Turbo-like Codes for the Block-Fading Channel
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We consider a single-input single-output block-fading channel with \(N_B\) fading blocks [1]. The received signal can be compactly expressed in the matrix form
\[
Y = \sqrt{\rho} H X + Z
\]
where \(Y \in \mathbb{C}^{N_B \times L}, X \in \mathbb{C}^{N_B \times L}, H = \text{diag}(h_1,\ldots,h_{N_B}) \in \mathbb{C}^{N_B \times N_B}\) and \(Z \in \mathbb{C}^{N_B \times L}\). The b-th block fading coefficient is denoted by \(h_b\) and the noise \(z_a\) is i.i.d. proper complex Gaussian, with components \(\sim \mathcal{N}_C(0,1)\). We assume normalized fading, such that \(E[|h_b|^2] = 1\). Therefore, the average signal-to-noise ratio (SNR) is \(\rho\) and the instantaneous SNR on block \(b\) is given by \(|h_b|^2\rho\).

The collection of all possible transmitted codewords \(X\) forms a coded modulation scheme over \(\mathcal{X}\). We study schemes \(M(C,\mu,X)\) obtained by concatenating a binary linear code \(C\) of length \(N_B LM\) and rate \(r\) bit/symbol with a memoryless one-to-one symbol mapper \(\mu: \mathbb{F}_2^L \rightarrow \mathcal{X}\), with \(M = \log_2 |\mathcal{X}|\). The resulting coding rate (in bit/complex dimension) is given by \(R = rM\).

We define the SNR reliability function \(d_B^*(R)\) as the maximum achievable SNR exponent of error probability for codes in a given family of interest [3]. Namely, we define
\[
d_B^*(R) = \sup_{C,\rho} \lim_{L \to \infty} -\log P_e(C,\rho) \quad \text{at SNR} \quad \rho = \frac{R}{M}
\]
where \(P_e(C,\rho)\) is the error probability of a given coding scheme \(C\), and the supremum is taken over all coding schemes in the family \(\mathcal{C}\). For discrete signal sets and for bit-interleaved coded modulation (BICM) [2] we have the following results:

**Theorem 1** Consider the block-fading channel (1) with i.i.d. Rayleigh fading and input signal set \(\mathcal{X}\) of cardinality \(2^M\). The SNR reliability function of the channel is upperbounded by the Singleton bound
\[
d_B^*(R) \leq \delta_B(R) = 1 + \frac{1}{N_B} \left( 1 - \frac{R}{M} \right)
\]
The random coding SNR exponent of the coding ensemble \(M(C,\mu,X)\) defined previously, with block length \(L(R)\) satisfying \(\lim_{L \to \infty} L(R) = M\) and rate \(R\), is lowerbounded by \(\beta N_B M \log(2) \left( 1 + \frac{1}{L(R)} \right)\) when \(0 \leq \beta < \frac{1}{M \log(2)}\) and by \(\delta_B(R) - 1 + \min\left\{1, \beta M \log(2) \left[ N_B \left( 1 - \frac{R}{M} \right) - \delta_B(R) + 1 \right] \right\}\) when \(\frac{1}{M \log(2)} \leq \beta < \infty\). Furthermore, the SNR random coding exponent of the associated BICM channel satisfies the same lower bounds.

**Corollary 1** The SNR reliability function of the block-fading channel with inputs \(X\) and of the associated BICM channel is given by \(d_B^*(R) = \delta_B(R)\) for all \(R \in (0,M)\), except for the \(N_B\) discontinuity points of \(d_B(R)\), i.e., for the values of \(R \) for which \(N_B(1-R/M)\) is an integer.

Fig. 1 shows \(\delta_B(R)\) (Singleton bound) and the random coding lower bounds for \(\beta M \log(2) = 1/2\) and \(\beta M \log(2) = 2\), in the case \(N_B = 8\) and \(M = 4\). It can be observed that as \(\beta\) increases, the random coding lower bound coincides over a larger and larger support with the Singleton upper bound. However, in the discontinuity points it will never coincide.

We say that a code ensemble \(M(C,\mu,X)\) is good if for \(L \to \infty\) its error probability converges to the outage probability, while if it shows a fixed SNR gap (independent of \(L\)) we say that \(M(C,\mu,X)\) is weakly good. In [4] we provide a sufficient condition for weak goodness based on asymptotic multivariate weight enumerators.

We consider the coded modulation family \(M(C,\mu,X)\) of block-wise concatenated codes (BCC) (see Fig. 2). The binary linear outer code \(C^O \in \mathbb{F}_2^{L \times O}\) of rate \(r_O\) is partitioned into \(N_B\) blocks of length \(L_O/N_B\). The blocks are separately interleaved and fed to \(N_B\) binary inner encoders \(C^I \in \mathbb{F}_2^{L \times I}\) of rate \(r_I\). Finally, the output of each component inner code is mapped onto a sequence of signal components in \(\mathcal{X}\) by the modulator mapping \(\mu\). The proposed BCCs significantly outperform conventional serial and parallel turbo codes in the block-fading channel. Differently from the AWGN and fully-interleaved fading cases, iterative decoding performs very close to ML on the block-fading channel, even for relatively short block lengths. Moreover, at constant decoding complexity per information bit, BCCs are shown to be weakly good, while standard block codes obtained by trellis-termination of convolutional codes have a gap from outage that increases with the block length: this is a different and more subtle manifestation of the so-called “interleaving gain” of turbo-like codes.

**REFERENCES**


