ANALYSIS OF SEMIDEFINITE PROGRAMMING RELAXATION APPROACH FOR MAXIMUM LIKELIHOOD MIMO DETECTION

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ABSTRACT

Many signal processing applications reduce to solving integer least square problems, e.g., Maximum Likelihood (ML) detection, which is NP-hard. Recently semidefinite programming (SDP) approach has been shown to be promising approach to combinatorial problems. SDP methods have been applied to the communications problem, e.g., [1], [2], [3]. But so far no theoretical analysis of the algorithm is shown and the evaluation of the SDP approach for detection is based only on simulation results. In this paper, we theoretically evaluate bounds for the SDP approach. We also establish relationship between the exact maximum/minimum value of the objective function to the SDP relaxed (approximate) maximum/minimum value of the objective function.

1. INTRODUCTION

In many communications systems the optimal receiver structure is maximum likelihood sequence detector (MLSD). However the complexity of the MLSD is exponential in the number of antennas. This led to find low complexity (approximate) solution of the MLSD problem, e.g., MIMO-DFE and V-BLAST are some of them. Recently, the SDP approach has become popular in solving combinatorial optimization problems. In [1,8], a CDMA detection problem is solved by the SDP approach using interior point method and they obtained very close approximation to the ML performance. Similarly [2,3], also describe the SDP method to solve the detection problem for the CDMA case. In [2], the authors shows the relationship of the SDP to LMMSE and SAGE detectors. In this paper using the analysis of [4,5], we are able to give bounds on the SDP method. Furthermore, following the analysis of [6] we also show the relationship between exact maximum/minimum value of the objective function to the relaxed problem's maximum/minimum value of the objective function.

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2. SIGNAL MODEL

We consider a typical flat fading MIMO transmission model

$$y = Hx + n \tag{1}$$

where usually the vectors x, n, y and the matrix H are given in the equivalent baseband, and hence are complex valued. H is the channel matrix, x is the vector of transmitted symbols, each chosen from same finite alphabet A and n is an additive white Gaussian noise vector. H is of dimension $M \times N$. H matrix contains complex transfer coefficients between transmit and receive antennas. Eq (1) can equivalently be written as

$$\begin{pmatrix} Ry \\ Iy \end{pmatrix} = \begin{pmatrix} RH & -IH \\ IH & RH \end{pmatrix} \begin{pmatrix} Rx \\ Ix \end{pmatrix} + \begin{pmatrix} Rn \\ In \end{pmatrix}, \tag{2}$$

where R and I denotes real and imaginary parts respectively, which gives an equivalent $2M$ dimensional real model of the form

$$y_r = H_r x_r + n_r, \tag{3}$$

For the sake of simplicity we consider BPSK case and assume that H_r is known at the receiver. The ML detection problem reduces to

$$\bar{q} = min \ \ ||y_r - H_r x_r||^2$$

$$subject \ \ to \ \ x_r \in \{-1, 1\}^{2N} \tag{4}$$

which can be further written as

$$\bar{q} = max \ \ x_r^T(-H_r^T H_r)x_r + 2c_r^T x_r$$

$$subject \ \ to \ \ x_r \in \{-1, 1\}^{2N} \tag{5}$$

where $c_r^T = y_r^T H_r$. Let $\bar{H} = -H_r^T H_r$, and including a redundant dummy variable, u_{2N+1}, we can express the above equation as,

$$\begin{pmatrix} x_r^*, & u_{2N+1}^* \end{pmatrix} = max_{x_r, u_{2N+1}} \begin{pmatrix} x_r^T & u_{2N+1} \end{pmatrix} \begin{pmatrix} \bar{H} & c \\ c^T & 0 \end{pmatrix}$$

$$\cdot \begin{pmatrix} x_r \\ u_{2N+1} \end{pmatrix}$$

$$subject\ to \begin{pmatrix} x_r \\ u_{2N+1} \end{pmatrix} \in \{-1, 1\}^{2N+1},\ u_{2N+1} = 1 \quad (6)$$

Since the cost function is symmetric, $u_{2N+1} = 1$ need not to be maintained explicitly. We reformulate the above equation as

$$x^* = arg\ max\ x^T Q x$$

$$subject\ to\ x \in \{-1, 1\}^{2N+1} \quad (7)$$

where

$$Q = \begin{pmatrix} \bar{H} & c \\ c^T & 0 \end{pmatrix}\ and\ x = \begin{pmatrix} x_r^T & u_{2N+1}^T \end{pmatrix}$$

The above problem is NP-hard. To present semidefinite relaxation, we consider a reformulation of the Boolean quadratic program. Since $x^T Q x = Tr(Qxx^T)$, Tr(.) is trace operator. The above problem is equivalent to the following Quadratic program (QP).

$$max\ Tr(QX)$$

$$subject\ to\ X = xx^T = rank(1),\ x \in R^{2N+1}$$

$$diag(X) = e,\ Tr(X) = 2N + 1 = b, \quad (8)$$

where e is all ones vector and $diag(X)$ is vector composed of diagonal elements of matrix X. Due to the constraint $X = xx^T$, the above problem is non-convex optimization problem. If the rank(1) constraint is removed from the above equation, we obtain the following relaxed problem:

$$max\ Tr(QX) = \sum_{i,j} c_{ij} x_i x_j$$

$$subject\ to\ X \geq 0$$

$$diag(X) = e,\ Tr(X) = 2N + 1 = b, \quad (9)$$

where $X \geq 0$ means that X is symmetric and positive semidefinite (PSD). The above problem is known as the primal semidefinite programming relaxation of the Boolean problem.

3. FRAMEWORK FOR THE ANALYSIS OF SDP APPROACH

Following seminal work of Goemans and Williamson [4], see also [5,6], some very interesting relaxations can be defined by allowing x_i to be multidimensional vector v_i of unit Euclidean norm, i.e.,

$$max\ \sum_{i,j} c_{ij} v_i^T v_j$$

$$subject\ to\ v_i^T v_i = 1,\ and\ v_i \in R^{2N+1}, \quad (10)$$

the above resulting optimization problem is the relaxation of the SDP because the objective function reduces to the objective function of the SDP in case of vectors lying in a 1-dimensional space. Since $X = V^T V$, v_i is the i^{th} column of V, if X is symmetric PSD. Choosing a random vector r uniformly from the unit sphere S^{2N} and let the solution be given as

$$\bar{x_i} = 1\ if\ v_i^T r \geq 0$$

$$\bar{x_i} = -1\ if\ v_i^T r < 0 \quad (11)$$

Choosing random r is same as choosing random hyperplane. Let X^* and v_i^* denote the optimal solution to the relaxation and obtain solution $\bar{x}$ as indicated above. Now we state a lemma, which is result of nontrivial observation in [4].

Lemma 1[4]:

$$P(\bar{x_i}\bar{x_j} = -1) = \frac{1}{\pi}\theta_{ij}$$

$$= \frac{1}{\pi} arcCos(v_i^{*^T} v_j^*) = \frac{1}{\pi} arcCos(x_{ij}^*) \quad (12)$$

i.e., the probability that the vector v_i and v_j are on the opposite sides of the hyperplane is exactly the proportion of the angle between v_i and v_j to π, i.e., $arcCos(v_i^T v_j)/\pi$. Now we are in position to derive the expected value of our solution. In this respect we have

$$E(\bar{x_i}\bar{x_j}) = (1 - \frac{1}{\pi} arcCos(x_{ij}^*)) - \frac{1}{\pi} arcCos(x_{ij}^*)$$

where E(.) is the expectation operator.

$$E(\bar{x_i}\bar{x_j}) = \frac{2}{\pi} arcSin(x_{ij}^*) \quad (13)$$

In deriving the above equation we have used the fact, $arcSin(x) + arcCos(x) = \pi/2$. Let $arcSin(X) = arcSin(x_{ij})$, and $\bar{X} = \bar{x_i}\bar{x_j}$. The expected value of our solution is

$$E(\sum_{i,j} c_{ij}\bar{x_i}\bar{x_j}) = \frac{2}{\pi} \sum_{i,j} c_{ij} arcSin(x_{ij}^*), \quad (14)$$

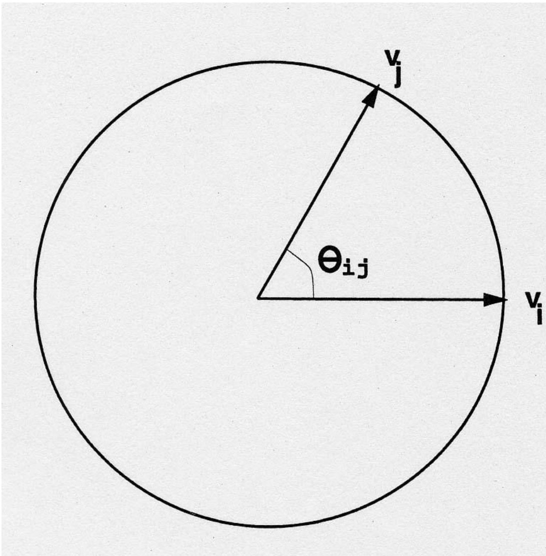

Fig. 1. Two vectors on unit sphere.

which can be further written as

$$E[Tr(Q\bar{X})] = \frac{2}{\pi} Tr(Q.arcSin(X)^*) \quad (15)$$

$$E[Tr(Q\bar{X})] \geq \frac{2}{\pi} Tr(QX^*). \quad (16)$$

In deriving the above inequality we used the fact that

$$arcSin(X) = X + \frac{1}{2}\frac{X^3}{3} + \frac{1.3}{2.4.5}X^5 \dots, \quad (17)$$

$$|x_{ij}| \leq 1 \;\; for \;\; all \;\; i,j$$

which implies that $arcSin(X) \geq X$. The value of the SDP relaxation is

$$\sum_{i,j} c_{ij} x^*_{ij} = Tr(QX^*) \geq optimal \;\; value \quad (18)$$

(note that X is a matrix).

$$E[Tr(Q\bar{X})] \geq \frac{2}{\pi} Tr(QX^*) \geq \frac{2}{\pi}.optimal \;\; value \quad (19)$$

Inequality (19) shows a lower bound on the expected value of the objective function for SDP relaxation.
The upper bound proceeds as follows:
Using eq (17) we have the following inequality,

$$arcSin(X) \leq X + \frac{1}{2.3} \sum_{j=1}^{2N+1} x^3_{jj} I + \frac{1.3}{2.4.5} \sum_{j=1}^{2N+1} x^5_{jj} I + \dots \quad (20)$$

where I is identity matrix. Knowing that $x_{jj} = 1$, we can write the above equation as

$$arcSin(X) \leq X + (2N+1)(\frac{1}{2.3} + \frac{1.3}{2.4.5} + \dots)I \quad (21)$$

where

$$arcSin(1) = \frac{\pi}{2} = 1 + \frac{1}{2.3} + \frac{1.3}{2.4.5} + \dots \quad (22)$$

Hence we can write

$$arcSin(X) \leq X + (2N+1)(\frac{\pi}{2} - 1)I \quad (23)$$

putting the above inequality in eq (15) and rearranging, we get

$$E[Tr(Q\bar{X})] \leq \frac{2}{\pi} Tr(QX^*) + (2N+1)(\frac{\pi}{2} - 1)Tr(Q) \quad (24)$$

The above inequality is the upper bound on the expected value of the SDP relaxation.

It is shown in [5] (we state the result without proof) that maximum and minimum value of the objective function is given by

$$\bar{q}(Q) = max_{X \geq 0} \;\; \frac{2}{\pi} Tr(Q.arcSin(X)), \;\; diag(X) = e \quad (25)$$

$$\underline{q}(Q) = min_{X \geq 0} \;\; \frac{2}{\pi} Tr(Q.arcSin(X)), \;\; diag(X) = e \quad (26)$$

where e is all ones vector. Having stated the above theorem we are now in position to prove the following theorem. Before proving it, we define the following terms: Let $\bar{q} = \bar{q}(Q)$, $\underline{q} = -\bar{q}(-Q)$ be the maximal and minimal objective value of x^TQx in the feasible set of the Quadratic program (QP). Let $\bar{p} = \bar{p}(Q)$ and $\underline{p} = -\bar{p}(-Q)$ be the maximal and minimal value of the objective function, x^TQX in the feasible set of SDP.
The dual of our SDP (eq (9)), can be written as [7]

$$min \;\; z^Te + yb$$

$$subject \;\; to \;\; Z + yI = Q, \;\; Z \geq 0 \quad (27)$$

where I is identity matrix, Z (slack variable) is a diagonal matrix with $diag(Z) = z$. If the primal has the finite optimal value, so as its dual, with the same objective value, i.e., there is no duality gap between the primal and the dual. The minimal objective value of the dual SDP is denoted by

$\underline{p} = e^T\underline{z} + b\underline{y}$, where $\underline{z}, \underline{y}$ are minimal values of the variables z and y respectively.

Theorem 1:
1. $\bar{q} - \underline{p} \geq \frac{2}{\pi}(\bar{p} - \underline{p})$
2. $\bar{p} - \underline{q} \geq \frac{2}{\pi}(\bar{p} - \underline{p})$
3. $\bar{p} - \underline{p} \geq \bar{q} - \underline{q} \geq \frac{4-\pi}{\pi}(\bar{p} - \underline{p})$

Proof: We know that

$$\frac{\pi}{2}\bar{q} \geq Tr(Q.arcSin(X)). \tag{28}$$

This can further be written as

$$\frac{\pi}{2}\bar{q} \geq Tr((Q - \underline{Z} - \underline{y}I + \underline{Z} + \underline{y}I).arcSin(X))$$

$$\frac{\pi}{2}\bar{q} \geq Tr((Q - \underline{Z} - \underline{y}I).X) + Tr((\underline{Z} + \underline{y}I).arcSin(X)),$$

where we have used the fact that $Sin(X) \geq X$. The above inequality can further be written as (after bit of algebra)

$$= Tr(QX) - \underline{z}^T diag(X) - \underline{y}b + \underline{z}^T\frac{\pi}{2}diag(X) + \underline{y}\frac{\pi}{2}b, \tag{29}$$

where we have used the fact that $diag(arcSin(X)) = \frac{\pi}{2}diag(X)$ and $diag(X)$ is the vector composed of diagonal elements of X. Knowing that $diag(X) = e$, we have

$$\frac{\pi}{2}\bar{q} \geq Tr(QX) + \underline{z}^T e(\frac{\pi}{2} - 1) + \underline{y}b(\frac{\pi}{2} - 1)$$

$$\frac{\pi}{2}\bar{q} \geq Tr(QX) + (\frac{\pi}{2} - 1)(\underline{z}^T e + \underline{y}b) \tag{30}$$

$$\frac{\pi}{2}\bar{q} \geq Tr(QX) + (\frac{\pi}{2} - 1)\underline{p}. \tag{31}$$

Let X converge to $\bar{X}$, then $Tr(QX) \to \bar{p}$. Putting $\bar{p}$ in the above inequality and rearranging gives

$$\bar{q} - \underline{p} \geq \frac{2}{\pi}(\bar{p} - \underline{p}), \tag{32}$$

hence we have proved the first inequality. The second inequality can be proved by replacing Q with $-Q$. The third inequality can be proved by adding the first two inequalities. Having proved Theorem1, we can quite straight forwardly prove the following theorem.

Theorem 2: Let $\hat{x}$ be generated above from $X = \bar{X}$. Then

$$\frac{\bar{q} - E_u q(\hat{x})}{\bar{q} - \underline{q}} \leq \frac{\pi}{2} - 1 \tag{33}$$

Proof: In order to prove the above theorem we have corollary [6], which is following

$$lim_{X \to \bar{X}} E_u q(\hat{X}) = lim_{X \to \bar{X}} \frac{2}{\pi}Tr(Q.arcSin(X))$$

$$\geq \frac{2}{\pi}\bar{p} + (1 - \frac{2}{\pi})\underline{p} \tag{34}$$

Using the Theorem 1, we can write

$$\frac{\bar{q} - E_u q(\hat{x})}{\bar{q} - \underline{q}} \leq \frac{\bar{q} - \frac{2}{\pi}\bar{p} - (1 - \frac{2}{\pi})\underline{p}}{\bar{q} - \underline{q}}$$

$$\leq \frac{\bar{q} - \frac{2}{\pi}\bar{p} - (1 - \frac{2}{\pi})\underline{p}}{\bar{q} - (1 - \frac{2}{\pi})\bar{p} - \frac{2}{\pi}\underline{p}}$$

$$\leq \frac{\bar{p} - \frac{2}{\pi}\bar{p} - (1 - \frac{2}{\pi})\underline{p}}{\bar{p} - (1 - \frac{2}{\pi})\bar{p} - \frac{2}{\pi}\underline{p}}$$

$$= \frac{(1 - \frac{2}{\pi})(\bar{p} - \underline{p})}{\frac{2}{\pi}(\bar{p} - \underline{p})}$$

$$= \frac{\pi}{2} - 1, \tag{35}$$

hence proving the theorem.

4. CONCLUSIONS

There have been some works by some authors to apply SDP approach to MLSD. The performance evaluation of the SDP approach were based on simulations. In this paper, we have shown the upper and lower bound for the SDP relaxation method. We also showed relationship between the maximum/minimum of the exact objective function to the maximum/minimum of the relaxed problem. Besides the theoretical analysis, we also carried out simulations to evaluate the performance of the SDP relaxation approach. The simulations are performed for two transmit and two receive antennas. Fig (2) shows the performance in terms of BER of the SDP relaxation approach and second order cone programming (SOCP) approach. It is clear from the figure that the performance is very close to the exact ML.

5. REFERENCES

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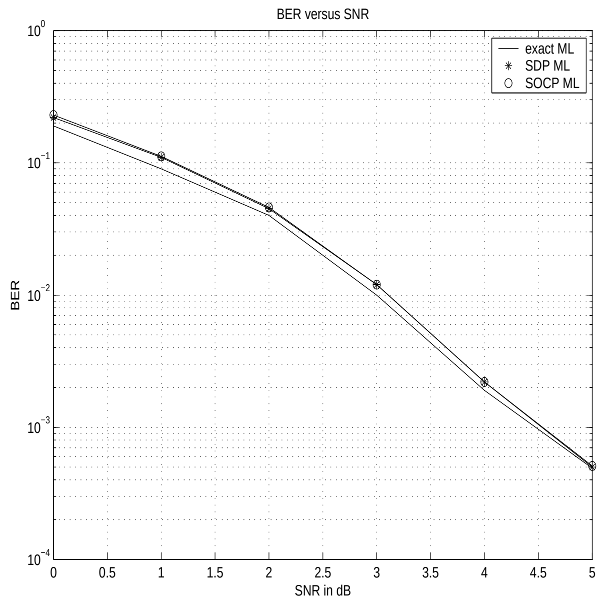

Fig. 2. Av. BER for N=2, M=2 vs $SNR(dB)$.

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