DIVERSITY-MULTIPLEXING TRADEOFF OF SIMPLIFIED RECEIVERS FOR FREQUENCY-SELECTIVE MIMO CHANNELS

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ABSTRACT
Since the introduction of the Diversity-Multiplexing Tradeoff (DMT) by Zheng and Tse for ML reception in frequency-flat MIMO channels, some results have been obtained also for the DMT of frequency-selective MIMO channels and for the DMT of suboptimal receivers such as linear (LEs) and decision-feedback equalizers (DFEs) for frequency-selective SIMO channels or frequency-flat MIMO channels. In this paper we extend these results to the case of linear receivers for frequency-selective MIMO channels. We consider infinite-length and FIR equalizers in standard single-carrier systems, and un constrained equalizers in cyclic prefix systems. For linear equalizers, the diversity gain suffers significantly in the absence of any Channel State Information at the Transmitter (CSIT), since only a part of the receive spatial diversity is available (normally \( n\) inputs and outputs will be mainly thought of as corresponding to multiple antennas). After a receive (Rx) filter (possibly noise whitening), we sample the received signal to obtain a discrete-time system at symbol rate. Wherever we denote the vectors containing the SIMO impulse response coefficients \( \mathbf{h}_i = [h_{i0} \cdots h_{iL}]^T \) and the overall coefficient vector \( \mathbf{h} = [h_0^T \cdots h_n^T]^T \). Assume the energy normalization \( \sum_i R_{hh} = n_r \), with \( R_{hh} = \mathbb{E}\left\{ hh^H \right\} \). By default we shall assume the i.i.d. complex Gaussian channel model: \( \mathbf{h} \sim \mathcal{CN}(0, \frac{1}{L+1} R_{hh}) \) so that spatio-temporal diversity of order \( n_r n_t (L+1) \) is available (which is the case from the moment \( R_{hh} \) is nonsingular). The average per Rx antenna SNR is \( \rho = \frac{\sigma^2}{\sigma_n^2} \). In this paper we consider full channel state information at the Rx (CSIR) and usually none (otherwise antenna selection) at the Tx (CSIT).

Whereas in non-fading channels, probability of error \( P_e \) decreases exponentially with SNR, for a given symbol constellation, in fading channels the probability of error taking channel statistics into account behaves as \( P_e \sim \rho^{-d} \) for large SNR \( \rho \), where \( d \) is the diversity order. On the other hand, at high SNR the channel capacity increases with SNR as log \( \rho \), which can be achieved with adaptive modulation and coding (AMC) on the basis of the long-term SNR (slow feedback), not to be confused with the instantaneous SINR (fast feedback). In [1] it was shown however that both benefits at high SNR cannot be attained simultaneously and a compromise has to be accepted: the “diversity-multiplexing tradeoff” (DMT). In [1] the frequency-flat MIMO channel was considered. These results were extended to the frequency-selective SISO channel in [2] and the frequency-selective MIMO channel in [3], see also [4],[5]. In [6], it was shown for the frequency-selective SIMO channel that a Zero Forcing (ZF) or Minimum Mean Squared Error (MMSE) Decision-Feedback Equalizer (DFE) with unconstrained feedforward filter allows to attain the optimum diversity and similar results for the MIMO frequency-flat channel case, with a linear MIMO prefilter and a MMSE MIMO DFE appears in...
We get for the Matched Filter Bound (MFB) of stream $i$

$$\text{MFB}_i = \rho \left( \|h_i\|^2 + \|h_i\|^2 \right) = \sum_{k=0}^{L-0.5} (\|h_{i,k}\|_2^2 - \int_{-0.5}^{0.5} \|h_i(f)\|_2^2 df)$$

where e.g. $\|h_{i,k}\|_2^2 = h_{i,k}^* h_{i,k}$. The MFB corresponds to Maximum Ratio Combining (MRC) of all energy in the spatio-temporal channel. The MFB is a close approximation for the performance of Maximum Likelihood Sequence Detection (MLSD). In practice one is often forced to resort to suboptimal Rx's when the delay spread and or the constellation size get large. Two popular classes of suboptimal Rx's are linear and decision-feedback equalizers (LE and DFE). Both types of equalizers are in fact linear estimators of the transmitted symbol sequence, one is based on the received signal only whereas the other is also based on the past detected symbols.

The goal of these suboptimal Rx's is to transform the frequency-selective channel into a frequency-flat channel the performance of which depends on the Signal-to-Interference-plus-Noise Ratios (SINRs) at its outputs. For Mutual Information (C) purposes, the channel-equalizer cascade is treated as an AWGN channel, hence $C = \sum C_i = \sum_{n=1}^{\infty} \log(1 + \text{SINR}_n)$. In the MIMO multichannel context considered here, a zero-forcing (ZF) LE (or DFE) only exists if $n_r \geq n_s$, and is not unique for $n_r > n_s$ since $h_i(f)$ has a non-empty orthogonal complement. Among all the ZF equalizers, there is one that will minimize the noise enhancement (MSE), which hence can be called the MMSE-ZF design. For a DFE, which has a feedforward and a feedback filter, this non-uniqueness already arises for the $n_r = n_s$ case. To simplify notation, we shall henceforth refer to the MMSE-ZF design as the ZF design. Introduce

$$\delta = \begin{cases} 0 & \text{MMSE-ZF design}, \\ 1 & \text{MMSE design}. \end{cases}$$

For a MMSE design, we need to introduce the following extended transfer function(s):

$$h_c[z] = \left[ h[i][\delta] \right] = \left[ h_r[z] \cdots h_r[z] \right].$$

The description of the LE requires the following orthogonalized SIMO transfer functions:

$$h_c^o[z] = \left[ h_i[z] \right] = \left[ h_r[z] \right]$$

where e.g. $P_{h_c}^o = I - P_{h_c}^o$, and $h_c^o[z]$ is obtained from $h_c[z]$ by removing column $i$, namely $h_c[z]$. For DFEs, the (ordered) Gram-Schmidt orthogonalization is required:

$$h_c^o[z] = \left[ h_i[z] \right] = \left[ h_i^o[z] \right]$$

For infinite-length non-causal (feedback) filters, we get the following SINR results

- $\text{MFB}_i = \rho \left( \|h_i(f)\|_2^2 \right) df = \rho \left( \|h_i(f)\|_2^2 \right) df - \delta$
- $\text{SINR}_{DFE,i} = \rho \exp \left[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \log(\|h_i(f)^o\|_2^2) df \right] - \delta$
- $\text{SINR}_{DFE,i} = \rho \left[ \left[ h_i^o(f) \right]^{-1} df \right] - \delta$

with inequalities

$$\text{SINR}_{DFE,i} \leq \text{SINR}_{DFE,i} \leq \text{MMF}_i \leq \text{SINR}_i$$

where the last inequality holds for either LE or DFE, and comparison between DFE and LE or MFB assume the same Tx antenna ordering. For the case of MMSE design, the SINR here corresponds to the SINR computed correctly (SINR = $\text{MSE} - 1$) which might be more easily interpreted in terms of Unbiased MMSE (UMMSE) design [8].

$\rho_i \|h_i\|^2$
3. OUTAGE-RATE TRADEOFF

The SINR, random due to its dependence on the random channel $\mathbf{h}$. In [21], it was demonstrated that at high SNR outage only depends on the SINR distribution behavior near zero (this was also observed in [1]). This result is quite immediate. Indeed, let us introduce the normalized SINR $\gamma$ through $\text{SNR}_{i} = \gamma$ and consider the dominating term in the cumulative distribution function (cdf) of a $\gamma$:

$$\text{Prob}\{\gamma \leq \epsilon\} = c \epsilon^k$$

(8)

for small $\epsilon > 0$. Then the outage probability for a certain outage threshold $\alpha$ is

$$\text{Prob}\{\text{SNR} \leq \alpha\} = c \left(\frac{\alpha}{\rho}\right)^k$$

(9)

from which we see that $k$ is the diversity order.

Now consider outage in terms of outage capacity. Since at high SNR the SINR will tend to be proportional to $\rho$, the mutual information $C = \log(1 + \text{SNR})$ in a single stream will tend to be $\log \rho$. So consider the rate $R = r \log \rho$ (in nats, assuming natural logarithm) where $r \in [0, 1]$ is the normalized rate. Then the outage probability at high SNR is

$$P_o = \text{Prob}\{C < R\} = \text{Prob}\{\log(1 + \text{SNR}) < r \log(\rho')\}$$
$$= \text{Prob}\{\rho \gamma < \rho' - 1\} = \text{Prob}\{\gamma < \frac{1}{\rho(1-r)} - \frac{1}{\rho'}\}$$
$$= \text{Prob}\{\gamma < \frac{1}{\rho^{1-r}k}\}, \text{ for } r > 0$$

(10)

Hence for the SISO system with the SINR considered, we get for $r \in (0, 1]$:

$$d(r) = (1 - r)k$$

(11)

where $d(r)$ is the diversity(order)-rate tradeoff. The case $r = 0$ (fixed rate) requires separate investigation.

The mutual information for the frequency-selective MIMO channel with white Gaussian input is $C = \int \frac{1}{2} \log \det(I_{n_t} + \rho \mathbf{h}^H(f) \mathbf{h}(f)) df = \sum_{i=1}^{n_s} \log(1 + \text{SNR}_{\text{MMSE},i})$

which reconfirms the canonical character of the MMSE DFE Rx. The optimal diversity-multiplexing (or outage-rate) tradeoff (DMT) for such channel has been shown in [3],[4],[5] to be given by the piecewise linear curve that connects the points

$$(d, r) = \left(\left((L+1)n_r - r\right)(n_s - r), r\right), \quad r = 0, 1, \ldots, n_r.$$  

(12)

In the MIMO case, in which $n_s \leq n_r$ streams get transmitted, the normalized rate $r = \frac{R}{\log \rho}$ (at high SNR $\rho$) can indeed go up to $n_r$. Even though the diversity order has been introduced here in terms of outage probability only, it also applies to frame error rate for any spatial multiplexing scheme with non-vanishing determinant (since then the probability of error in the case of no outage decreases exponentially with SNR).

4. OUTAGE ANALYSIS OF SUBOPTIMAL RECEIVER SINRS

A suboptimal Rx transforms the channel-Rx cascade into a set of $n_t$ parallel SISO channels, each characterized by their SINR. In the case of a ZF DFE with unconstrained (non-causality, length) filters, these SINRs are independent, but in other cases (especially the LE case) they are dependent as we shall see. A perfect outage of stream $i$ occurs when $\text{SNR}_{i} = 0$. For the MFB, this can only occur if $h_i = 0$. For a suboptimal Rx however (or also the MI), the SINR can vanish for any $h$ on the Outage Manifold $\mathcal{M}_i = \{\mathbf{h}: \text{SNR}(\mathbf{h}) = 0\}$. At fixed rate $R$, the diversity order is the codimension of (the tangent subspace of) the outage manifold, assuming this codimension is constant almost everywhere and assuming a channel distribution with finite positive density everywhere (e.g. Gaussian with non-singular covariance matrix).

For example, for the MFB (which only depends on $h_i$) the outage manifold is the origin, the codimension of which is the total size of $h_i$. The codimension is the (minimum) number of complex constraints imposed on the complex elements of $h$ by putting $\text{SNR}_i(h) = 0$. Some care has to be exercised with complex numbers. Valid complex constraints (which imply two real constraints) are such that their number becomes an equal number of real constraints if the channel coefficients were to be real. A constraint on a coefficient magnitude however, which is in principle only one real constraint, counts as a valid complex constraint (at least if the channel coefficient distributions are insensitive to phase changes). An actual outage occurs whenever $h$ lies in the Outage Shell, a (thin) shell containing the outage manifold. The thickness of this shell shrinks as the rate increases.

In the MIMO case with suboptimal Rxs, two cases can be considered [17], depending on whether the $n_t$ streams are the result of joint encoding ($j-\text{enc}$ schemes) or separate encoding ($s-\text{enc}$ schemes), with ensuing joint or separate decoding. The UMTS HSDPA PARC scheme is an example of a $s-\text{enc}$ scheme, but with streamwise fast feedback of $C_i$ knowledge to the Tx. In absence of any (fast) CSIT, a $s-\text{enc}$ scheme will distribute the total rate $R$ evenly ($R/n_t$) over the $n_t$ streams. The outage probabilities of $j-\text{enc}$ and $s-\text{enc}$ schemes are respectively

$$P_o^{j-\text{enc}}(R) = \text{Prob}\left(\sum_{i=1}^{n_t} \log(1 + \text{SNR}_i) < R\right)$$
$$P_o^{s-\text{enc}}(R) = \text{Prob}\left(\sum_{i=1}^{n_t} \log(1 + \text{SNR}_i) < \frac{R}{n_t}\right).$$

(13)

The outage manifold for a $s-\text{enc}$ scheme is given by $\mathcal{M} = \bigcup_{i=1}^{n_s} \mathcal{M}_i$.

5. LINEAR EQUALIZATION (LE) IN SINGLE CARRIER CYCLIC PREFIX (SC-CP) SYSTEMS

The diversity of LE for SC-CP systems has been studied in [11] for the SISO case with i.i.d. Gaussian channel elements, fixed rate $R$ and block size $N = L+1$. The LE DMT for SIMO SC-CP systems appears in [16]. Consider a block of $N$ symbol periods preceded by a cyclic prefix (CP) of length $L$ (as a result of the CP insertion, actual rates are reduced by a factor $\frac{N}{N-L}$, which is ignored here in what follows). The channel input-output relation over one block can be written as

$$\mathbf{y} = \mathbf{h} \mathbf{a} + \mathbf{v}$$

(14)
where $Y = Y_k = [y^T_{k-1} y^T_{k-1} \cdots y^T_{k-N+1}]^T$ etc. and $H$ is a banded block-circulant matrix (see (13) in [16]). Now apply an $N$-point DFT (with matrix $F_N$) to each subchannel received signal, then we get

$$
F_{N,n} Y = F_{N,n} H F_{N,n}^{-1} A + F_{N,n} V
$$

where $F_N = F_N \otimes I_{n_0}$ (Kronecker product: $A \otimes B = [a_iB]$), $\mathcal{H} = \text{blockdiag}(h(0), \ldots, h(N-1))$ with $h(n)$, the $n \times n_0$ channel transfer function at tone $n$: $f_n = \frac{n}{N}$, at which we have

$$
u_n = h(f_n) \eta_n + w_n.
$$

The $\eta_n$ components are i.i.d. and independent of the i.i.d. $w_n$ components with $\sigma^2_\eta = N \sigma^2_\varepsilon$, $\sigma^2_w = N \sigma^2_\varepsilon$. A ZF ($\delta = 0$) or MMSE ($\delta = 1$) LE produces per tone $\tilde{x} = (h^H h + \delta f_n)^{-1} h^H u$ from which $\tilde{a}_i$ with components $\tilde{a}_i$ is obtained after IDFT with

$$
\text{SINR}_{\text{ZF-LE},i} = \rho \left( \frac{1}{N} \sum_{n=0}^{N-1} \parallel \tilde{a}_i(f_n) \parallel^2 \right)^{-1} - \delta.
$$

We first focus on the ZF case:

$$
\text{SINR}_{\text{ZF-LE},i} = \rho \left( \frac{1}{N} \sum_{n=0}^{N-1} \parallel h^H f_n \parallel^2 \right)^{-1}.
$$

Using the exponential equality notation, $\parallel \cdot \parallel$ (meaning same high SNR diversity order), and $R = \log \rho\rho$, then

$$
P_{o-\text{enc}}(r) = \mathbb{P} \left( \bigcup_{n=0}^{N-1} \left\{ \log (1 + \text{SINR}_{\text{ZF-LE},i}) < \frac{R}{n_0} \right\} \right)
$$

$$
\approx \mathbb{P} \left( \log (1 + \text{SINR}_{\text{ZF-LE},1}) < \frac{R}{n_0} \right)
$$

$$
\approx \mathbb{P} \left( \text{SINR}_{\text{ZF-LE},1} < \rho^{\frac{R}{n_0}} \right)
$$

$$
\approx \mathbb{P} \left( \rho \parallel h^H_{1}(0) \parallel^2 < \rho^{\frac{R}{n_0}} \right)
$$

where the second equality is due to the fact that the $\text{SINR}_{\text{ZF-LE},i}$ are identically distributed (regardless of dependence) and the last equality is a property of harmonic averages of identically distributed quantities $\parallel h^H_{1}(f_n) \parallel$. Hence

$$
d_{o-\text{enc}}^{ZF-LE} (r) = (n_r - n_s + 1)(1 - \frac{r}{n_0}), \quad r \in [0, n_s].
$$

Note that the outage manifold $\mathcal{M}^{ZF-enc}_{p}$ is the collection of $h$ for which $\parallel h^H_{i}(f_n) \parallel = 0$ for any $i$ or $n$. To dig in more deeply, simplify notation $h^H_{i}(f_n) = h_i$ (any particular $n$), introduce the normalized vectors $h_i = h_i/\parallel h_i \parallel$, then $\parallel h_i \parallel^2 = \parallel h_i^H h_i \parallel^2 \parallel h_i \parallel^2$ where $\parallel h_i \parallel$ and $h_i$ are independent. In the case $n_s = 2$, then $\parallel h_i \parallel^2 = \sin^2 \theta = 1 - \parallel h_i^H h_i \parallel^2$ where $\theta$ is the “angle” between $h_i$ and $h_i$. The diversity order of $\parallel h_i \parallel^2$ is the minimum of the diversity orders of $\parallel h_i \parallel^2$ and $\parallel h_i \parallel^2$, and hence is the diversity order of $\parallel h_i^H h_i \parallel^2$ which is $n_s = n_s + 1$. Now, $\parallel h_i \parallel^2 = 0$ for any $i$ whenever $\exists x \in C^{n_s}, \parallel x \parallel = 1$: $\exists f \in [0, 1]: h(f) x = 0$. In spite of the ambiguity on $f$, the DMT is again as in (20) (see also [16]).

6. OPTIMIZING THE NUMBER OF TX STREAMS $n_s$

From (20) it is clear that at lower rates $r$ it is beneficial to activate a smaller number $n_s$ of streams. Also, by reducing the number of active streams, the case $n_s > n_r$ can trivially be handled. As a result, one can find from (20) the optimal $n_s(r)$ which varies from $n_s(1) = 1$ for $r = 0$ (giving $d(n_s(r), r) = n_s$ for $r = 0$ to $n_s(r) = \min(n_s(r), n_r)$ for $r = \min(n_s(r), n_r)$).

7. ANTENNA SUBSET SELECTION CSIT

By ordering the vector channels $h_i$ as in [4], the diversity order of a reduced set of the first $n_s$ vector channels gets boosted by a factor $n_s - n_r + 1$ (see [4]). As a result the diversity in the DMT curve of (20) gets multiplied by this factor:

$$
d(r) = (n_r - n_s + 1)(n_s - n_r + 1)(1 - \frac{r}{n_0}), \quad r \in [0, n_s].
$$

After again optimizing over $n_r$ as a function of $r$, we get in particular a maximal diversity of $d(0) = n_s n_r$ and $r$ can go up to $n_r$ which itself can go up to $\min(n_r, n_s)$. In this way substantial diversity boosts and a simplified Tx scheme are obtained, especially for the case $n_s > n_r$. For small rates $r$, the diversity obtained is only lightly reduced compared to the optimal (flat channel) MIMO DMT.

8. OFDM

In order to attain the CP MIMO system with LE DMT in an OFDM approach, no coding across tones is required since the frequency-selectivity diversity does not get exploited with LE. The use of redundant linear precoding can remedy this completely however! One such instance is zero padding (ZP) for which it has been shown in [9] for SISO systems that a LE allows to attain full diversity. The case of more general redundant linear precoding for SIMO systems is considered in [10]. A linear precoder for SIMO OFDM has a redundancy equal to $p$ if it mixes $N - p$ symbols over the $N$ tones. In [10] it has been shown that as long as the precoding introduces a redundancy greater than the maximum degree of singularity ($L$) that the channel $\mathcal{H}$ can suffer (without becoming completely zero), then LE allows full diversity (and hence DMT). Indeed, the FIR SIMO channel can show at most zeros at $L$ tones. If more zeros would appear, that would mean that the whole channel impulse response is zero.

In the MIMO case, $\det(h^H(f) h(f))$ can show at most $n_s L$ zeros. Hence the precoder should introduce at least this much redundancy to allow a LE to enjoy full diversity.

9. NON-CAUSAL INFINITE LENGTH LINEAR EQUALIZER

For the infinite length (ZF) LE case, $\mathcal{M} = \{h : \exists x \in C^{n_r}, \parallel x \parallel = 1, \exists f \in [0, 1]: h(f) x = 0\}$. In spite of the ambiguity on $f$, the DMT is again as in (20) (see also [16]).

10. FIR LINEAR EQUALIZATION

Consider now the use of an FIR LE of length $N$. For MIMO channels, there exist indeed FIR equalizers for FIR channels, due to the Bezout identity, as long as $N \geq \frac{L + 1 - n_s}{n_r - n_s}$. The LE is based on a banded block Toeplitz input-output matrix

$$
Y = Y_k = [y^T_{k-1} y^T_{k-1} \cdots y^T_{k-N+1}]^T
$$
which can be obtained by starting from a block circulant $H$ (as in the CP case) of size $N+L$ and removing the top $L$ block rows. We obtain for a certain equalizer delay and stream

$$\text{SINR}_{\text{FIR}} = \frac{\rho}{\|H_{\perp} H + \frac{\delta}{\rho} I\|_2} = \frac{\rho}{\lambda_{\min} + \frac{\delta}{\rho} \|V_d\|^2}$$

(22)

where $e$ is a standard unit vector containing all zeroes except for a 1 in the position corresponding to the delay and stream index, and we introduced the SVD $H_{\perp} H = VAV^H = \sum_{i=1}^{\lambda_{\min}} \lambda_i V_i V_i^H$. The outage manifold is determined (again) by $\lambda_{\min} = 0$. Singularity of $H_{\perp} H$ occurs whenever $H$ loses full column rank. This occurs whenever a linear combination of the SIMO channels has subchannels with a zero in common, or $\mathcal{M} = \{h : \exists x \in \mathbb{C}^r, \|x\| = 1, \exists z_0 \in \mathbb{C} : h[z_0] x = 0\}$ $|z_0| = 1$. 

11. LE: THE MMSE CASE

The regularization provided in the MMSE case has no effect as soon as $r > 0$. As a result the MMSE DMTs coincide with those of the corresponding ZF designs for $r > 0$. For $r = 0$ however, full diversity $n_r n_r (L+1)$ is obtained for appropriate block/FIR dimensions (for detailed investigations see e.g. [18] for the frequency-flat MIMO case and [1] for the frequency-selective SISO CP case).

Acknowledgment

Eurecom’s research is partially supported by its industrial partners: BMW, Bouygues Telecom, Cisco Systems, France Télécom, Hitachi Europe, SFR, Sharp, ST Microelectronics, Swisscom, Thales. The research work leading to this paper has also been partially supported by the European Commission under the ICT research network of excellence NewCom++ of the 7th Framework programme and by the French ANR project APOGEE.

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