LINEAR PRECODING AND DFE EQUALIZATION
ACHIEVE THE DIVERSITY VS MULTIPLEXING
OPTIMAL TRADEOFF

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ABSTRACT
The use of multiple transmit (TX) and receive (RX) antennas allows
to transmit multiple signal streams in parallel and hence to
increase communication capacity. We have previously introduced
simple convolutive linear precoding schemes that spread transmitted
symbols in time and space, involving spatial spreading, delay
diversity and possibly temporal spreading. In this paper we show
that the use of the classical MIMO DFE Equalizer for this system
allows to achieve the optimal diversity versus multiplexing trade-
off introduced in [1].

1. INTRODUCTION
The $N_{tx} \times N_{rx}$ MIMO system is essentially described by
\[ y_k = H a_k + v_k = H T(q) b_k + v_k \]
where the white noise power spectral density matrix is $S_{w}(z) = \sigma_q^2 I$, and $q^{-1} b_k = b_{k-1}$. We consider the case of channel state
information being absent at the transmitter (TX) and perfect at
the receiver (RX). The linear precoding considered here (introduced in [2] and further analyzed in [3]) consists of a modi-
fication of VBLAST, obtained by inserting a square matrix pre-
filter $T(z)$ before inputting the vector signal $b_k$ into the channel
$H$. The $N_{tx}$ signal components of $b_k$ are called streams or lay-
ers. The suggested prefilter is $T(z) = D(z)Q$ where $D(z) = \text{diag} \{ 1, z^{-1}, \ldots, z^{-N_{tx} - 1} \}$. $Q$ is unitary $Q^H Q = I$:
\[ Q = \frac{1}{\sqrt{N_{tx}}} \begin{bmatrix} 1 & \theta_1 & \cdots & \theta_1^{N_{tx} - 1} \\
1 & \theta_2 & \cdots & \theta_2^{N_{tx} - 1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \theta_{N_{tx}} & \cdots & \theta_{N_{tx}}^{N_{tx} - 1} \end{bmatrix}, \]
where the $\theta_i$ are the roots of $\theta^{N_{tx}} - j = 0$, $j = \sqrt{-1}$.
Every symbol stream $m$ ($b_{m,k}$) passes through the equivalent SIMO
channel $\sum_{i=1}^{N_{tx}} z^{-i-1} H_{i,m} Q_{i,m}$ which now has memory due to the
delay diversity introduced by $D(z)$. It is important that the different
columns $H_{b,i}$ of the channel matrix get spread out in time to get
full diversity (the streams just pass through a linear
combination of the columns, as in VBLAST, which offers limited
diversity). The delay diversity only becomes effective by the intro-
duction of the spatial spreading matrix $Q$, which has equal magni-
tude elements for uniform diversity spreading (a specific choice for $Q$
exists for maximum coding gain in case of QAM symbols [3]).
We can see that each symbol stream has the same Matched Filter
Bound (MFB), which is proportional to the channel Frobenius norm, hence full diversity is exploited. Also, since the prefilter
$T(z)$ is parity and transforms the white stream $b_k$ into the white stream $a_k$, no loss in ergodic capacity is incurred. In what
follows we denote the overall channel by $G(z) = H T(z)$.

2. CONVENTIONAL MIMO DFE RECEIVER
Consider the classical MIMO decision feedback equalizer, in which the symbol vectors $b_k$ are processed sequentially in time (see Fig. 1).
The output of the matched filter is $G(z) = z_k = G^H(q) y_k$.
\[ y_k \rightarrow G^H(q) \rightarrow z_k \]
where the feedback filter $B(z) = \sum_{i \geq 1} B_i z^{-i}$ is such that $B(z) = I + B(z)$ is causal, monic and minimum phase. Different designs
of Rx are possible (MMSE, MMSE ZF ...), we consider here the
MMSE design.

2.1. MMSE Conventional MIMO DFE Rx
The MMSE linear symbol vector estimate (MMSE linear equalizer
output) verifies
\[ \hat{b}_k = \text{MMSE} = S_{yy}(q) S_{yn}(q) y_k \\
= \sigma_q^2 G^H(q) (\sigma_q^2 G(q) G^H(q) + \sigma_q^2 I)^{-1} y_k \\
= \rho (\rho G^H(q) G(q) + I)^{-1} G^H(q) y_k \\
= \mathbf{R}^{-1}(q) z_k \]
where $R(z) = G(z)G(z) + 1/z$ and $p = \sigma^2_e = \sigma^2_z$.

Then $b_k = b_{k-m_m} - b_{k-m_{m_m}} = R_k^{-1}z_k - b_{k-m_{m_m}}$. (5)

Due to the orthogonality principle of the MMSE estimate we have

$$S_{bb}(z) = S_{bb}(z) - \sigma^2 z R^{-1}(z).$$

Consider the minimum and maximum phase factorization of $R(z)$ (see [4]). Let $B(z)$ be the unique causal, monic ($B(\infty) = 1$) minimum phase factor of $R(z)$, then

$$R(z) = B^*(z)M B(z).$$

where $M$ is a constant positive definite hermitian matrix.

Then $b_k = B^{-1}(q)M^{-1}B^{-1}(q)z_k - b_{k-m_{m_m}}$.

By choosing $F(q) = M^{-1}B^{-1}(q)$, we get

$$F(q)z_k = M^{-1}B^{-1}(q)z_k - b_{k-m_{m_m}}$$

$$= B(q)b_k - B(q)b_k$$

$$= \bar{B}(q)b_k + \epsilon_k$$

where $\bar{S}_{ee}(z) = \sigma^2_z B(z)R^{-1}(z)B^*(z) = \sigma^2 M^{-1}$.

$\bar{B}(z) = B(z)^{-1}$ is tightly related to the MIMO prediction error filter $P(z)$ of the spectrum $R(z)$. $B(q)R(q)P(z)$ is Constant Matrix. Indeed, $P(z) = B^0(z)$ obviously. The following theorem gives $B(z)$ in the case of a flat MIMO channel.

**Theorem 1**: For a frequency-flat MIMO channel the feedback filter is

$$B(z) = T(z)^{-1}L(z),$$

with the corresponding

$$M = QH^T DQ.$$ (10)

where $L$ and $D$ result from the LDU triangular matrix decomposition of $H^H + 1/p = L D L^H$.

**Proof**:

We need to show that $B(z) = QH^T D(z)^{-1}L(z)D(z)Q$ is a minimum phase causal monic filter and verifies $B^{-1}(z)R(z)B^{-1}(z) = M$. $L(z)$ is upper triangular with unit diagonal, then due to the diagonal structure of $D(z), D(z)L(z)D(z)$ is a monic causal filter. $Q$ is unitary, hence $B(z)$ is also a causal monic filter. Det $B(z) = \det L^H = 1$, which shows that $B(z)$ is minimum phase. To complete the proof of the theorem it is sufficient to verify that $B^{-1}(z)R(z)B^{-1}(z) = QH^TDQ = M$.

2.2. Unbiased MMSE Conventional MIMO DFE Rx

$F(q)z_k - \bar{B}(q)b_k$ is a biased estimate of $b_k$, since

$$F(q)z_k - \bar{B}(q)b_k = [M^{-1}B^{-1}(q)G^*(q)G(q) - \bar{B}(z)]b_k$$

$$= (I - \frac{1}{p}M^\infty) b_k + \bar{e}_k,$$ (11)

where

$$\bar{e}_k = M^{-1}B^{-1}(q)G^*(q)\psi_k - \frac{1}{p}M^{-1}(B^{-1}(q) - I)b_k.$$ (12)

The covariance of $\bar{e}_k$ is

$$C_{ee} = \frac{1}{\sigma^2_z} \bar{F}^T[M^{-1}(\sigma^2_zB^{-1}(q)G^*(q)G(q) - \bar{B}(z)]b_k$$

$$= \frac{1}{\sigma^2_z} \bar{F}^T[M^{-1}(\sigma^2_zB^{-1}(q)G^*(q)G(q) - \bar{B}(z))\bar{B}(z)]M^{-1}$$

$$= \frac{1}{\sigma^2_z} \bar{F}^T[M^{-1}(\sigma^2_zB^{-1}(q)G^*(q)G(q) + \rho^{-1}I)B^{-1}(z)]M^{-1}$$

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The feedforward UMMSSE filter is

$$\tilde{F}(q) = (I - \frac{1}{\rho}M^{-1})^{-1}M^{-1}B^{-1}(q) = (M - \frac{1}{\rho}I)^{-1}B^{-1}(q),$$

whereas the corresponding feedback filter is

$$\tilde{B}(q) = (I - \frac{1}{\rho}M^{-1})^{-1}(B(q) - I).$$ (15)

The output of the DFE is then

$$\tilde{b}_k = \tilde{F}(q)z_k - \tilde{B}(q)b_k$$

$$= b_k + \epsilon_k$$,

where $\epsilon_k = \sigma^2_z M^{-1}M^{-1} - \sigma^2_z \rho^{-1}M^{-2}.$

3. DIVERSITY VS MULTIPLEXING TRADEOFF

In [1], Zheng and Tse introduced the diversity versus multiplexing tradeoff. In what follows, we study the diversity vs multiplexing tradeoff achieved by the Conventional MIMO DFE equalizer, applied to our linearly precoded system. We consider a transmission over a large frame of length $T (T >> N_{tx})$. As the delay introduced by $T(q)$ is $N_{tx} - 1$, then the number of symbol vectors $b_k$ transmitted over the frame duration is $T - N_{tx} + 1$ (padded by $N_{tx} - 1$ transmitted zeros at the end of each frame). As the considered frame size is large ($T >> N_{tx}$), we can then neglect the effect on the rate that results for the loss of $N_{tx} - 1$ symbol periods.

**Theorem 2**: In the case of a frequency-flat channel and $N_{tx} = 2^n \leq N_{tx}$ (in integer), the use of a weighted minimum distance detector and QAM constellations allows the Unbiased MMSE design of a Conventional MIMO DFE Rx to achieve the diversity vs multiplexing tradeoff given by $d^*(r)$ (see [1]). $d^*(r)$ is given by the piecewise-linear function connecting the points $(k, d^*(k))$, $k = 0, 1, \ldots, p$, where

$$d^*(k) = (p - k)(q - k)$$

with $p = \min\{N_{tx}, N_{rx}\}$ and $q = \max\{N_{tx}, N_{rx}\}$. This theorem shows that the MMSE design allows to attain the optimal diversity vs. multiplexing tradeoff derived in [1].

**Proof**: We consider the Unbiased MMSE Conventional MIMO DFE Rx. The special symbol $\delta$ denotes the exponential equality, i.e., we write $f(\rho) = \rho^b$ to denote

$$\lim_{\rho \to \infty} \frac{\ln f(\rho)}{\ln \rho} = b.$$ (18)

The proof of Theorem 2 is structured in three steps. In step 1 we characterize the frame(block) error probability in term of the first symbol error probability. In step 2 we derive a lower bound on the first symbol error probability. Finally, in step 3, we characterize the behavior of the error probability for large SNR and derive the diversity versus multiplexing tradeoff. However for the lack of
In this step of the proof, we derive a lower bound on the first symbol error rate. We denote by $E_k$ the event of making an error when detecting the $k^{th}$ symbol vector $b_k$ ($E_k^c$ is the complement of the event or the event where no error is made when detecting the $k^{th}$ symbol vector). Whenever there is an error on any of the detected symbols, the frame is said to be in error. $P_e$ denotes the frame error probability.

Step 1:
The symbol vectors of the transmitted frame are detected sequentially using the DFERx. We denote by $P_{e,k}$ the event of making an error when detecting the $k^{th}$ symbol vector $b_k$ ($E_k^c$ is the complement of the event or the event where no error is made when detecting the $k^{th}$ symbol vector) whenever there is an error on any of the detected symbols, the frame is said to be in error. $P_e$ denotes the frame error probability.

Using the following expansion

$$P_e = P(T - N_{e} + 1)$$

(Step 2:)

In this step of the proof, we derive a lower bound on the first symbol error probability for a fixed channel realization $P(E_1|H)$.

We use the weighted minimum distance detector at the output of the unbiased MMSE MIMO MIMO DFE (16). An error occurs if there is $b'_k \neq b_k$, and we decide $b'_k$ for transmitted $b_k$.

For an error to occur we need to have

$$||b'_k - b_k||_2^2 = \frac{1}{\sigma_{\nu}'^2}$$

where $\sigma_{\nu}'^2 = \sigma^2 M^{-1}(I + \frac{1}{\rho} M^{-1})$ and $M = Q^H \cdot DQ$.

The diagonal part of the $DQ$ decomposition of $H^H H + \frac{1}{\rho}$ (verifies $\det(D) = \det(H^H H + \frac{1}{\rho})$, see section 2.1).

We denote by $\Delta b = b_k - b'_k$, then (23) is equivalent to

$$\Delta b^H (I + \frac{1}{\rho} M^{-1}) \Delta b \leq 2 \Re \{ \Delta b^H (I + \frac{1}{\rho} M^{-1}) \Delta b \}$$

Let $\Delta c = Q \Delta b$ and $\bar{v}_1 = \frac{1}{\sigma_{\nu}'^2} \Delta c^H (I + \frac{1}{\rho} M^{-1}) \Delta c$

The channel mutual information is $I(H) = \ln \det (I + \rho H^H H) = \ln \det (\rho D)$. $D$ is diagonal, then using the Jensen’s inequality we get that

$$\frac{1}{\sigma_{\nu}'^2} \Delta c^H (\rho D) \Delta c \geq (\prod_{i=1}^{N_s} \frac{1}{\rho} D_i) |\Delta c|^2 \frac{\sigma_{\nu}'^2}{\sigma_{\nu}'^2}$$

We consider a scheme where the transmitted rate varies with the SNR.

The different component of $b_k$ comes from the same QAM constellation of size $(2M)^2 = \rho \frac{\sigma_{\nu}'^2}{\sigma_{\nu}'^2}$, (r $\geq 0$) where $R(\rho) = \rho \ln \rho$ is the overall allocated rate and $M$ is a positive integer. The minimum distance of the constellation is $2d$, with $d^2 = \frac{3\rho^2}{2(\rho \frac{\sigma_{\nu}'^2}{\sigma_{\nu}'^2} - 1)}$.

For $i = 1, \ldots, N_{e}$, $\Delta b_i = 2d[l_i^H + j\rho \beta_i^H, l_i^H + \rho \beta_i^H] \in \{-2M + 1, -2M + 2, \ldots, 2M - 1\}$. Then

$$\Delta b^H (\Delta b) \leq N_{e} + d^2 ((2M - 1)^2 + (2M - 1)^2) \leq 8d^2 N_{e} \frac{\sigma_{\nu}'^2}{\sigma_{\nu}'^2}$$

In the other hand the choice of $Q$ ensures that $|\bar{v}_1|^2 \geq \gamma(H)$.

For a given channel realization, the error event $E_1$ is included in the event of equation (29), then

$$P(E_1|H) \leq P(|\bar{v}_1|^2 \geq \gamma(H)|H)$$.  

$\bar{v}_1$ can be written as

$$\bar{v}_1 = C_{\rho}^{1/2} \varepsilon_1$$

where $\bar{v}_1 = C_{\rho}^{1/2} \varepsilon_1 = C_{\rho}^{1/2} \varepsilon_1$ and $\varepsilon_1$ is spatially white.

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$$\frac{1}{\sigma_{\nu}'^2} \Delta c^H (\rho D) \Delta c \geq (\prod_{i=1}^{N_s} \frac{1}{\rho} D_i) |\Delta c|^2 \frac{\sigma_{\nu}'^2}{\sigma_{\nu}'^2}$$

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For $i = 1, \ldots, N_{e}$, $\Delta b_i = 2d[l_i^H + j\rho \beta_i^H, l_i^H + \rho \beta_i^H] \in \{-2M + 1, -2M + 2, \ldots, 2M - 1\}$. Then

$$\Delta b^H (\Delta b) \leq N_{e} + d^2 ((2M - 1)^2 + (2M - 1)^2) \leq 8d^2 N_{e} \frac{\sigma_{\nu}'^2}{\sigma_{\nu}'^2}$$

In the other hand the choice of $Q$ ensures that $|\bar{v}_1|^2 \geq \gamma(H)$.

For a given channel realization, the error event $E_1$ is included in the event of equation (29), then

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For $i = 1, \ldots, N_{e}$, $\Delta b_i = 2d[l_i^H + j\rho \beta_i^H, l_i^H + \rho \beta_i^H] \in \{-2M + 1, -2M + 2, \ldots, 2M - 1\}$. Then

$$\Delta b^H (\Delta b) \leq N_{e} + d^2 ((2M - 1)^2 + (2M - 1)^2) \leq 8d^2 N_{e} \frac{\sigma_{\nu}'^2}{\sigma_{\nu}'^2}$$

In the other hand the choice of $Q$ ensures that $|\bar{v}_1|^2 \geq \gamma(H)$.
Equation (30) becomes now
\[ P(E_i|\mathbf{H}) \leq P(||\mathbf{v}_i^*||_2 \geq \sqrt{\gamma(H)} - \gamma_i|\mathbf{H}). \] (32)
\[ \mathbf{v}_i^* \] has an unbiased Gaussian distribution, with covariance that is majorized by the identity
\[ \mathbf{E}\mathbf{v}_i^*\mathbf{v}_i^*H \leq \mathbf{E}\mathbf{v}_i^*\mathbf{v}_i^*H + \mathbf{E}\mathbf{v}_i^*\mathbf{v}_i^*H = \mathbf{E}\mathbf{v}_i\mathbf{v}_i^*H \leq \mathbf{I} \] (33)
Denote \( \mathbf{n}_i = (\mathbf{E}\mathbf{v}_i\mathbf{v}_i^*H)^{-1/2}\mathbf{v}_i^* \), as \( (\mathbf{E}\mathbf{v}_i\mathbf{v}_i^*H)^{1/2} \leq \mathbf{I} \) we can write the following inequality
\[ ||\mathbf{n}_i||_2 \geq ||\mathbf{v}_i^*||_2 \] (34)
where \( \mathbf{n}_i \) follows the unbiased Gaussian distribution with covariance identity.

The error probability is then majorized by
\[ P(E_i|\mathbf{H}) \leq P(||\mathbf{v}_i^*||_2 \geq \sqrt{\gamma(H)} - \gamma_i|\mathbf{H}) = P(||\mathbf{n}_i||_2 \geq \sqrt{\gamma - \gamma_i}|\mathbf{H}), \] (35)
where \( \gamma_2(H) = (\sqrt{\gamma(H)} - \gamma_i)^2 \).

Step 3:
In step 3 we seek to study the behavior of the error probability for large SNR, in order to derive the diversity versus multiplexing tradeoff achieved by our scheme.

We define \( p = \max\{N_{sx}, N_{tx}\} \), \( q = \max\{N_{sx}, N_{tx}\} \) and \( \lambda_i, i = 1, \ldots, p \), to be the nonzero eigenvalues of \( \mathbf{H}^H\mathbf{H} \) sorted in the increasing order.

We continue in the foot steps of [1] and use the following variable change \( \lambda_i = \rho^{-\alpha_i} \). At high SNR we have \( (1 + \rho \lambda_i) = \rho^{1-\alpha_i} \), where \( (z)^+ \) denotes \( \max(0, z) \). In the other hand the mutual information verifies \( I(\mathbf{H}) = \sum_{i=1}^{p} \ln(1 + \rho \lambda_i) \), hence \( e^{I(\mathbf{H})} = \rho^{\sum_{i=1}^{p}(1-\alpha_i)^+} \).

In [1], it was shown that for an allocated rate \( r \ln \rho \), the outage probability is
\[ P(outage) = P(\sum_{i=1}^{p}(1-\alpha_i)^+ \leq r) = \rho^{-d_{out}(r)}, \] (36)
where \( d_{out}(r) \) is given by the piecewise-linear function connecting the points \((k, d_{out}(k)), k = 0, 1, \ldots, p\), where
\[ d_{out}(k) = (p-k)(q-k). \] (37)
It was also shown in the same paper that any scheme with rate \( R(\rho) = r \ln \rho \) has an error probability that verifies
\[ P_e \geq \rho^{-d_{out}(r)}, \] (38)
\[ d_{out}(r) = d^+(r) \] is also called the optimal tradeoff curve.

Let \( \epsilon \) be a small real positive number \( \epsilon > 0 \). We define the \textit{outage}, event for \( \sum_{i=1}^{p}(1-\alpha_i)^+ \leq r + \epsilon \). The complement event of \textit{outage}, is denoted as \textit{no outage}.

Then the following relation is verified
\[ \{\text{outage}\} \cup E_1 = \{\text{outage}_e\} \cup \{\text{no outage}_e\} \cap E_1. \] (39)

A upper bound on \( P(E_1) \) can then be derived as
\[ P(E_1) \leq P(\{\text{outage}_e\} \cup E_1) = P(\text{outage}_e) + P(E_1, \text{no outage}_e). \] (40)

For (36) we conclude that
\[ P(\text{outage}_e) = P(\sum_{i=1}^{p}(1-\alpha_i)^+ \leq r + \epsilon) = \rho^{-d_{out}(r + \epsilon)}, \] (41)

We want to characterize \( P(E_1, \text{no outage}_e) \). By applying the Chernoff bound to (35), and for any \( \lambda > 0 \), we get
\[ P(E_1, \text{no outage}_e) = \frac{1}{\lambda} \int_{\text{no outage}_e} P(\sum_{i=1}^{p}(1-\alpha_i)^+ \geq r - \lambda \epsilon) e^{-\lambda \gamma_2(H)} f(\mathbf{H}) d\mathbf{H}, \] (42)
For any realization of the channel with \textit{no outage}_e verifies \( \sum_{i=1}^{p}(1-\alpha_i)^+ > r + \epsilon \), then
\[ P(E_1, \text{no outage}_e) \leq \int_{\text{no outage}_e} \gamma_2(\rho) f(\mathbf{H}) d\mathbf{H} \leq \gamma_2(\rho), \] (43)
where \( \gamma_2(\rho) = 2^N_r e^{-\frac{1}{2} \ln(2\pi e N_r)} \).

For any \( \epsilon > 0 \) the following property is verified \( \lim_{\rho \to 0} \frac{\gamma_2(\rho)}{\ln \rho} = -\infty \). Hence for any finite \( y \) we have \( P(E_1, \text{no outage}_e) \leq \rho^{-y} \), and by consequence
\[ P(E_1, \text{no outage}_e) \leq \rho^{-d_{out}(r + \epsilon)}. \] (44)
Combining this result with (40) and (41) leads to
\[ P(E_1) \leq \rho^{-d_{out}(r + \epsilon)}, \] (45)
which is valid for any \( \epsilon > 0 \), we have then
\[ P(E_1) \leq \rho^{-d_{out}(r)} = \rho^{-d^+(r)}. \] (46)

Using (22), we end with an upper bound to the frame error probability
\[ P_e \leq \rho^{-d^+(r)}. \] (47)
The lower bound of (38), allows us to finally conclude that our scheme attain the optimal diversity vs multiplexing tradeoff, or
\[ P_e \leq \rho^{-d^+(r)}. \] (48)

4. REFERENCES