On the Diversity-Multiplexing Tradeoff for Frequency-Selective MIMO Channels

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***Abstract*— We have previously (ISIT'05) introduced the optimal Diversity versus Multiplexing Tradeoff (DMT) for a FIR frequency-selective i.i.d. Rayleigh MIMO channel. This tradeoff is the same as for a frequency-flat MIMO channel with the larger of the number of receive or transmit antennas being multiplied by the delay spread. In this paper we provide alternative proofs and insights into this result. In particular, we consider the ordered LDU decomposition instead of the usual eigen decomposition of the channel Gram matrix. Popular approaches for frequency-selective channels use OFDM techniques in order to exploit the diversity gain due to frequency selectivity. We show that the minimum number of subcarriers that need to be involved in space-frequency coding to allow achieving the optimal tradeoff is the delay spread times the smaller of the number of transmit or receive antennas, thus answering a question that was open hitherto. Although the no-CSIT/full-CSIR case is considered here, we propose an alterative DMT interpretation based on negligible CSIT. This CSIT allows to exploit the ordered LDU decomposition.**

I. INTRODUCTION

Consider a linear modulation scheme and single-carrier transmission over a Multiple Input Multiple Output (MIMO) linear channel with additive white noise, as shown in Fig. 1. The multiple (subchannel) outputs will be mainly thought of as corresponding to multiple receive antennas. After a Rx filter (possibly noise whitening), we sample the received signal to obtain a discrete-time system at symbol rate[1]. After stacking the samples corresponding to multiple subchannels in column vectors, the discrete-time communication system is described by

$$\mathbf{y}_k = \mathbf{H}(q)\,\mathbf{a}_k + \mathbf{v}_k = \sum_{l=0}^{L-1} \mathbf{H}_l \mathbf{a}_{k-l} + \mathbf{v}_k\,, \qquad (1)$$

where $\mathbf{H}(q) = \sum_{l=0}^{L-1} \mathbf{H}_l\, q^{-l}$, $\; q^{-1}x_k = x_{k-1}$ (q^{-1} is the unit sample delay operator). The coefficients $\mathbf{H}_l$ are $N_r \times N_t$ matrices. L is the channel delay spread. We introduce the SNR variable $\rho = \frac{P}{N_t \sigma_v^2} = \frac{\sigma_a^2}{\sigma_v^2}$. We consider the i.i.d. Rayleigh channel model in which the entries of $\mathbf{H}_l$, $l = 0, \ldots, L-1$ are i.i.d. Gaussian : $\mathbf{H}_l^{rt} \sim \mathcal{CN}(0,1)$.

[1] In the case of additional oversampling with integer factor, we would vectorize the samples to get a per antenna vector received signal sequence at symbol rate.

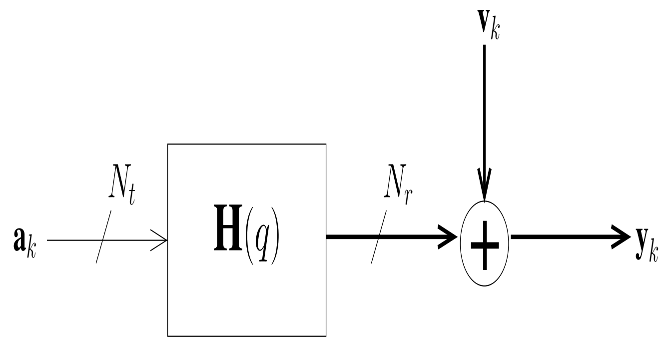

Fig. 1. MIMO channel model.

II. DIVERSITY AND OUTAGE BASICS

The SINR is random due to its dependence on the random channel $\mathbf{h}$. In [1], it was demonstrated that at high SNR outage only depends on the SINR distribution behavior near zero (this was also observed in [2]). This result is quite immediate. Indeed, let us introduced the normalized SINR γ through SINR $= \rho\,\gamma$ and consider the dominating term in the cumulative distribution function (cdf) of γ:

$$\text{Prob}\{\gamma \le \epsilon\} = c\,\epsilon^k \qquad (2)$$

for small $\epsilon > 0$. Then the outage probability for a certain outage threshold α is

$$\text{Prob}\{\text{SINR} \le \alpha\} = c\left(\frac{\alpha}{\rho}\right)^k = \left(\frac{\alpha}{g\,\rho}\right)^k \qquad (3)$$

from which we see that k is the diversity order and $g = c^{-1/k}$ is the coding gain (reduction in SNR required for identical outage probability). When γ is obtained as a combination of independent γ_i, we get the diversity orders k that are indicated in the table below.

γ	k
$\sum_i \gamma_i$	$\sum_i k_i$
$\max_i \gamma_i$	$\sum_i k_i$
$\min_i \gamma_i$	$\min_i k_i$
$\prod_i \gamma_i$	$\min_i k_i$

In [2], Zheng and Tse introduced a scenario of SNR-adaptive modulation and coding schemes (MCS) with hence varying diversity and spatial multiplexing (or normalized rate). Scheme $\mathcal{C}(\rho)$ is a family of codes (MCS) of block length T (one code for each SNR level), that supports a bit rate $R(\rho)$. This scheme is said to achieve *spatial multiplexing* r and diversity gain d if the data rate and the average error probability satisfy

$$\lim_{\rho\to\infty}\frac{R(\rho)}{\ln(\rho)} = r \quad , \quad \lim_{\rho\to\infty}\frac{\ln P_e(\rho)}{\ln(\rho)} = -d \; . \tag{4}$$

For each r, $d^*(r)$ is defined to be the supremum of the diversity order achieved over all possible schemes. The maximal diversity gain is defined by $d^*_{max} = d^*(0)$ and the maximal spatial multiplexing gain is $r^*_{max} = sup\{r : d^*(r) > 0\}$.
For a **Flat MIMO channel** ($L = 1$), with $T \geq N_t$, the optimal trade-off curve $d^*(r)$ (DMT) is given by the piecewise-linear function connecting the points $(k, d^*(k))$, $k = 0, 1, \ldots, q$, where

$$\begin{aligned} d^*(k) &= (p-k)(q-k), \\ q &= \min\{N_r, N_t\}, \\ p &= \max\{N_r, N_t\} \end{aligned}$$

as shown in Fig. 2.The optimal DMT can be achieved by a family of codes with non-vanishing determinant [3], such as e.g. the space-time spreading codes of [4].

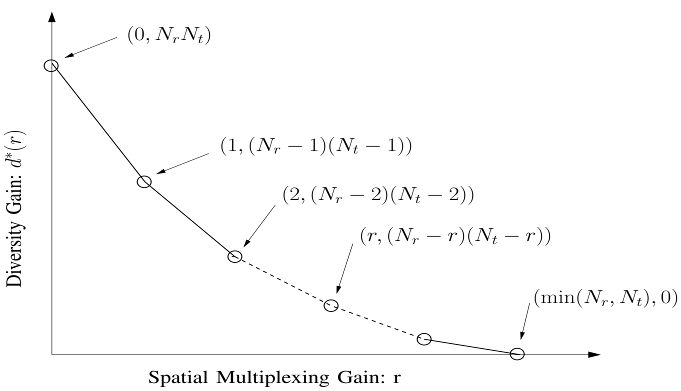

Fig. 2. Optimal Diversity vs. Multiplexing tradeoff (DMT) for freq.-flat MIMO ch.

For **SIMO/MISO frequency selective channels**, the optimal trade-off curve is given by the linear function $d^*(r) = L\,p\,(1-r)$ [5]. For SIMO, the DMT can be achieved by using QAM at the Tx and MMSE DFE at the Rx [10], see also [6] for the DMT of various SIMO linear and decision-feedback equalizers.

III. FLAT MIMO DMT VIA LDU

First of all, for STC schemes with non-vanishing determinant (see e.g. [3] or also [4]), we have $P_e(\rho) = \text{Prob}\{error\} = \text{Prob}\{\text{error,outage}\} + \text{Prob}\{\text{error, no outage}\} \doteq \text{Prob}\{\text{outage}\} = P_{out}(\rho)$ (due to the faster decay of Prob{error, no outage}), hence $d^*(r) = d^{out}(r)$ and an outage analysis suffices. In what follows, assume w.l.o.g. $N_r \geq N_t$ (otherwise replace $\mathbf{H}$ by $\mathbf{H}^\dagger$). Now introduce the LDU (Lower Diagonal Upper triangular factorization) [7] of the channel Gram matrix:

$$\mathbf{H}^H\,\mathbf{H} \;=\; L\,D\,L^H \qquad = \; (L\,D^{\frac{1}{2}})\,(L\,D^{\frac{1}{2}})^H \tag{5}$$

where $L = [L_{i,j}]$ has unit diagonal, $D = \text{diag}\,\{d_1, \ldots, d_q\}\,,\;\; d_i \geq 0$. The second factorization in (5) corresponds to the Cholesky decomposition. Let $\mathbf{H} = [\mathbf{h}_1 \cdots \mathbf{h}_q] = \mathbf{h}_{1:q}$, and introduce the projection matrices $P_{\mathbf{H}} = \mathbf{H}(\mathbf{H}^H\mathbf{H})^{\#}\mathbf{H}^H$, $P^\perp_{\mathbf{H}} = I - P_{\mathbf{H}}$. Then we can write

$$\left\{\begin{array}{l} d_{i+1} = \|P^\perp_{\mathbf{h}_{1:i}}\mathbf{h}_{i+1}\|^2 \\ L_{i+1,j+1} = \mathbf{h}^H_{i+1}P^\perp_{\mathbf{h}_{1:j}}\mathbf{h}_{j+1}/\|P^\perp_{\mathbf{h}_{1:j}}\mathbf{h}_{j+1}\|^2 \end{array}\right. \tag{6}$$

The Cholesky factorization of a Wishart matrix (such as $\mathbf{H}^H\,\mathbf{H}$) leads to

$$\left\{\begin{array}{l} d_{i+1} \sim \frac{\sigma^2}{2}\,\chi^2_{2(p-i)} \\ L_{i+1,j+1}\sqrt{d_{j+1}} \;\sim \mathcal{CN}(0, \sigma^2)\;, \;\; i > j \end{array}\right. \tag{7}$$

which is also known as Bartlett's decomposition [8]. Note that $\det(\mathbf{H}^H\mathbf{H}) = \det(D) = \prod_{i=1}^{q} d_i$. A proof of the DMT via the LDU decomposition has been provided in [9], in which the DMT for Flat MIMO has been extended to the partial CSIT case. The key starting point in [9] is the Matrix Determinant Expansion lemma:

$$\begin{aligned} &\det(I_q + \rho\,\mathbf{H}^H\mathbf{H}) \\ &= \; 1 + \textstyle\sum_{i=1}^q \rho^i \left(\sum_{(l_1,\ldots,l_i)\in(1,\ldots,q)} \det(D_{l_1<\cdots<l_i})\right) \; . \end{aligned} \tag{8}$$

A somewhat simpler proof can be obtained by considering the LDU decomposition with *pivoting* (ordering): order the columns of $\mathbf{H}$ (no influence on capacity) to obtain the columns of $\widetilde{\mathbf{H}} = \left[\widetilde{\mathbf{h}}_1 \cdots \widetilde{\mathbf{h}}_q\right]$ recursively:

$$\|P^\perp_{\widetilde{\mathbf{h}}_{1:i}}\widetilde{\mathbf{h}}_{i+1}\| = \max_{k\in(i+1,\ldots,q)} \|P^\perp_{\widetilde{\mathbf{h}}_{1:i}}\widetilde{\mathbf{h}}_k\|\;, \;\; i = 0, 1, \ldots, q{-}1\;. \tag{9}$$

This leads to the LDU with ordering: $\widetilde{\mathbf{H}}^H\,\widetilde{\mathbf{H}} = \widetilde{L}\,\widetilde{D}\,\widetilde{L}^H$, $\widetilde{d}_{i+1} = \|P^\perp_{\widetilde{\mathbf{h}}_{1:i}}\widetilde{\mathbf{h}}_{i+1}\|^2 \sim \chi^2_{2(p-i)(q-i)}$. Note that ordering modifies the marginal pdf's but not the joint pdf (apart from the support region). Observe that the diversity orders of the $\widetilde{d}_{i+1}$ correspond to the diversity orders in the breakpoints of the DMT curve. Also note that $\mathbf{H}$ is of rank i if $\widetilde{d}_i$ is not in outage but $\widetilde{d}_{i+1}$ is. Now, the actual quantity of interest is $I_q + \rho\,\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}} = L^{'}D^{'}L^{'H}$. However, at high SNR ρ, $\ln(d_i^{'}) \doteq \ln(\rho\,\widetilde{d}_i)$ if $\widetilde{d}_i$ is not in outage, whereas $\ln(d_i^{'}) \doteq 0$ otherwise. To be a bit more precise, consider

$$\begin{aligned} &P_{out}(\rho) \\ &= \;\text{Prob}\{\ln\det(I_q + \rho\,\widetilde{\mathbf{H}}^H\widetilde{\mathbf{H}}) \;<\; r\,\ln\rho\} \\ &= \;\text{Prob}\{\sum_{i=1}^{q}\ln d'_i \;<\; r\,\ln\rho\} \\ &\doteq \;\text{Prob}\{\ln d'_{k-1} + \ln d_k^{'} \;<\; (r{-}k{+}1)\,\ln\rho\},\; k{-}1 < r \leq k \end{aligned} \tag{10}$$

This development leads to the DMT, after some details that are omitted here for lack of space.

IV. DIVERSITY AND MULTIPLEXING FOR FREQUENCY SELECTIVE MIMO CHANNELS

Assume $T >> L$, then the mutual information per symbol period for white input is

$$\begin{aligned} I_T(\mathbf{H}) \approx I(\mathbf{H}) &= \oint \frac{dz}{2\pi j\, z} \ln\det(\mathbf{I} + \rho\, \mathbf{H}(z)\mathbf{H}^\dagger(z)) \\ &= \oint \frac{dz}{2\pi j\, z} \ln\det(\mathbf{I} + \rho\, \mathbf{H}^\dagger(z)\mathbf{H}(z)) \end{aligned} \tag{11}$$

where the approximation is explaine din more detail in [4], and we introduced the paraconjugate (matched filter): $\mathbf{H}^\dagger(z) = \mathbf{H}^H(1/z^*)$. For Single-Carrier Cyclic Prefix (SC-CP) or OFDM systems, it suffices to replace the integral by a sum over subcarriers. Now,

$$\begin{aligned} &\oint \frac{dz}{2\pi j\, z} \ln\det(\mathbf{I} + \rho\, \mathbf{H}^\dagger(z)\mathbf{H}(z)) \\ &\doteq \ln \oint \frac{dz}{2\pi j\, z} \det(\mathbf{I} + \rho\, \mathbf{H}^\dagger(z)\mathbf{H}(z)) \end{aligned} \tag{12}$$

since $\det(\mathbf{I} + \rho\, \mathbf{H}^\dagger(z)\mathbf{H}(z))$ is a FIR spectrum and for a FIR spectrum $S(z)$, it was shown in [10] that

$$c \,\ln(\oint \frac{dz}{2\pi j\, z} S(z)) \leq \oint \frac{dz}{2\pi j\, z} \ln S(z) \leq \ln(\oint \frac{dz}{2\pi j\, z} S(z)) \tag{13}$$

where c only depends on the FIR length. This states that for a FIR spectrum, a prediction error variance fades like the corresponding variance. As in the frequency-flat case, at high SNR outage is determined by the behavior of the Gram matrix $\mathbf{H}^\dagger(z)\,\mathbf{H}(z)$. So consider again the LDU factorization: $\mathbf{H}^\dagger(z)\,\mathbf{H}(z) = L(z)\,D(z)\,L^\dagger(z)$. Then

$$\begin{aligned} &\ln \oint \frac{dz}{2\pi j\, z} \ln\det(\mathbf{H}^\dagger(z)\mathbf{H}(z)) \\ &= \ln \oint \frac{dz}{2\pi j\, z} \ln\det(D(z)) \\ &= \sum_{i=1}^{q} \ln \oint \frac{dz}{2\pi j\, z} d_i(z) \end{aligned} \tag{14}$$

where $d_i(z) = \dfrac{\det(D_{1:i}(z))}{\det(D_{1:i-1}(z))}$ is IIR, but the $\det(D_{1:i}(z))$ are FIR and the numerator and denominator of $d_i(z)$ are strongly coupled. In particular,

$$\oint \frac{dz}{2\pi j\, z} d_i(z) = \oint \frac{dz}{2\pi j\, z} \mathbf{h}_i^\dagger(z)\, P^\perp_{\mathbf{h}_{1:i-1}(z)} \mathbf{h}_i(z) = \mathbf{h}_i^H\, A_i\, \mathbf{h}_i \tag{15}$$

where $P_{\mathbf{h}(z)} = \mathbf{h}(z)(\mathbf{h}^\dagger(z)\mathbf{h}(z))^{-1}\mathbf{h}^\dagger(z)$ and $\mathbf{h}_i = [\mathbf{h}_{i,0}^H\, \mathbf{h}_{i,1}^H \cdots \mathbf{h}_{i,L-1}^H]^H$. The matrix A_i is block Toeplitz, with block $(A_i)_{m,n} = I_p\, \delta_{m,n} - \oint \frac{dz}{2\pi j\, z} P_{\mathbf{h}_{1:i-1}(z)}\, z^{-(m-n)}$. nullity$(A_i) = i-1$ w.p. 1, indeed:

- $\mathbf{h}_i^H\, A_i\, \mathbf{h}_i = \oint \frac{dz}{2\pi j\, z} \mathbf{h}_i^\dagger(z)\, P^\perp_{\mathbf{h}_{1:i-1}(z)} \mathbf{h}_i(z)$
  $\in [0\,,\, \mathbf{h}_i^H \mathbf{h}_i] \;\Rightarrow\; \lambda_k(A_i) \in [0,1]$
- Null$(A_i) = $ Span$(\mathbf{h}_{1:i-1})$

Eigen decomposition: $A_i = \underbrace{V_i}_{pL\times(pL-i+1)} \underbrace{\Lambda_i}_{(pL-i+1)\times(pL-i+1)} \underbrace{V_i^H}_{(pL-i+1)\times pL}$ where V_i is unitary: $V_i^H V_i = I_{pL-i+1}$, Λ_i is diagonal. Now $\mathbf{h}_i^H\, A_i\, \mathbf{h}_i = \mathbf{h}_i^{'H}\, \Lambda_i\, \mathbf{h}_i^{'} = \sum_{k=1}^{pL-i+1} \lambda_k |\mathbf{h}_{i,k}^{'}|^2$ where $\mathbf{h}_i^{'} = V_i^H \mathbf{h}_i \sim \mathcal{CN}(0, \sigma^2 I_{pL-i+1})$. Note that A_i and hence Λ_i is random since function of $\mathbf{h}_{1:i-1}$. Hence we need the diversity order of $\lambda_{min} = \lambda_{pL-i+1}$.

Now, $\lambda_{min} = 0$ if $\min\{\|\mathbf{h}_{1,0}\|, \ldots, \|\mathbf{h}_{i-1,0}\|, \|\mathbf{h}_{1,L-1}\|, \ldots, \|\mathbf{h}_{i-1,L-1}\|\} = 0$.

And $\min\{\|\mathbf{h}_{1,0}\|, \ldots, \|\mathbf{h}_{i-1,0}\|, \|\mathbf{h}_{1,L-1}\|, \ldots, \|\mathbf{h}_{i-1,L-1}\|\}$ (all i.i.d.) has the same pdf as e.g. $\|\mathbf{h}_{1,0}\|$, so consider w.l.o.g. $\|\mathbf{h}_{1,0}\| = 0$. Now, if $\|\mathbf{h}_{1,0}\| = 0$, then $\lambda_{min} = 0$ with eigen vector $\mathbf{h}^H = [\mathbf{h}_{1,1}^H \cdots \mathbf{h}_{1,L-1}^H\; 0_{1\times p}]$.

To find the pdf of λ_{min} near 0, consider $\Delta\lambda_{min} = \mathbf{h}^H\, \Delta A_i \mathbf{h}$ with ΔA_i due to $\Delta\mathbf{h}_{1,0}$. It follows that

$$\Delta\lambda_{min} = \frac{|\mathbf{h}_{1,1}^H \mathbf{h}_{1,0}|^2}{\|\mathbf{h}_{1,0}\|^2 \sum_{k=1}^{L-1} \|\mathbf{h}_{1,k}\|^2} \sim \chi_2^2$$

hence div$(\mathbf{h}_i^H\, A_i\, \mathbf{h}_i = \sum_k \lambda_k |\mathbf{h}_{i,k}^{'}|^2) = $ div$(\|\mathbf{h}_i^{'}\|^2) = $ div$(\chi^2_{2(pL-i+1)})$ due to the diversity rules in the table in Section II.

Now, as in the frequency-flat case, consider the ordered LDU (ordering on variances of $d_i(z)$), then due to the same diversity rule:

$$\text{div}(\widetilde{d}_{i+1}(z)) = \text{div}(\chi^2_{2(pL-i)(q-i)})\;. \tag{16}$$

The behavior of $I(\mathbf{H})$ is characterized by $I(\mathbf{H}) \doteq \ln\det(\mathbf{I} + \rho\, \bar{\mathbf{H}}\bar{\mathbf{H}}^H)$, where $\bar{\mathbf{H}} = \begin{bmatrix} \mathbf{H}_0 \\ \vdots \\ \mathbf{H}_{L-1} \end{bmatrix}$ for $N_t \leq N_r$, $\bar{\mathbf{H}} = [\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_{L-1}]$ for $N_t \geq N_r$. The optimal trade-off curve $d^*(r)$ is given by the piecewise-linear function connecting the points $(k, d^*(k))$, $k = 0, 1, \ldots, p$, where

$$\begin{aligned} &d^*(k) = (Lq-k)(p-k), \\ &\text{with } p = \min\{N_r, N_t\},\;\; q = \max\{N_r, N_t\}\;. \end{aligned} \tag{17}$$

which is the DMT of the equivalent frequency-flat MIMO channel $\bar{\mathbf{H}}$. For e.g. $N_t \leq N_r$, the DMT is the same as for a flat MIMO channel with $N_t' = N_t$ and $N_r' = LN_r$.

A. Outage Manifolds Analysis

An intuitive explanation of the DMT can be obtained as follows. Consider a parameterization of FIR channels of rank $k \leq q = N_t \leq N_r = p$

$$\underbrace{\underbrace{\mathbf{H}(z)}_{q\times p}}_{\text{FIR}-L} = \underbrace{\underbrace{\overline{\mathbf{H}}(z)}_{q\times k}}_{\text{FIR}-L}\; \underbrace{[\, I_k \quad \underbrace{\overline{\overline{\mathbf{H}}}}_{k\times(p-k)}\,]}_{\text{constant}}\; \underbrace{\mathcal{P}}_{\text{permutation}}\;. \tag{18}$$

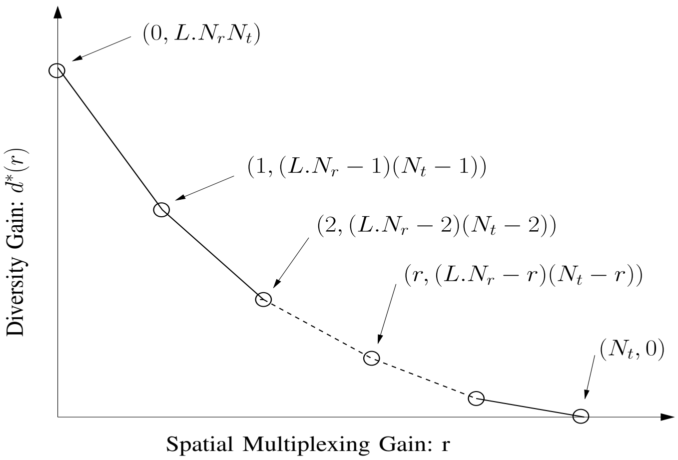

Fig. 3. Asymptotic diversity vs. multiplexing tradeoff for a frequency-selective channel (for the case $N_t \leq N_r$).

The number of degrees of freedom in the $q \times p$ rank-k FIR-L manifold is:

$$\underbrace{q\,k\,L}_{\overline{\mathbf{H}}} + \underbrace{(p-k)\,k}_{\overline{\overline{\mathbf{H}}}} = q\,p\,L - (q\,L-k)(p-k)\ . \tag{19}$$

To send at rate k, one needs to be guaranteed rank k. The diversity degree is the remaining number of degrees of freedom in $\mathbf{H}(z)$:

$$d^*(k) = q\,p\,L - (q\,p\,L - (q\,L-k)(p-k)) \;=\; (q\,L-k)(p-k)\ . \tag{20}$$

B. Min Blocklength / Min Number of OFDM Subcarriers

For a SISO/SIMO/MISO FIR channel of length L and OFDM transmission, the full diversity order is obtained by jointly coding over at least L subcarriers. Hence some may expect this to continue to hold in the MIMO case. However, for a MIMO OFDM approach using coding over L independent OFDM subcarriers (frequency-spacing of $\frac{1}{L}$ or not): we get $d(r) = L(q-r)(p-r) \;\leq\; d^*(r)$ (in the case of frequency-spacing of $\frac{1}{L}$, the channel transfer function at the L subcarriers is i.i.d. and the DMT result follows from the transmission over parallel i.i.d. channels in [2]). In any case, coding over only L subcarriers is suboptimal in the MIMO case. The difference (suboptimality) in diversity is $d^*(r) - d(r) = (L-1)\,r\,(p-r)$, it peaks at $r = \frac{p}{2}$. For large L and $N_r = N_t$, $\;d^*(p/2) \approx 2\,d(p/2)$.

In the MIMO case, the minimum blocklength in time domain, or the minimum number of subcarriers to be coded jointly in an OFDM approach is

$$q\,L \;=\; \min\{N_t, N_r\}\;L\ . \tag{21}$$

The position of the subcarriers used only influences the coding gain, not the diversity order. The above result follows from observing that

$$\det(\mathbf{I} + \rho\,\mathbf{H}(z)\mathbf{H}^\dagger(z)) = g^\dagger(z)\,g(z) \tag{22}$$

for some scalar FIR spectral factor $g(z)$ of length $q\,L$.

C. Case $N_t > N_r$

Note that even though we consider to be in the no CSIT case, the whole DMT approach in fact assumes that the receive SNR is known at the transmitter. Indeed, the whole DMT idea is one of adaptive modulation and coding, though adapting only to the average SNR, and hence requires some minimal feedback. At the very high SNR considered in the DMT analysis, the capacity becomes unbounded. So, in a duplex transmission system, feedback consisting of a finite amount of bits per symbol period or even per transmission block constitutes a negligible perturbation of the capacity, and hence can be assimilated to the no CSIT case. This is the point of view we shall follow in this subsection.

Straightforward space-time coding techniques use $N_t\,L$ as blocklength in time or number of subcarriers in frequency. Tx antenna ordering CSIT can be used to limit transmission to the N_r best transmit antennas when $N_t > N_r$. If indeed we assume the channel column pivoting order to be known at the Tx, then to transmit at rate $r = k$, the Tx will only use the columns (Tx antennas) $\widetilde{\mathbf{h}}_{1:k+1}$ with diversity order that of $\widetilde{d}_{k+1}$ (weakest): $\;(p-k)(q-k)$.

So, ordering CSIT is especially handy when $N_t > N_r$ ($\widetilde{d}_i = 0,\ i > N_r$), it allows to simplify the space-time coding to the N_r ($< N_t$) Tx antennas scenario. Note: the ordering is based on the global spatio-frequency SIMO channel power $\oint \frac{dz}{2\pi j\,z} d_i(z)$ after Gram-Schmidt orthogonalization, not to be confused with per subcarrier ordering.

V. CONCLUDING REMARKS

In this paper we introduced first an alternative proof of the DMT of frequency-flat MIMO channels by using the ordered LDU factorization instead of the basic LDU or eigen decompositions. Then we extend the use of this ordered LDU factorization to the frequency-selective channel. Some remarks are in order.

$\mathbf{H}(z)$ and $\mathbf{H}^\dagger(z)$ have the same capacity, but transmission can only be done from the transmit side $\Rightarrow$ ordering CSIT can be handy (MIMO Tx selection diversity); e.g. case 4×2: with ordering CSIT, one can apply the Golden code instead of using double Alamouti.

The existing diversity-rate tradeoff: defined at high SNR, and only focuses on diversity order and not on coding gain/SNR offset. To observe the frequency-selective MIMO DMT, one needs to go to very high SNR (e.g. 50dB, due to bad coding gain of products of fading variables of equal div. order). Hence, work at finite SNR required.

Correlation of fading variables only influences (decreases) the coding gain, not the diversity order.

Extension to include temporal diversity: when coding gets performed over multiple data blocks in which the channel varies with non-singular covariance of the temporal variation, then the diversities (at any multiplexing/rate) of the blocks simply add up.

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