a $2\times 2L$ matrix which is given by:

$$\mathbf{F}=\left.\begin{bmatrix} \frac{\partial\tau_1}{\partial x} & \cdots & \frac{\partial\tau_L}{\partial x} & \frac{\partial\sigma_1^2}{\partial x} & \cdots & \frac{\partial\sigma_L^2}{\partial x} \\ \frac{\partial\tau_1}{\partial y} & \cdots & \frac{\partial\tau_L}{\partial y} & \frac{\partial\sigma_1^2}{\partial y} & \cdots & \frac{\partial\sigma_L^2}{\partial y} \end{bmatrix}\right|_{x=x_0,y=y_0} \tag{7}$$

Note that LDP vector θ will be defined differently in the next section which will result in a different $\mathbf{F}$ matrix. If we check Eq. (6), for local identifiability of $\mathbf{r}$, $\mathbf{J}_\mathbf{r}$ must be full rank (rank 2). For $\mathbf{J}_\mathbf{r}$ to be full rank, it is required that $\mathbf{J}_\theta$ must have at least rank 2. For the path amplitude variances, they are mostly modeled by distance dependent attenuation which is accompanied by a path-loss coefficient (isotropic model). In that case $\sigma_i^2=\frac{k}{\tau_i^\gamma}$ where k is a positive constant depending on the propagation speed of the wave, antenna gains, etc and γ is the path-loss coefficient ($\gamma\geq 2$). In such a condition, σ_i^2 is just a function of τ_i. So only τ_i carries position dependent information. On the other hand we can consider σ_i^2 itself as a position dependent parameter (anisotropic model). For example in a given position it might be a function of the surrounding geography which will cause reflections, refractions and so on. It is obvious that in that case each path will carry 2 distinct information about position instead of 1. Also note that by chain rule, we have $\frac{d\sigma_i^2}{dx}=\frac{d\sigma_i^2}{d\tau_i}\frac{d\tau_i}{dx}=\eta_i\frac{d\tau_i}{dx}$ where $\eta_i=-k\gamma\tau_i^{-(\gamma+1)}$ for the isotropic model. We can say that $\mathbf{F}$ is a generic matrix. Hence it is full rank (rank 2) with probability 1 for the anisotropic case. For the isotropic modeling, $rank(\mathbf{F})=\min(2,L)$ due to the chain rule. Therefore it is never possible to achieve the local identifiability of $\mathbf{r}$ for the isotropic case when $L=1$. Now we will consider each of these cases separately.

A. Anisotropic Path Amplitude Variances

If we turn back to the discussion about $\mathbf{J}_\theta$, in the Rayleigh fading case the channel estimates have zero mean because $E\mathbf{b}(t)=\mathbf{0}$. Hence the second term in Eq. (5) vanishes. The covariance matrix of the channel estimates $\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}$ can be easily obtained from Eq. (3) and given by $\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}=\mathbf{P}_\tau\mathbf{C}_b\mathbf{P}_\tau^H+\sigma_v^2\,\mathbf{I}$, σ_v^2 being the channel estimation error variance. $\mathbf{C}_b$ is a diagonal matrix with entries $[\sigma_1^2,\sigma_2^2,\cdots,\sigma_L^2]$. The diagonal structure of $\mathbf{C}_b$ comes from the uncorrelated scattering assumption of the paths. So for the GML technique with Rayleigh fading, the FIM is:

$$[\mathbf{J}_\theta]_{ij} = tr\left(\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}^{-1}\frac{\partial \mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}}{\partial \theta_{\mathbf{i}}}\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}^{-1}\frac{\partial \mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}}{\partial \theta_{\mathbf{j}}}\right). \quad (8)$$

After this assumption, we can derive the elements of the FIM by using Eq. (8). We can explicitly obtain the inverse of the covariance matrix by using Woodbury's matrix identity. By exploiting the orthogonality of the pulse matrix $\mathbf{P}_\tau$, inverse covariance matrix is obtained as:

$$\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}{}^{-1} = \sigma_v^{-2}\mathbf{I} - \sigma_v^{-2}\sum_{i=1}^{L}\frac{\sigma_i^2}{e_p\,\sigma_i^2 + \sigma_v^2}\mathbf{p}_{\tau_i}\mathbf{p}_{\tau_i}^H. \quad (9)$$

For the preparation of the computation of the FIM entries, we first compute the partial derivatives of the covariance matrix with respect to the parameters as follows:

$$\frac{\partial \mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}}{\partial \sigma_i^2} = \mathbf{p}_{\tau_i}\mathbf{p}_{\tau_i}^H, \quad \frac{\partial \mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}}{\partial \tau_i} = -\sigma_i^2\left(\mathbf{p}'_{\tau_i}\mathbf{p}_{\tau_i}^H + \mathbf{p}_{\tau_i}\mathbf{p}'^H_{\tau_i}\right) \quad (10)$$

where $\mathbf{p}'_\tau = \begin{bmatrix} p'(\tau_s - \tau) \\ p'(2\tau_s - \tau) \\ \vdots \\ p'(N_\tau \tau_s - \tau) \end{bmatrix}$ and $p'(n\tau_s - \tau)$ being defined as: $p'(n\tau_s - \tau) = \left.\frac{dp(t)}{dt}\right|_{t=n\tau_s-\tau}$. With these partial derivatives, and by using Eq. (8), (9) and the assumption that $\mathbf{p}_{\tau_i}^H\mathbf{p}_{\tau_j} = \delta_{ij}e_p$, we obtain the FIM entries:

$$\begin{aligned}\mathbf{J}_{\tau_i,\tau_i} &= tr\left(\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}^{-1}\frac{\partial \mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}}{\partial \tau_{\mathbf{i}}}\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}^{-1}\frac{\partial \mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}}{\partial \tau_{\mathbf{i}}}\right) \\ &= \sigma_i^4\sigma_v^{-4}\left(tr\left(\mathbf{B_i B_i}\right) + c_i^2\, tr\left(\mathbf{C_i C_i}\right) - 2c_i\, tr\left(\mathbf{B_i C_i}\right)\right)\end{aligned}$$

where

$$\begin{aligned}&\mathbf{B_i} = \left(\mathbf{p}'_{\tau_i}\mathbf{p}_{\tau_i}^H + \mathbf{p}_{\tau_i}\mathbf{p}'^H_{\tau_i}\right), \mathbf{C_i} = \left(\alpha\mathbf{p}_{\tau_i}\mathbf{p}_{\tau_i}^H + e_p\mathbf{p}_{\tau_i}\mathbf{p}'^H_{\tau_i}\right) \\ &\alpha = \mathbf{p}_{\tau_i}^H\mathbf{p}'_{\tau_i} = a + jb, \quad c_i = \frac{\sigma_i^2}{e_p\,\sigma_i^2 + \sigma_v^2}.\end{aligned} \quad (11)$$

However we recognize that $\alpha = jb$ (a turns out to be 0). However we omit the proof due to lack of space. Also when the pulse is real or symmetric around its center, b also becomes 0 resulting in $\alpha = 0$. After doing the algebra, we obtain the result as follows:

$$\mathbf{J}_{\tau_i,\tau_i} = \frac{2\,\sigma_i^4\sigma_v^{-2}(e_p\,e_d - b^2)}{e_p\,\sigma_i^2 + \sigma_v^2}, \quad (12)$$

where $\mathbf{p}'^H_{\tau_i}\mathbf{p}'_{\tau_i} = e_d$. Evidently information is higher for stronger paths. By using the same methodology we continue to calculate the rest of the FIM.

$$\mathbf{J}_{\sigma_i^2,\sigma_i^2} = tr\left(\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}^{-1}\frac{\partial \mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}}{\partial \sigma_{\mathbf{i}}^2}\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}^{-1}\frac{\partial \mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}}{\partial \sigma_{\mathbf{i}}^2}\right) = \left(\frac{e_p}{e_p\sigma_i^2 + \sigma_v^2}\right)^2. \quad (13)$$

And $\mathbf{J}_{\sigma_i^2,\tau_i} = \mathbf{J}_{\tau_i,\sigma_i^2}$ turns out to be 0. For $L > 1$, we have the cross terms of the FIM for different paths, e.g. $\mathbf{J}_{\tau_\mathbf{i},\tau_\mathbf{j}}$ for $i \neq j$. Due to the non-overlapping pulse assumption, all the entries of the FIM corresponding to different paths result in 0. Proof is simple but omitted here due to lack of space. After having completely derived the FIM for the LDP vector, we can check the conditions to have at least rank 2 to achieve the local identifiability of $\mathbf{r}$. We will first investigate the case when $L = 1$. In this case we have two LDPs namely τ_1 and σ_1^2. For $L = 1$, FIM has the following structure:

$$\mathbf{J}_\theta = \begin{bmatrix} \mathbf{J}_{\tau_1,\tau_1} & 0 \\ 0 & \mathbf{J}_{\sigma_1^2,\sigma_1^2} \end{bmatrix} \quad (14)$$

Obviously to achieve a rank of 2, the diagonals of the matrix must be nonzero. As can be seen from Eq. (13), $\mathbf{J}_{\sigma_1^2,\sigma_1^2}$ is always positive. For $\mathbf{J}_{\tau_1,\tau_1}$, the following condition must hold: $e_p\ e_d \neq b^2$. We can also state in the following form: $|\mathbf{p}_\tau^H\mathbf{p}'_\tau|^2 \neq \|\mathbf{p}_\tau\|^2\ \|\mathbf{p}'_\tau\|^2$. Note that we have not used τ_1, but instead we just used τ because the statement is independent of the delay. What we observe is that local identifiability of $\mathbf{r}$ depends on the pulse shape and its derivative for $L = 1$. By using the Cauchy-Schwarz inequality we have: $\|\mathbf{p}_\tau\|^2\ \|\mathbf{p}'_\tau\|^2 \geq |\mathbf{p}_\tau^H\mathbf{p}'_\tau|^2$. So unless one vector is a scalar multiple of the other vector (pulse shape and its derivative), the equality never holds making the matrix rank 2 (full rank in this case). This is an important result because local identifiability of $\mathbf{r}$ can be achieved with only 1 path. Another thing to emphasize is that if the pulse is real or symmetric, then α and consequently b becomes 0. In this case $\mathbf{J}_{\tau_1,\tau_1}$ is always nonzero. Hence local identifiability of $\mathbf{r}$ is achieved without any constraints in this case. We can easily extend the investigation for $L > 1$. Moreover it is also possible to extract the CRBs for the estimation of the elements of the LDP vector θ. For that purpose diagonal entries of the inverse of $\mathbf{J}_\theta$ must be computed. For $L > 1$, FIM is still a diagonal matrix ($2L \times 2L$). Hence now local identifiability of $\mathbf{r}$ is guaranteed without any constraints on the pulse shape for $L > 1$. Computing the CRBs is quite easy for a diagonal matrix and given by:

$$\begin{aligned} E(\tau_i - \hat{\tau}_i)^2 &\geq \frac{1}{\mathbf{J}_{\tau_i,\tau_i}} = \frac{1}{8\pi^2W^2SNR_i}\left(1 + \frac{1}{SNR_i}\right), \\ E(\sigma_i^2 - \hat{\sigma_i^2})^2 &\geq \frac{1}{\mathbf{J}_{\sigma_i^2,\sigma_i^2}} = \sigma_i^4\left(1 + \frac{1}{SNR_i}\right)^2, \end{aligned}$$

where $SNR_i = \frac{\sigma_i^2\,e_p}{\sigma_v^2}$ is the signal to noise ratio (SNR) of the i^{th} path, and W is the the effective bandwidth of the pulse given by: $W = \frac{\sqrt{e_d/e_p}}{2\pi}$. One remark we can make is that due to the non-overlapping pulse assumption, the CRBs for path i only depend on the parameters of the i^{th} path. And also estimation of the delay becomes easier with the increasing bandwidth, and as expected higher SNR makes the estimation easier for all parameters.

B. Isotropic Path Amplitude Variances

Now as explained before, we consider the path variances as distance dependent. Hence $\sigma_i^2 = \frac{k}{\tau_i^\gamma}$. As τ_i and σ_i^2 are coupled now, we will apply chain rule to derive the elements of the FIM. In fact we do not need the entries explicitly for local identifiability analysis. As we told before, for $L = 1$, it is impossible to achieve local identifiability (only possible for $L > 1$). To distinguish the entries from the anisotropic case, we will use the notation $\mathbf{J'}_{\tau_i,\tau_i}$ for example. By using Eq. (8) again we have:

$$\mathbf{J'}_{\tau_i,\tau_i} = tr\left(\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}{}^{-1}\frac{\partial \mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}}{\partial \sigma_i^2}\frac{d\sigma_i^2}{d\tau_i}\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}{}^{-1}\frac{\partial \mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}}{\partial \sigma_i^2}\frac{d\sigma_i^2}{d\tau_i}\right)$$

where $\frac{d\sigma_i^2}{d\tau_i} = -k\gamma\tau_i^{-(\gamma+1)} = \eta_i$. Hence $\mathbf{J'}_{\tau_i,\tau_i} = \eta_i^2\mathbf{J'}_{\sigma_i^2,\sigma_i^2}$, and $\mathbf{J'}_{\sigma_i^2,\tau_i} = \mathbf{J'}_{\tau_i,\sigma_i^2} = \eta_i\,\mathbf{J'}_{\sigma_i^2,\sigma_i^2}$. Hence it is not possible to calculate the CRBs for the LDPs due to the rank deficiency of the FIM. Therefore we change the strategy here. The rank deficiency results from the fact that parameters are coupled (σ_i^2 is a function of τ_i). Therefore for the LDP, only delays will be accounted, FIM will consist of only delays and their CRBs will be calculated. CRBs for the estimation of σ_i^2's will be calculated by the transformation of parameters technique [4]. We obtain easily:

$$\mathbf{J'}_{\tau_i,\tau_i} = \mathbf{J}_{\tau_i,\tau_i} + \eta_i^2\mathbf{J}_{\sigma_i^2,\sigma_i^2}. \tag{15}$$

The calculation of the CRBs is now straightforward:

$$E(\tau_i - \hat{\tau_i})^2 \geq \frac{1}{\mathbf{J'}_{\tau_i,\tau_i}} = \frac{1}{\mathbf{J}_{\tau_i,\tau_i} + \eta_i^2\mathbf{J}_{\sigma_i^2,\sigma_i^2}}, \tag{16}$$

$$E(\sigma_i^2 - \hat{\sigma_i^2})^2 \geq \frac{1}{\mathbf{J}_{\sigma_i^2,\sigma_i^2} + \mathbf{J}_{\tau_i,\tau_i}/\eta_i^2}, \tag{17}$$

where $\mathbf{J}_{\tau_i,\tau_i}$ and $\mathbf{J}_{\sigma_i^2,\sigma_i^2}$ are given by Eq. (12) and (13) respectively in the anisotropic case. We see that the information is higher than the anisotropic case for both of the parameters, and this is an expected result. The reason is that, now not only delay, but also the path power carries information about the delay and vice versa which makes the estimation of the parameters easier.

IV. CRB Analysis for the Deterministic Case

Now in this section we model the path amplitudes as deterministic unknowns which does not depend on delays, instead a genuine function of position (anisotropic modeling). We turn back to the channel model in Eq. (3) and write the complex path amplitude of path i in polar form as $A_i(t) = a_i(t)e^{j\phi_i(t)}$ where we assume that the phase ϕ_i's are deterministic unknowns. In this situation the LDP vector is: $\theta = [\tau_1,\ \tau_2,\ \cdots\ \tau_L,\ a_1,\ a_2,\ \cdots\ a_L]^T$. As we now have deterministic path amplitudes, mean of the channel estimates is not zero and given by $\mu = \mathbf{P}_\tau\mathbf{b}(t)$. We also have a different covariance matrix which is $\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}} = \sigma_v^2\mathbf{I}$. Under these conditions, the computation of the FIM matrix will be different. If we check Eq. (5), unlike the Rayleigh fading case, now the first term vanishes because the covariance matrix is not a function of the LDP vector elements, hence its derivatives with respect to these elements are zero. The second term involving the mean now remains which is a function of the LDP vector elements. So the FIM is given as:

$$[\mathbf{J}_\theta]_{ij} = 2\Re\left(\left[\frac{\partial\mu}{\partial\theta_i}\right]^H\mathbf{C}_{\hat{\mathbf{h}}\hat{\mathbf{h}}}^{-1}\left[\frac{\partial\mu}{\partial\theta_j}\right]\right). \tag{18}$$

We need the following partial derivatives for the computation of the FIM entries:

$$\frac{\partial\mu}{\partial\tau_i} = -a_i\ e^{j\phi_i}\mathbf{p'}_{\tau_i},\ \ \frac{\partial\mu}{\partial a_i} = e^{j\phi_i}\mathbf{p}_{\tau_i}. \tag{19}$$

With these partial derivatives, the entries of the FIM can be computed as:

$$\mathbf{J}_{\tau_i,\tau_i} = \frac{2}{\sigma_v^2}\Re\left(\left[\frac{\partial\mu}{\partial\tau_i}\right]^H\left[\frac{\partial\mu}{\partial\tau_i}\right]\right) = \frac{2}{\sigma_v^2}\ a_i^2\ e_d,$$
$$\mathbf{J}_{a_i,a_i} = \frac{2}{\sigma_v^2}\Re\left(\left[\frac{\partial\mu}{\partial a_i}\right]^H\left[\frac{\partial\mu}{\partial a_i}\right]\right) = \frac{2e_p}{\sigma_v^2},$$

where $\mathbf{J}_{\tau_i,a_i} = 0$. For $L > 1$, the cross terms again all turn out to be equal to 0 for $i \neq j$. For $L = 1$, the FIM is given as follows:

$$\mathbf{J}_\theta = \frac{2}{\sigma_v^2}\begin{bmatrix} a_i^2\ e_d & 0 \\ 0 & e_p \end{bmatrix}. \tag{20}$$

Clearly it is always rank 2. Obviously for any L, FIM is always full rank (rank $2L$) which guarantees the local identifiability of $\mathbf{r}$. For the CRBs we have:

$$E(\tau_i - \hat{\tau_i})^2 \geq \frac{1}{\mathbf{J}_{\tau_i,\tau_i}} = \frac{1}{8\pi^2W^2SNR_i}, \tag{21}$$

$$E(a_i - \hat{a_i})^2 \geq \frac{1}{\mathbf{J}_{a_i,a_i}} = \frac{a_i^2}{2SNR_i}. \tag{22}$$

where now $SNR_i = a_i^2e_p/\sigma_v^2$. Hence we see that estimating the delay is easier now than in the Rayleigh fading case.

If we extend the results to the isotropic modeling of the path amplitude a_i's, we obtain similar results as we obtained in the Rayleigh fading case in the sense that estimation of the parameters become easier than their anisotropic counterparts.

V. Conclusion

After the analysis what we have seen is that local identifiability of the position vector depends on the number of paths (L) and the modeling of the path amplitudes as well. For the Rayleigh fading case, for the anisotropic modeling, local identifiability of the position vector can be achieved even for $L = 1$. However for the isotropic modeling, at least two paths are required ($L \geq 2$) for local identifiability. The difference stems from the fact that if we consider the isotropic modeling, only the delay parameter (τ) carries distinct information about position. On the other hand each path carries two distinct information about position for the anisotropic modeling. For the deterministic path amplitude case there is the same reasoning again. As a result what is obtained is parallel to the case in which local identifiability can be achieved with 2 TOA information.

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